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Midterm Exam in TMA4225 Foundations of Analysis

English
Thursday, October 2, 2008
Hours: 14.15–16
Aids: Code B – All printed and handwritten aids permitted. Approved calculator permitted.

Solutions

Problem 1 Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that $|f|$ is measurable. Does it follow that f is measurable? If yes, prove it. If no, give a counterexample.
Solution. *The answer is no. Here is a counterexample: Let $\mathcal{N} \subset \mathbf{R}$ be a non-measurable set, and consider $f = \chi_{\mathcal{N}^c} - \chi_{\mathcal{N}}$. Then $\{f < 0\} = \mathcal{N}$, a non-measurable set. On the other hand, since $|f|(x) = 1$ for all $x \in \mathbf{R}$, $|f|$ is clearly measurable. Thus $|f|$ is measurable and f is not.*

Problem 2 Let $f\colon \mathbf{R} \to \mathbf{R}$ be a non-decreasing function, i.e., $x \leq y \implies f(x) \leq f(y)$. Show that f is measurable.
Solution. *Let $c \in \mathbf{R}$ and set $E_c = \{f < c\}$. We consider 3 cases: (i) $E_c = \emptyset$ (ii) $E_c = \mathbf{R}$ (iii) $\emptyset \neq E_c \neq \mathbf{R}$.*
In case (iii) set $M = \sup E_c$. If $x \in E_c$ and $y < x$, then $f(y) \leq f(x) \leq c$, so $y \in E_c$, too. Thus $E_c = (-\infty, M)$ or $E_c = (-\infty, M]$ (according to as whether $M \in E_c$ or not).
So all in all we have the following 4 possibilities for $\{f < c\}$: $\emptyset$, $\mathbf{R}$, $(-\infty, M)$, $(-\infty, M]$. These are all measurable, hence f is measurable.

Problem 3 Let $E_1, \ldots, E_n$ be measurable subsets of $[0,1]$, and suppose that each $x \in [0,1]$ belongs to at least k of these sets. Prove that $\max_j m(E_j) \geq \frac{k}{n}$.
[*Hint:* Consider the function $f(x) = \sum_{j=1}^n \chi_{E_j}(x)$.]
Solution. *Since each $x \in [0,1]$ belongs to at least k of the sets, we have $f(x) \geq k$ for all $x \in [0,1]$. Thus:*

$$\sum_{j=1}^n m(E_j) = \sum_{j=1}^n \int_{[0,1]} \chi_{E_j}(x)\,dx = \int_{[0,1]} f(x)\,dx \geq k\,.$$

It follows that we must have $m(E_j) \geq \frac{k}{n}$ for at least one $j \in \{1, 2, \ldots, n\}$, i.e, $\max_j m(E_j) \geq \frac{k}{n}$.

Problem 4 Let E be a bounded subset of $[0,\infty)$, i.e., $E \subset [0,M]$ for some $M > 0$. Show that if $m_*(E) = 0$ and $F = \{x^2 \colon x \in E\}$, then $m_*(F) = 0$.

Solution. *Let $\epsilon > 0$ be given. Since $m_*(E) = 0$, we can find intervals $[a_j, b_j]$ such that $E \subset \cup_{j=1}^\infty [a_j, b_j]$ and $\sum_{j=1}^\infty (b_j - a_j) < \epsilon$. We may assume $a_j \ge 0$ and $b_j \le M$ for all j. Then $F \subset \cup_{j=1}^\infty [a_j^2, b_j^2]$ and hence*

$$m_*(F) \le \sum_{j=1}^\infty (b_j^2 - a_j^2) = \sum_{j=1}^\infty (b_j - a_j)(b_j + a_j) \le 2M \sum_{j=1}^\infty (b_j - a_j) < 2M\epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we must have $m_*(F) = 0$.

Problem 5 Let $\{E_k\}_{k=1}^\infty$ be a sequence of measurable sets, and define

$$\liminf_{k\to\infty} E_k = \cup_{n=1}^\infty (\cap_{k=n}^\infty E_k), \quad \limsup_{k\to\infty} E_k = \cap_{n=1}^\infty (\cup_{k=n}^\infty E_k)\,.$$

Show that

$$m(\liminf_{k\to\infty} E_k) \le \liminf_{k\to\infty} m(E_k)$$

and

$$m(\limsup_{k\to\infty} E_k) \ge \limsup_{k\to\infty} m(E_k) \quad \text{if } m(\cup_{k=1}^\infty E_k) < \infty.$$

Solution. *We first prove $m(\liminf_{k\to\infty} E_k) \le \liminf_{k\to\infty} m(E_k)$. Set $F_n = \cap_{k=n}^\infty E_k$. Then $F_1 \subset F_2 \subset F_3 \subset \cdots$, so by Corollary 3.3(i) on page 20 we have $m(\cup_{n=1}^\infty F_n) = \lim_{n\to\infty} m(F_n) = \sup_{n\to\infty} m(F_n)$. Also, $F_n \subset E_k$ for all $k \ge n$, so $m(F_n) \le m(E_k)$ for all $k \ge n$, i.e., $m(F_n) \le \inf_{k\ge n} m(E_k)$. This gives*

$$\begin{aligned} m(\liminf_{k\to\infty} E_k) &= m(\cup_{n=1}^\infty (\cap_{k=n}^\infty E_k)) = m(\cup_{n=1}^\infty F_n) = \sup_n m(F_n) \\ &\le \sup_n (\inf_{k\ge n} m(E_k)) = \liminf_{k\to\infty} m(E_k)\,. \end{aligned}$$

Next we prove $m(\limsup_{k\to\infty} E_k) \ge \limsup_{k\to\infty} m(E_k)$. Set $F_n = \cup_{k=n}^\infty E_k$. Then $F_1 \supset F_2 \supset F_3 \supset \cdots$, so since $m(F_1) = m(\cup_{k=1}^\infty E_k) < \infty$, we get by Corollary 3.3(ii) on page 20 that $m(\cap_{n=1}^\infty F_n) = \lim_{n\to\infty} m(F_n) = \inf_{n\to\infty} m(F_n)$. Also, $F_n \supset E_k$ for all $k \ge n$, so $m(F_n) \ge m(E_k)$ for all $k \ge n$, i.e., $m(F_n) \ge \sup_{k\ge n} m(E_k)$. This gives

$$\begin{aligned} m(\limsup_{k\to\infty} E_k) &= m(\cap_{n=1}^\infty (\cup_{k=n}^\infty E_k)) = m(\cap_{n=1}^\infty F_n) = \inf_n m(F_n) \\ &\ge \inf_n (\sup_{k\ge n} m(E_k)) = \limsup_{k\to\infty} m(E_k)\,. \end{aligned}$$

Problem 6 Compute

$$\lim_{n\to\infty} \int_0^1 n \sin(\frac{x^2}{n})\, dx \,.$$

State which theorem(s) you are using and justify your steps.

Solution. *We plan to use the Bounded Convergence Theorem (BCT) (Theorem 1.4, page 56). We will also make use of the following elementary facts from calculus:* $|\sin t| \le |t|$ *for all* $t \in \mathbf{R}$ *and* $\lim_{t\to 0} \frac{\sin(at)}{t} = a$ *for* $a \in \mathbf{R}$.

Set $f_n(x) = n\sin(\frac{x^2}{n})$. *Then* $|f_n(x)| = n|\sin(\frac{x^2}{n})| \le n\frac{x^2}{n} = x^2 \le 1$ *for* $x \in [0,1]$ *and* $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\sin(\frac{x^2}{n}) = \lim_{n\to\infty} \sin(\frac{x^2}{n})/\frac{1}{n} = \lim_{t\to 0} \frac{\sin(x^2 t)}{t} = x^2$ *for all* x. *All the functions* f_n *are clearly measurable (being continuous), the integration takes place on a set* $E = [0,1]$ *of finite measure, and they are bounded by a common constant* $M = 1$ *on* E. *So all the conditions of the BCT are fulfilled, and we get:*

$$\lim_{n\to\infty} \int_0^1 n\sin(\frac{x^2}{n})\, dx \overset{BCT}{=} \int_0^1 \lim_{n\to\infty} n\sin(\frac{x^2}{n})\, dx = \int_0^1 x^2\, dx = \frac{1}{3}\,.$$