Excision in K-theory
Talk at the Institut Mittag-Leffler April 2006

Bjørn Ian Dundas (joint with Harald Kittang)

April 11, 2006

Brave New Algebraic Geometry with singularities?

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- How do we assemble local information?
- In particular: how do we paste vector bundles? (whose geometric interpretation in BNAG is still up for grabs)

We even want to consider what can be said while allowing singularities [and so the questions are not accessible through motivic methods even in OAG].

Topological K-theory

Topological K-theory is a cohomology theory and satisfies excision. If

$$\begin{array}{ccc} X_{12} & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_0 \end{array}$$

is a (homotopy) cocartesian square of spaces, then

$$\begin{array}{ccc} K(X_{12}) & \longleftarrow & K(X_1) \\ \uparrow & & \uparrow \\ K(X_2) & \longleftarrow & K(X_0) \end{array}$$

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Algebraic K-theory

In AG one would consider a pushout square like

$$\begin{array}{ccc} X_{12} & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_0 \end{array}$$

in schemes, and ask whether the corresponding square of algebraic K-theory spectra were homotopy cartesian.

Let us look at the affine case. Let

$$\mathcal{A} = \left\{ \begin{array}{ccc} A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_{12} \end{array} \right\}$$

be a cartesian square of rings (corresponding in the commutative case to a cocartesian square

$$Spec(\mathcal{A}) = \left\{ \begin{array}{ccc} Spec(A_0) & \longleftarrow & Spec(A_1) \\ \uparrow & & \uparrow \\ Spec(A_2) & \longleftarrow & Spec(A_{12}) \end{array} \right\}$$

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do 4 examples

Examples

- **The disjoint union:**

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A_1 \times A_2 & \longrightarrow & A_1 \\
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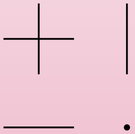

Examples (cont.)

- **The cusp:**

$$\begin{array}{ccc} k[x,y]/x^2-y^3 & \xrightarrow[]{\substack{x\mapsto t^3\\ y\mapsto t^2}} & k[t] \\ \downarrow & & \downarrow \\ k & \longrightarrow & k[t]/t^2 \end{array}$$

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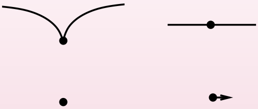

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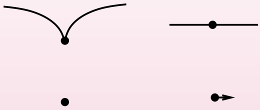

- **The Rim square:**

$$\begin{array}{ccc} \mathbf{Z}[C_p] & \longrightarrow & \mathbf{Z}[\zeta_p] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{F}_p \end{array}$$

Bass' K_1 and Milnor's theorem

Bass' program for higher K-theory rested on getting this right: he realized that the Whitehead groups K_1 could be built out of "clutching functions" so that excision seemed secure.

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Bass' K_1 and Milnor's theorem

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Theorem [Milnor]

Let $\mathcal{A}$ be a cartesian square of rings where the vertical maps are surjections. Then the corresponding square of categories of finitely generated modules $\mathcal{P}_{\mathcal{A}}$ is a fiber square.

Excision in low dimensions

In the end Milnor establishes that if all maps in the cartesian square are surjections you get a Mayer-Vietoris sequence

$$K_2(A_1) \oplus K_2(A_2) \to K_2(A_{12}) \to$$
$$K_1(A_0) \to K_1(A_1) \oplus K_1(A_2) \to K_1(A_{12})$$
$$\to K_0(A_0) \to K_0(A_1) \oplus K_0(A_2)$$

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(without the K_2 and with half of the surjectivity this is in Bass).

The projective line

Non affine example:

$$\begin{array}{ccc} \mathcal{P}_{\mathbf{P}^1_k} & \longrightarrow & \mathcal{P}_{k[t]} \\ \downarrow & & \downarrow \\ \mathcal{P}_{k[t^{-1}]} & \longrightarrow & \mathcal{P}_{k[t,t^{-1}]} \end{array}$$

is a fiber square

Fact:

$$\begin{array}{ccc} K(\mathbf{P}^1_k) & \longrightarrow & K(k[t]) \\ \downarrow & & \downarrow \\ K(k[t^{-1}]) & \longrightarrow & K(k[t,t^{-1}]) \end{array}$$

is homotopy cartesian.

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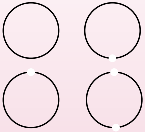

is a fiber square (the vector bundles on $\mathbf{P}^1_k$ are gotten by gluing together two bundles over affine lines).

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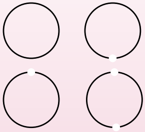

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is homotopy cartesian.

However, Swan showed that there was no way this could continue: there is no functor K_3 extending this sequence:

Example: the coordinate axes

$$\begin{array}{ccc} k[x,y]/xy & \longrightarrow & k[x] \\ \downarrow & & \downarrow \\ k[y] & \longrightarrow & k \end{array}$$

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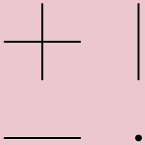

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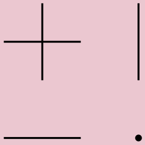

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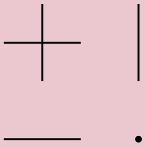

With $k = \mathbf{Z}$ you get that $K_2 k[x,y]/xy \cong K_2 k \oplus \mathbf{Z}$ (where the last factor is generated by $[x_{12}^{x}, x_{21}^{y}] \in K_2 = ker\{St \to E\}$ if I remember correctly).

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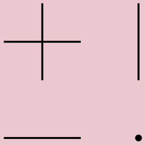

With $k = \mathbf{Z}$ you get that $K_2 k[x,y]/xy \cong K_2 k \oplus \mathbf{Z}$. But then the fact that the surjections are split and functoriality of K_3 forces the boundary map in

$$K_3(k[x]) \oplus K_3(k[y]) \to K_3(k) \to \\ K_2(k[x,y]/xy) \to K_2(k[x]) \oplus K_2(k[y])$$

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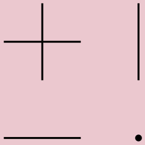

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to be zero, but the infinite cyclic factor of $K_2(k[x,y]/xy)$ can't map injectively into $K_2(k[x]) \oplus K_2(k[y]) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

Damage control

So, Milnor's theorem is not enough: that the square of exact categories is a fiber square does not guarantee that its K-theory will be homotopy cartesian.
Can one control this breakdown of excision?

First step in this direction came around 1980 when Guin-Waléry and Loday showed (with a slight reinterpretation) that if

$$\mathcal{A} = \left\{ \begin{array}{ccc} A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow f_1 \\ A_2 & \xrightarrow{f_2} & A_{12} \end{array} \right\}$$

is a cartesian square of Q-algebras with all maps surjective and $I_j = ker\{f_j\}$, then the first nonvanishing obstruction group satisfies

$$\pi_2 ifiK(\mathcal{A}) \cong I_1/I_1^2 \otimes_{A_0^o \otimes A_0} I_2/I_2^2 \cong \pi_1 ifiHC(\mathcal{A})$$

where *ifi* is the iterated fiber of a cube of spectra, and *HC* is cyclic homology.

The KABI-conjecture

It was soon conjectured (at least by Weibel, and perhaps by others) that if A_0 is a Q-algebra, then

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for all q .

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for all q [give a brief history of the KABI-conjecture].

Resolving the classical case

The conjecture remained unresolved until recently when Cortiñas proved it using the techniques of Suslin and Wodzicki [Ann. Math. 92].

Resolving the classical case

Theorem [Cortiñas, Inv. math. 06]

If $\mathcal{A}$ is a cartesian square of Q-algebras with vertical maps surjective, then the trace map induces an equivalence of spectra

$$ifiK(\mathcal{A}) \xrightarrow{\sim} ifiHC^-(\mathcal{A}) \xleftarrow{\sim} S^1 \wedge ifiHC(\mathcal{A}).$$

Considering a map of squares as a cube, Cortiñas theorem says that both the cubes

$$K(\mathcal{A}) \to HC^-(\mathcal{A}), \text{ and } S^1 \wedge HC(\mathcal{A}) \to HC^-(\mathcal{A})$$

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The connection between cyclic homology and negative cyclic homology derives from the fact that periodic cyclic homology does satisfy excision ($HP(\mathcal{A})$ is homotopy cartesian).

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Example, the cusp

Let k be a field of characteristic 0

$$\begin{array}{ccc} k[x,y]/x^2-y^3 & \xrightarrow[y\mapsto t^2]{x\mapsto t^3} & k[t] \\ \downarrow & & \downarrow \\ k & \longrightarrow & k[t]/t^2 \end{array}.$$

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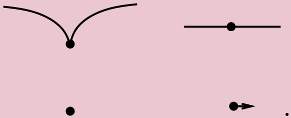

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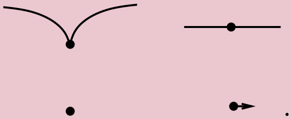

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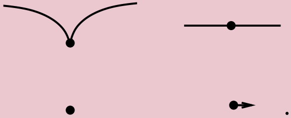

The fiber of $Kk \to K(k[t]/t^2)$ is known in terms of HC, and so modulo knowing Kk the calculation of the K-theory of $C = k[x,y]/x^2-y^3$ is reduced to HC-calculations [GRW, Crelle 89]:

$$K_0C = \mathbf{Z} \oplus k$$
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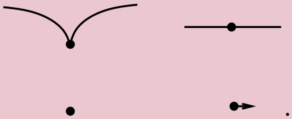

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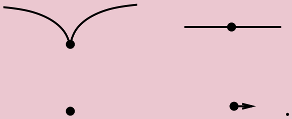

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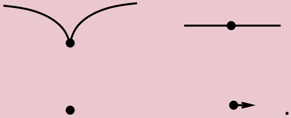

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$$\begin{aligned} K_0C &= \mathbf{Z} \oplus k \\ K_1C &= k^* \oplus \Omega^1_{k/\mathbf{Q}} \\ K_2C &= K_2k \oplus k[t]/t^2 \oplus \Omega^2_{k/\mathbf{Q}} \\ K_3C &= K_3k \oplus k[t]/t^2 \otimes_k \Omega^1_{k/\mathbf{Q}} \oplus \Omega^3_{k/\mathbf{Q}} \end{aligned}$$

Example, the cusp

Let k be a field of characteristic 0

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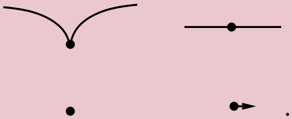

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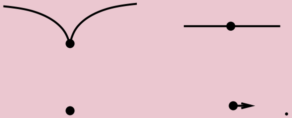

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Since topological cyclic homology is open to attack with tools from stable homotopy theory, this makes calculations of K-theory possible. For instance, Geisser and Hesselholt have calculated the K-theory of the coordinate axes $\mathbf{F}_p[x,y]/xy$.

Extending to connective **S**-algebras

Since simplicial rings lie densely in connective **S**-algebras, one would expect that these results followed for connective **S**-algebras once one extended the results from discrete rings to simplicial rings.

Indeed this is true, and such a proof has been worked out. However, there is a quicker proof, which I will present shortly.

Reason for wanting the long proof: Simplicial rings give hope of uncovering the right homotopical conditions on the categories of modules (surjectivity is not the right condition).

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Note:

The circulating preprint only claims this after profinite completion, but using Goodwillie's model for integral TC and the techniques of Suslin/Wodzicki and Geisser/Hesselholt, Kittang has worked out the integral statement.

Proof

Let F be the homotopy fiber of the p-completion of the cyclotomic trace $K \to TC$.
The theorem is equivalent to the claim that $F(\mathcal{A})$ is homotopy cartesian, that is, F satisfies excision.

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Lemma

Under the assumptions of the theorem,

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- What replaces the “Rim squares” in S-algebras?
- What does it tell us about vector bundles over S-algebras?
- Is a study of “combinatorial” S-algebras possible? (e.g., what are BN Stanley-Reisner rings”?)