Topology: some definitions.

A *topology* is a family of sets $\mathcal{U}$ closed under finite intersection and arbitrary unions, that is if

if $U, U' \in \mathcal{U}$, then $U \cap U' \in \mathcal{U}$

if $\mathcal{I} \subseteq \mathcal{U}$, then $\bigcup_{U \in \mathcal{I}} U \in \mathcal{U}$.

Note that the set $X = \bigcup_{U \in \mathcal{U}} U$ is a member of $\mathcal{U}$. We say that $\mathcal{U}$ is a *topology on* X, that $(X, \mathcal{U})$ is a *topological space*. Frequently we will even refer to X as a topological space when U is evident from the context.
The members of $\mathcal{U}$ are called the *open sets* of X with respect to the topology $\mathcal{U}$. A subset C of X is *closed* if the complement $\{x \in X | x \notin C\}$ is open.
A subset of X is called a *neighborhood* of $x \in X$ if it contains an open set containing x.
If $(X, \mathcal{U})$ is a topological space, a subfamily $\mathcal{B} \subseteq \mathcal{U}$ is a *base for the topology* $\mathcal{U}$ if for each $x \in X$ and each neighborhood V there is a $U \in \mathcal{B}$ such that $x \in U \subseteq V$.
This is equivalent to the condition that each member of $\mathcal{U}$ is a union of members of $\mathcal{B}$. Conversely, given a family of sets $\mathcal{B}$ closed under intersection, then $\mathcal{B}$ is a base for the topology on $X = \bigcup_{U \in \mathcal{B}} U$ given by declaring the open sets to be arbitrary unions from $\mathcal{B}$.
Example: the real line gets the usual topology if we take the open intervals (possibly empty) to be a basis for the topology.
Excercise: The real line has a countable basis for its topology: Can you prove that the set containing only the intervals (a, b) when a and b varies over the rational numbers is a basis for the usual topology on the real numbers?
A topological space $(X, \mathcal{U})$ is *Hausdorff* if for any two $x, y \in X$ there exist disjoint neighborhoods x and y.
Excercise: The real line is Hausdorff.

Continuous maps

A *continuous map* (or simply a map)

$$f\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$$

is a function $f\colon X \to Y$ such that for every $V \in \mathcal{V}$ the inverse image $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is in $\mathcal{U}$.
Excercise: Prove that a continuous map on the real line is just what you expect.
A *homeomorphism* is a continuous map $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ with a continuous inverse, i.eȧ continuous map $g\colon (Y,\mathcal{V}) \to (X,\mathcal{U})$ with $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in X$ and $y \in Y$.

Subspaces and quotient spaces

Let $(X,\mathcal{U})$ be a topological space. A *subspace* of $(X,\mathcal{U})$ is a subset $A \subset X$ with the topology given letting the open sets be $\{A \cap U | U \in \mathcal{U}\}$.
Let $(X,\mathcal{U})$ be a topological space, and consider an equivalence relation $\sim$ on X. The *quotient space* space with respect to the equivalence relation is the set $X/\sim$ with the *quotient topology*. The quotient topology is defined as follows: given letting the open sets be the subsets $V \subseteq X/\sim$ such that

Compact spaces

A *compact space* is a space $(X,\mathcal{U})$ with the following property:in any set $\mathcal{V}$ of open sets covering X (i.e. $\bigcup_{V\in\mathcal{V}} V = X$) there is a finite subset that also covers X.
We list without proof the results

> If $f\colon X \to Y$ is a continuous map and X is compact, then $f(X)$ is compact.
>
> A subset of $\mathbf{R}^n$ is compact iff it is closed and of finite size.

References

John L. Kelley, General Topology
James R. Munkres, Topology
Nicolas Bourbaki, Topologie générale
Per Holm og Jon Reed, Topologi