



Contact during the exam:

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Final Exam in TMA4225 Foundations of Analysis

English

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Hours: 09.00–13.00

Aids: Code B – All printed and handwritten aids permitted. Approved calculator permitted.

Grades will be announced no later than: December 22, 2008.

Solutions

Problem 1

a) Let $f_n(x) = \frac{1}{n}\chi_{[-n,n]}(x)$ for $x \in \mathbf{R}$, $n \in \mathbf{N}$. Show that f_n converges uniformly to 0 on \mathbf{R} .

Solution. For a given $\epsilon > 0$ choose an $N > 1/\epsilon$. Then for $n > N$ we have $|f_n(x)| \leq 1/n < 1/N < \epsilon$, showing uniform convergence to 0 (since N does not depend on x).

b) Is there a Lebesgue integrable function g defined on \mathbf{R} such that $f_n(x) \leq g(x)$ for all $x \in \mathbf{R}$, $n \in \mathbf{N}$? Justify your answer.

Solution. The answer is 'no'. If such a g existed, we would have $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n = \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n$ by the Dominated Convergence Theorem. But in our case

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n = \lim_{n \rightarrow \infty} \int_{-n}^n 1/n = \lim_{n \rightarrow \infty} (1/n \cdot 2n) = \lim_{n \rightarrow \infty} 2 = 2,$$

whereas $\int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n = \int_{-\infty}^{\infty} 0 = 0$; thus $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n$.

Problem 2 Let X be a set, and A and B two subsets of X . Explain why $|\chi_B - \chi_A|$ is the characteristic function of some subset $C \subset X$, and express C in terms of A and B .

Solution. Set $f = |\chi_B - \chi_A|$. The only possible values for f are 0 and 1, hence it is the characteristic function of some set. Now $f(x) = 1$ if and only if $f(x) \neq 0$ if and only if $\chi_B(x) \neq \chi_A(x)$, that is, when $x \in A \setminus B$ or $x \in B \setminus A$. Thus $C = (A \setminus B) \cup (B \setminus A) = A \Delta B$ (symmetric difference).

Problem 3 Suppose f is nonnegative and integrable on \mathbf{R}^d . For each $y > 0$, let $E_y = \{x \in \mathbf{R}^d : f(x) > y\}$. Show that

$$\int_{\mathbf{R}^d} f(x) dx = \int_0^\infty m(E_y) dy.$$

Justify your steps, and indicate which theorems you are using.

[Hint: Consider the subgraph of f , given by $\{(x, y) \in \mathbf{R}^d \times \mathbf{R} \mid 0 < y < f(x)\}$.]

Solution. As in the proof of Corollary 3.8 on page 85 we consider the function $F(x, y) = y - f(x)$ on $\mathbf{R}^d \times \mathbf{R}$. By Corollary 3.7, p. 85, the functions $F_1(x, y) = y$ and $F_2(x, y) = f(x)$ are measurable on $\mathbf{R}^d \times \mathbf{R}$, hence so is $F = F_1 - F_2$. Denoting the subgraph of f by A , it follows that $A = \{F < 0\} \cap \{F_1 > 0\}$ is measurable. Further, with notation as on p. 75: $A^y = \{x \in \mathbf{R}^d : f(x) > y\} = E_y$ for each fixed $y > 0$, and $A_x = \{y \in \mathbf{R} \mid 0 < y < f(x)\} = (0, f(x))$, for each fixed $x \in \mathbf{R}^d$.

By Theorem 3.2 on page 81, since χ_A is measurable and non-negative, the integral $m(A) = \int_{\mathbf{R}^d \times \mathbf{R}} \chi_A(x, y) dx dy$ can be computed as an iterated integral in any order, thus¹:

$$\begin{aligned} m(A) &= \int_{\mathbf{R}^d \times \mathbf{R}} \chi_A(x, y) dx dy = \int_{\mathbf{R}} \left(\int_{\mathbf{R}^d} \chi_A(x, y) dx \right) dy = \int_{\mathbf{R}} \left(\int_{\mathbf{R}^d} \chi_{A^y}(x) dx \right) dy \\ &= \int_{\mathbf{R}} m(A^y) dy \stackrel{\text{see footnote}^1}{=} \int_0^\infty m(A^y) dy = \int_0^\infty m(E_y) dy \end{aligned}$$

and also

$$\begin{aligned} m(A) &= \int_{\mathbf{R}^d \times \mathbf{R}} \chi_A(x, y) dx dy = \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}} \chi_A(x, y) dy \right) dx = \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}} \chi_{A_x}(y) dy \right) dx \\ &= \int_{\mathbf{R}^d} m(A_x) dx = \int_{\mathbf{R}^d} f(x) dx \end{aligned}$$

hence

$$\int_{\mathbf{R}^d} f(x) dx = \int_0^\infty m(E_y) dy.$$

¹Note that $A^y = \emptyset$ for $y \leq 0$.

Problem 4 Let $f: [0, \infty) \rightarrow \mathbf{R}$ be defined by

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t > 0 \\ 1, & t = 0. \end{cases}$$

a) Set $a_n = \int_{n\pi}^{(n+1)\pi} f(t) dt$ for $n \geq 0$. Prove that the series $\sum_{n=0}^{\infty} a_n$ converges by showing that: (i) The series is alternating, (ii) $|a_n| \geq |a_{n+1}|$ for $n \geq 0$, (iii) $\lim_{n \rightarrow \infty} a_n = 0$.

Use this to show that the improper Riemann integral $\int_{[0, \infty)}^{\mathcal{R}} f(t) dt \stackrel{\text{def}}{=} \lim_{b \rightarrow \infty} \int_0^b f(t) dt$ exists.

[Hint: Show that $\frac{2}{(n+1)\pi} \leq |a_n| \leq \frac{2}{n\pi}$ for $n \geq 1$.]

Solution. Set $I_n = [n\pi, (n+1)\pi]$. The function f is of constant sign on each I_n , but changes sign when going from I_n to I_{n+1} . This implies that the series is alternating and that $|\int_{I_n} f(t) dt| = \int_{I_n} |f(t)| dt$ for all n . Further, for $n \geq 1$:

$$\begin{aligned} |a_n| &= \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin t}{t} \right| dt \geq \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin t| dt = \frac{1}{(n+1)\pi} \int_0^\pi \sin t dt = \frac{2}{(n+1)\pi} \\ |a_n| &= \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin t}{t} \right| dt \leq \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} |\sin t| dt = \frac{1}{n\pi} \int_0^\pi \sin t dt = \frac{2}{n\pi} \end{aligned}$$

This gives $|a_n| \geq \frac{2}{(n+1)\pi} \geq |a_{n+1}|$ and $\lim a_n = 0$. Thus the series converges by the alternating series test, i.e., the limit $L = \lim_{n \rightarrow \infty} \int_0^{n\pi} f(t) dt = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} a_i$ exists.

To finish the proof, we need to show that $\int_0^b f(t) dt$ is close to L for all sufficiently large b (and not just for b of the form $n\pi$); i.e., we must find a b_0 such that $|\int_0^b f(t) dt - L|$ is small for all real numbers $b \geq b_0$. To this end, let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} \int_0^{n\pi} f(t) dt = L$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \int_{n\pi}^{(n+1)\pi} f(t) dt = 0$, we can find a natural number N such that $|\int_0^{n\pi} f(t) dt - L| < \epsilon/2$ and $|\int_{n\pi}^{(n+1)\pi} f(t) dt| = \int_{n\pi}^{(n+1)\pi} |f(t)| dt < \epsilon/2$ for $n \geq N$. If $b \geq N\pi$, then $b \in I_m$ for some $m \geq N$, and we get:

$$\begin{aligned} \left| \int_0^b f(t) dt - L \right| &= \left| \int_0^{m\pi} f(t) dt + \int_{m\pi}^b f(t) dt - L \right| \leq \left| \int_0^{m\pi} f(t) dt - L \right| + \int_{m\pi}^b |f(t)| dt \\ &\leq \left| \int_0^{m\pi} f(t) dt - L \right| + \int_{m\pi}^{(m+1)\pi} |f(t)| dt < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

since $m \geq N$. Thus $|\int_0^b f(t) dt - L| < \epsilon$ for $b \geq b_0$ with $b_0 = N\pi$, and N as above.

b) Show that the function f is *not* Lebesgue integrable over $[0, \infty)$.

Solution. For all N we have

$$\int_0^\infty |f| \geq \int_0^{N\pi} |f| = \sum_{n=0}^{(N-1)} \int_{I_n} |f| \geq \sum_{n=0}^{(N-1)} \frac{2}{(n+1)\pi},$$

and since $\sum_{n=0}^{(N-1)} \frac{2}{(n+1)\pi} \rightarrow \infty$ as $N \rightarrow \infty$, we get $\int_0^\infty |f| = \infty$, i.e., f is not Lebesgue integrable.

Problem 5

a) Let $b > 0$ and define $f: [0, b] \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}}, & x \in (0, b] \\ 0, & x = 0. \end{cases}$$

Prove f is Lebesgue integrable, and compute the value of the Lebesgue integral $\int_0^b f(x) dx$.

[Hint: Consider the sequence $f_n = f\chi_{[\frac{1}{n}, b]}$, and use the Monotone Convergence Theorem.

You may use the Fundamental Theorem of Calculus to evaluate any Riemann integral you encounter along the way.]

Solution. With $f_n = f\chi_{[\frac{1}{n}, b]}$ we have $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in [0, b]$ and $0 \leq f_n \leq f_{n+1}$ for all n . Since all functions involved clearly are measurable, the Monotone Convergence Theorem implies that $\lim_{n \rightarrow \infty} \int_0^b f_n = \int_0^b \lim_{n \rightarrow \infty} f_n$; that is: $\lim_{n \rightarrow \infty} \int_{1/n}^b f = \int_0^b f$. The integral $\int_{1/n}^b f$ can be computed as an ordinary Riemann integral (since f is continuous on $[1/n, b]$, and the interval $[1/n, b]$ is bounded):

$$\int_{1/n}^b f = \int_{1/n}^b x^{-\frac{1}{2}} dx = 2(b^{\frac{1}{2}} - n^{-\frac{1}{2}})$$

and so

$$\int_0^b f = \lim_{n \rightarrow \infty} \int_{1/n}^b f = \lim_{n \rightarrow \infty} 2(b^{\frac{1}{2}} - n^{-\frac{1}{2}}) = 2b^{\frac{1}{2}}$$

b) Define $g: [0, 1] \rightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} x^{\frac{3}{2}} \sin(\frac{1}{x}), & x \in (0, 1] \\ 0, & x = 0. \end{cases}$$

Show that g' is integrable on $[0, 1]$ and that $g(x) = \int_0^x g'(t) dt$ for $x \in [0, 1]$. Conclude that g is absolutely continuous, hence of bounded variation, on $[0, 1]$.

[Hint: To prove $g(x) = \int_0^x g'(t) dt$, use an argument similar to that in a), but this time in conjunction with the Dominated Convergence Theorem.]

Solution. We have:

$$g'(x) = \begin{cases} \frac{3}{2}x^{\frac{1}{2}} \sin(\frac{1}{x}) - x^{-\frac{1}{2}} \cos(\frac{1}{x}), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

This gives $|g'(x)| \leq |\frac{3}{2}x^{\frac{1}{2}} \sin(\frac{1}{x})| + |x^{-\frac{1}{2}} \cos(\frac{1}{x})| \leq \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ for $x \in (0, 1]$, hence, with f as in part a), $|g'(x)| \leq \frac{3}{2}x^{\frac{1}{2}} + f(x)$ for all $x \in [0, 1]$. The first term, $\frac{3}{2}x^{\frac{1}{2}}$, is integrable on $[0, 1]$ (being continuous), and the second term, $f(x)$, is integrable by part a). It follows that g' is integrable on $[0, 1]$.

Now fix an $x \in [0, 1]$ and set $\psi_n(t) = g'\chi_{[1/n, x]}(t)$ for $t \in [0, x]$. Then $|\psi_n| \leq |g'|$ on $[0, x]$ for all n , and

$\lim_{n \rightarrow \infty} \psi_n(t) = g'(t)$ for all $t \in [0, x]$. Since g' is integrable on $[0, x]$, and since all the ψ_n are dominated by $|g'|$, it follows by the Dominated Convergence Theorem that

$$\lim_{n \rightarrow \infty} \int_0^x \psi_n = \int_0^x \lim_{n \rightarrow \infty} \psi_n.$$

But $\int_0^x \psi_n = \int_0^x g' \chi_{[1/n, x]} = \int_{1/n}^x g' = g(x) - g(1/n)$ and $\lim_{n \rightarrow \infty} \psi_n = g'$ on $[0, x]$, so this gives

$$\begin{aligned} \lim_{n \rightarrow \infty} (g(x) - g(1/n)) &= \int_0^x g'(t) dt \\ g(x) &= \int_0^x g'(t) dt, \end{aligned}$$

where in the last equality we used that $\lim_{n \rightarrow \infty} g(1/n) = g(0) = 0$, which follows from the continuity of g at 0.

Since we have exhibited g as an indefinite integral of its derivative, Theorem 14 on page 110 in Royden tells us that g is absolutely continuous, hence of bounded variation, on $[0, 1]$

c) Define $h: [0, 1] \rightarrow \mathbf{R}$ by

$$h(x) = \begin{cases} x \sin(\frac{1}{x}), & x \in (0, 1] \\ 0, & x = 0. \end{cases}$$

Show that h is *not* of bounded variation on $[0, 1]$.

Solution. Set $x_n = \frac{2}{(2n+1)\pi}$. Then

$$\begin{aligned} \sum_{n=1}^N |h(x_n) - h(x_{n-1})| &= \sum_{n=1}^N \left| \frac{2}{(2n+1)\pi} (-1)^n - \frac{2}{(2n-1)\pi} (-1)^{(n-1)} \right| \\ &= \frac{2}{\pi} \sum_{n=1}^N \left(\frac{1}{2n+1} + \frac{1}{2n-1} \right) \rightarrow \infty \quad \text{as } N \rightarrow \infty, \end{aligned}$$

hence h is not of bounded variation.

d) Consider the Cantor-Lebesgue function $F: [0, 1] \rightarrow [0, 1]$ as defined in Exercise 2(d) on page 38 in the textbook. You may use the following facts: F is a non-decreasing, uniformly continuous function which maps $[0, 1]$ onto $[0, 1]$; in addition, F is differentiable a.e. with $F' = 0$ a.e.

Is F of bounded variation? Is F absolutely continuous? Justify your answers.

Solution. Of bounded variation? Yes, since F is non-decreasing.

Absolutely continuous? No. Absolute continuity on $[0, 1]$ has two requirements: 1) F' is integrable on $[0, 1]$. 2) $F(x) - F(0) = \int_0^x F'(t) dt$ for all $x \in [0, 1]$. Here the first requirement is fulfilled since $F' = 0$ a.e., but the second one is not since $F(1) - F(0) = 1 \neq \int_0^1 F'(x) dx = \int_0^1 0 dx = 0$.