Borel-fixed Simplicial Complexes and Regularity in the Exterior Algebra

Thesis for the Master of Science degree
in Algebra

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Bergen, May 26, 2005
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Chapter 1

Introduction

In this brief introduction I will try to explain, in a non-technical way, what I have done in this thesis. The concepts will be properly introduced in the main part of the text. Many proofs in the thesis (especially in chapter 3) are very combinatorial and technical, so for convenience, they are followed by examples which illustrate the steps in the proofs.

Abstract simplicial complexes are combinatorial structures, but they can be described algebraically via the Stanley-Reisner correspondence. To each simplicial complex Δ on the vertex set $[n] = \{1, ..., n\}$ this correspondence assigns a quotient $k[\Delta]$ of the polynomial ring $S = k[x_1, ..., x_n]$ and a quotient $k\{\Delta\}$ of the exterior algebra E on n variables. The Hilbert functions of these algebraic structures are uniquely determined by the number of faces of different dimensions in Δ. These numbers are collected in the so-called *f-vector* of Δ. Also, the same information is encoded in what we call the *h-vector* of Δ. So these concepts are equivalent.

The motivation for the work in this thesis is that we want to describe these Hilbert functions, which is the same as describing the f- and h-vectors of simplicial complexes. We find a new proof for a description of the class of *Cohen-Macaulay* simplicial complexes.

A simplicial complex is called *Cohen-Macaulay for k* if its Stanley-Reisner ring $k[\Delta]$ is a Cohen-Macaulay ring. A theorem by Eagon and Reiner says that a simplicial complex Δ is Cohen-Macaulay if and only if the Stanley-Reisner ring $k[\Delta^*]$ of its *Alexander dual simplicial complex* has linear free resolution as an S-module. By a theorem of Aramova, Avramov and Herzog, this happens if and only if the exterior Stanley-Reisner algebra $k\{\Delta^*\}$ has linear resolution as an E-module.

The quotient $k\{\Delta^*\}$ is defined by the monomial ideal I_{Δ^*}, and if we use the reverse lexicographic order and pass to the *generic initial ideal* $\text{Gin}(I_{\Delta^*})$ of I_{Δ^*}, we get a monomial ideal with a very nice combinatorial structure. As any monomial ideal corresponds to a simplicial complex, this gives us via the Stanley-Reisner correspondence a new simplicial complex $(\Delta^*)^s$. This complex is again Alexander dual to some complex $((\Delta^*)^s)^*$, which we will denote by Δ_s. Because of the combinatorial properties of the generic initial ideals, this complex is relatively easy to study. It is well known that an ideal and its generic initial ideal have the same Hilbert function, and this implies that Δ and Δ_s have the same h-vector.

Chapter 4 in this thesis is devoted to a proof of the fact that, when we use the reverse lexicographic order, the *Castelnuovo-Mumford regularity* of a graded ideal in the exterior algebra equals the regularity of its generic initial ideal. For ideals in the symmetric algebra, this is the Bayer-Stillman theorem. A proof for ideals in the exterior algebra was published by Aramova and Herzog [2], but this was unknown to both my supervisor and myself until very recently.

So, the point is: A simplicial complex Δ is Cohen-Macaulay if and only if $k\{\Delta^*\}$ has linear resolution. It is a consequence of the theorem we prove in chapter 4 that $k\{\Delta^*\}$ has linear resolution if and only if $k\{(\Delta^*)^s\} = E/\operatorname{Gin}(I_{\Delta^*})$ has linear resolution, and this again is equivalent to $((\Delta^*)^s)^* = \Delta_s$ being Cohen-Macaulay. This motivates us to study simplicial complexes Δ with the property that the ideal I_{Δ^*} corresponding to the Alexander dual can occur as a generic initial ideal. This is the class of *Borel-fixed* simplicial complexes.

The process described above is equivalent to a process introduced by Gil Kalai called *algebraic shifting*, defined in a different way. We will not go into details about this.

In chapter 3 I prove that a Borel-fixed complex is Cohen-Macaulay for all fields k if and only if it is pure. Furthermore, I find an interpretation of the h-vector of such complexes, thus giving a description of h-vectors of Cohen-Macaulay simplicial complexes, since the process of passing from Δ to Δ_s preserves Cohen-Macaulayness and the h-vector. This gives a new proof of the fact that the h-vector of a Cohen-Macaulay simplicial complex is an *M-vector*, meaning that there exists a *multicomplex* Γ such that this vector counts the number of monomials in Γ in each degree. We give a constructive proof, building the multicomplexes from the Borel-fixed simplicial complexes.

I then move on to study *higher Cohen-Macaulayness* of Borel-fixed simplicial complexes. This concept deals with the question of how many vertices one must remove from a simplicial complex in order to destroy either the dimension or the Cohen-Macaulayness. I derive a criterion for determining when such a complex is l-CM but not $(l+1)$-CM. Then I combine this criterion and our description of the h-vectors to find some information on the h-vector of 2-CM simplicial complexes. In fact, we discover that for 2-CM Borel-fixed simplical complexes, the h-vector is a (finite) *non-decreasing sequence*. Chapter 3 ends with a description of what happens when we reduce our requirements to the simplicial complexes, and study complexes corresponding to *stable* ideals. It is then more difficult to say someting about them in general.

Chapter 2

Preliminaries

In this chapter we will introduce the concepts we will be concerned with in this thesis, and we will explain the relevance of and connection between the next two chapters.

2.1 Simplicial Complexes

In this section we will define (abstract) simplicial complexes and study some of their properties. Simplicial complexes arise in topology when we study spaces by triangulation. One might say that topological simplicial complexes are constructed by gluing together simplices along their faces. By a simplex we mean the convex hull of n corner points, with dimension $n-1$. If one corner point, or *vertex*, is removed, the remaining $n-1$ vertices span a convex hull of dimension $n-2$.

We will only be concerned with abstract simplicial complexes, although it is sometimes useful to visualize these as topological complexes, at least for low dimensions. The basic reference for the material on simplicial complexes is [16].

Throughout, we will let $[n]$ denote the finite set $\{1, ..., n\}$, and k is some field.

Definition 2.1.1. [1] An *abstract finite simplicial complex* Δ on the vertex set $[n]$ is a collection of subsets of $[n]$ which is closed under inclusion. That is, if $X \in \Delta$ and $Y \subset X$, then $Y \in \Delta$. An element of Δ will be called a *face* of Δ. If d is the cardinality of a face X of Δ, we say that X has *dimension* $d-1$. The dimension of Δ is the maximal dimension of its faces.

Remark 2.1.2. The empty set $\emptyset$ is in every non-empty simplicial complex. Also, there is a difference between the empty complex $\emptyset$ and the smallest non-empty one $\{\emptyset\}$.

Thus if $n = 4$, a possible simplicial complex is $\Delta = \{\{1,3,4\}, \{1,3\}, \{1,4\}, \{3,4\}, \{1\}, \{3\}, \{4\}, \emptyset\}$. It has dimension 2. Note that the first element of Δ spans the entire complex in some sense; all the other elements of the complex have to be there in order to make the

[1] From now on we will suppress the words *abstract* and *finite*.

criterion in the definition true. From now on, if the faces $F_1, ..., F_t$ span the complex Δ in this way, we will write $\Delta = \langle F_1, ..., F_t \rangle$.

0-dimensional faces (1-subsets) in a simplicial complex will be called *vertices*. Faces of dimension 1 will be called *edges*. Faces that are maximal under inclusion (i.e. those that span the complex) will be called *facets*. A simplicial complex where all the facets are of the same dimension will be called *pure*. Note that 1-dimensional complexes have topological realization as graphs, and when we use the word "graph" in this text, we will always mean a finite graph with no loops and no multiple edges. Other graphs are not simplicial complexes.

Definition 2.1.3. The *f-vector* of Δ is $(f_{-1}, f_0, ..., f_{\dim \Delta})$, where f_i is the number of faces with dimension i. Here we think of the element $\emptyset$ as having dimension -1.

Example 2.1.4. If Δ is the n-simplex, that is, the collection of all subsets of $[n]$, then $f_{-1} = 1$ (as always whenever Δ is non-empty) and $f_i = \binom{n}{i+1}$ for $i = 0, ..., n-1$.

Definition 2.1.5. Given a simplicial complex Δ, the *Alexander dual* simplicial complex of Δ is

$$\Delta^* = \{F \subseteq [n] \mid F^c \notin \Delta\}$$

where F^c denotes the complement of F in $[n]$.

Remark 2.1.6. This is indeed a true duality: $F \in \Delta \Leftrightarrow F^c \notin \Delta^*$ is equivalent to $F \notin \Delta \Leftrightarrow F^c \in \Delta^*$, so $(\Delta^*)^* = \Delta$. Furthermore, the f-vector of Δ^* is uniquely determined by the f-vector of Δ. (This is rather obvious from the definition, and explained numerically in section 3.1.) So in particular, complexes with equal f-vectors have Alexander duals with equal f-vectors.

Example 2.1.7. Let $n = 4$ and let us look at the simplicial complex

$$\begin{aligned}\Delta &= \langle \{1,2,3\}, \{3,4\} \rangle \\ &= \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}\end{aligned}$$

This simplicial complex has dimension 2. Its facets are $\{1,2,3\}$ and $\{3,4\}$. The f-vector is $f(\Delta) = (f_{-1}, f_0, f_1, f_2) = (1,4,4,1)$. The topological realization of Δ is shown in figure 2.1. The shaded triangle shows that the triplet $\{1,2,3\}$ is a face of Δ. The inclusion-minimal subsets of $[n]$ which are not faces of Δ are $\{1,4\}$ and $\{2,4\}$. Thus the Alexander dual Δ^* is $\langle \{1,3\}, \{2,3\} \rangle$, the complex spanned by the complements of the inclusion-minimal subsets which are not in Δ.

We now construct an algebraic object associated to a simplicial complex. It is called the *Stanley-Reisner ring*.

Definition 2.1.8. Let Δ be a simplicial complex on $[n]$, and let k be a field. The *Stanley-Reisner ring* of Δ is the k-algebra

$$k[\Delta] = k[x_1, ..., x_n]/J_\Delta$$

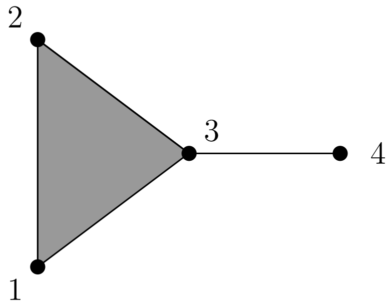

Figure 2.1: The simplicial complex Δ from example 2.1.7.

where J_Δ is the ideal generated by all monomials $x_{i_1}x_{i_2}\cdots x_{i_s}$ ($i_a \neq i_b$ when $a \neq b$) such that $\{i_1, ..., i_s\} \notin \Delta$. If $E = \bigwedge V$ is the exterior algebra of an n-dimensional k-vector space, we can construct a quotient of E by an ideal I_Δ defined in exactly the same way. This is called the *exterior* Stanley-Reisner algebra of Δ and is denoted $k\{\Delta\}$. This is explained in detail in definition 3.1.1.

Remark 2.1.9. This is a two-way correspondence: Given an ideal J in $S = k[x_1, ..., x_n]$ that is generated by square-free monomials, there is a unique simplicial complex Δ on $[n]$ such that $S/J = k[\Delta]$.

Theorem 2.1.10. *Let Δ be a simplicial complex on $[n]$ of dimension $d-1$. Then $k[\Delta]$ has the Hilbert series*

$$H_{k[\Delta]}(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}$$

Proof. This is corollary 4.1.8 in [6]. ☐

Definition 2.1.11. The finite sequence $h(\Delta) = (h_0, h_1, ..., h_d)$ which appears in the theorem is called the *h-vector* of Δ.

Theorem 2.1.12. *The f-vector and the h-vector of a simplicial complex Δ are related by:*

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \quad \text{and} \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i$$

for $0 \leq j \leq d$, where Δ is $(d-1)$-dimensional.

Proof. This is lemma 5.1.8 in [6]. ☐

So we see that all the information that is encoded in the f-vector is also encoded in the h-vector, and vice versa. Also, simplicial complexes with equal f-vectors have Stanley-Reisner rings with equal Hilbert functions.

Example 2.1.13. If Δ is the simplicial complex in example 2.1.7, we see that $J_\Delta \subset k[x_1, x_2, x_3, x_4]$ is the ideal (x_1x_4, x_2x_4). Note that the (minimal) generators of the ideal correspond exactly to the inclusion-minimal subsets of $[n]$ which are not in Δ. And as we pointed out in example 2.1.7, these correspond via complements to the facets of the Alexander dual Δ^*. Let us use the preceeding theorem to decide the h-vector of Δ.

$$\begin{aligned}
h_0 &= \sum_{i=0}^{0}(-1)^{0-i}\binom{3-i}{0-i} f_{i-1} = f_{-1} = 1 \\
h_1 &= \sum_{i=0}^{1}(-1)^{1-i}\binom{3-i}{1-i} f_{i-1} = -3f_{-1} + f_0 = -3 + 4 = 1 \\
h_2 &= \sum_{i=0}^{2}(-1)^{2-1}\binom{3-i}{2-i} f_{i-1} = 3f_{-1} - 2f_0 + f_1 = 3 - 2\cdot 4 + 4 = -1 \\
h_3 &= \sum_{i=0}^{3}(-1)^{3-i}\binom{3-i}{3-i} f_{i-1} = -f_{-1} + f_0 - f_1 + f_2 = -1 + 4 - 4 + 1 = 0
\end{aligned}$$

Thus the h-vector is $h(\Delta) = (h_0, h_1, h_2, h_3) = (1, 1, -1, 0)$.

Definition 2.1.14. A simplicial complex Δ is said to be *Cohen-Macaulay for k* if the corresponding Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay.

So in particular, the property of being Cohen-Macaulay depends on the characteristic of the field.

2.2 Monomial Orders and Initial Ideals

In this section we will introduce monomial orders and initial ideals corresponding to such orders. This is the starting-point of the theory of Gröbner bases, which is a useful tool in computations in commutative algebra and algebraic geometry. We will not go into that here. Eisenbud's textbook [8] is a good source for material on this subject. Green's notes [15] are a collection of much material on initial ideals, in particular on generic initial ideals.

Let V be a vector space of dimension n over a field k. $S = S(V)$ will denote the symmetric algebra over V, and if we choose a basis $\{x_1, ..., x_n\}$ of V, this is simply $S = k[x_1, ..., x_n]$, the polynomial ring on n variables over the field k. However, we will be concerned with "generic" coordinates later, so we will keep in mind that the basis consisting of the x_i is a chosen basis.

The definition of monomial orders can easily be made for finitely generated free modules over S. We will however only consider monomial orders for the ring S itself, and for the exterior algebra E.

Definition 2.2.1. A *monomial* in S is an expression of the kind $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, where $a = (a_1, ..., a_n) \in \mathbb{N}_0^n$. We will also use the notation x^a in this situation. A *monomial order* is a total order $>$ on the set of monomials of S such that if x^a, x^b and $n \neq 1$ are monomials of S, then $x^a > x^b$ implies $nx^a > nx^b > x^b$. We will always assume that our orders satisfy $x_1 > x_2 > \cdots > x_n$. A *term* is a monomial multiplied with a scalar.

Example 2.2.2. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ and consider x^a and x^b in the most commonly used orders:

- *The lexicographic order:* $x^a > x^b$ if and only if $a_i > b_i$ for the minimal index i with $a_i \neq b_i$.
- *The degree lexicographic order:* $x^a > x^b$ if and only if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and $a_i > b_i$ for the minimal index i with $a_i \neq b_i$.
- *The (degree) reverse lexicographic order:* $x^a > x^b$ if and only if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and $a_i < b_i$ for the maximal index i with $a_i \neq b_i$.

So there is a difference: In the first two orders, the ordering of the monomials of degree 2 in $k[x_1, x_2, x_3]$ is

$$x_1^2 > x_1x_2 > x_1x_3 > x_2^2 > x_2x_3 > x_3^2,$$

while in the last order, it is

$$x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2.$$

In the "reverse" order, monomials are dragged down by having a lot "in the end". Note also the difference that the "degree" orders take care of one degree at a time, while the pure lexicographic order has x_n as its smallest monomial, x_n^2 as the next, and infinitely many smaller than x_{n-1}. This makes the pure lexicographic order useless for induction purposes in S.

Definition 2.2.3. Given a monomial order $>$ and an element f in $S = k[x_1, ..., x_n]$. The *initial term* of f is the greatest term of f with respect to the order $>$. It is denoted $\mathrm{in}_>(f)$, or, when it is clear which order we are using, $\mathrm{in}(f)$. Given an ideal $J \subset S$, the *initial ideal* of J, denoted $\mathrm{in}_>(J)$ or simply $\mathrm{in}(J)$, is the ideal generated by all elements $\mathrm{in}(f)$, where f runs through J. Clearly, $\mathrm{in}(J)$ is a monomial ideal.

Lemma 2.2.4. *Given a monomial order on S, if $f, g \in S$, then*

$$\begin{aligned} \mathrm{in}(fg) &= \mathrm{in}(f)\,\mathrm{in}(g) \\ \mathrm{in}(f+g) &\leq \max(\mathrm{in}(f), \mathrm{in}(g)) \end{aligned}$$

Proof. Let $f = \sum c_a x^a$ and $g = \sum d_b x^b$ where all c_a, d_b are coefficients in the ground field k. Then

$$fg = \sum c_a d_b x^a x^b$$

By the property of monomial orders given in definition 2.2.1, for every b, $\mathrm{in}(f)x^b$ is larger than any other $x^a x^b$ with non-zero c_a. Likewise, for all a, $x^a\,\mathrm{in}(g)$ is larger than any other

$x^a x^b$ with non-zero d_b. Therefore the largest monomial $x^a x^b$ with non-zero coefficients c_a and d_b is $\text{in}(f)\,\text{in}(g)$.

The sum of the polynomials is

$$f + g = \sum c_a x^a + \sum d_b x^b$$

Obviously, $\text{in}(f+g)$ is the largest of all the monomials which appear in the sum, unless $\text{in}(f) = \text{in}(g)$ and $c_{\text{in}(f)} = -d_{\text{in}(g)}$. In this case, $\text{in}(f+g)$ is actually smaller. □

The main reason why we are interested in initial ideals is the following:

Theorem 2.2.5. *If $J \subset S$ is a homogeneous ideal, then J and $\text{in}(J)$ have the same Hilbert function. Of course, so do S/J and $S/\text{in}(J)$.*

Proof. This is proposition 1.11 in [15], or rather, the remark following it. The point is that if we consider a basis $\{x^{a_1}, ..., x^{a_t}\}$ for the vector space $\text{in}(J)_d$ of homogeneous elements of $\text{in}(J)$ of some degree d, we can construct a basis of J_d which is in a one-to-one correspondence to that basis by finding elements f_{a_i} in J_d with $\text{in}(f_{a_i}) = x^{a_i}$ for all i in the range $1 \leq i \leq t$. □

Basically the same things can be done for the exterior algebra. This is explained in [3]. Let V still be an n-dimensional vector space over k.

$$E = \bigwedge V = \bigoplus_{k=0}^{n} \wedge^k V$$

will denote the exterior algebra over V. If we choose a basis $\{e_1, ..., e_n\}$ for V, then E is the 2^n-dimensional vector space over k with basis

$$\{e_a = e_{a_1} \wedge ... \wedge e_{a_k} \mid a = \{a_1, ..., a_n\} \subseteq [n]\}^2 \qquad (2.1)$$

and skew-commutative product $\wedge$. For simplicity of notation we will write $fg = f \wedge g$.

The basis elements from (2.1) will be called *monomials* in E. The notion of monomial orders carries over from the symmetric case. We must, however, take into consideration that the monomials of E are zero-divisors: $e_a \wedge e_b = 0$ if and only if $a \cap b \neq \emptyset$. Therefore we require from a monomial order that $e_a > e_b$ implies $ne_a > ne_b > e_b$ when the products are non-zero. A *term* is again a monomial multiplied with a scalar.

The lexicographic and reverse lexicographic orders are defined in the same way as in S. We will mainly deal with the reverse lexicographic order. These orders take a much simpler form in the exterior algebra, since the exponents of the e_i's are either 1 or 0.

Definition 2.2.6. Given a monomial order in E and an element

$$f = \sum_a f_a e_a \in E, \quad f_a \in k,$$

[2] Unless otherwise stated, we will always mean $a_1 < a_2 < \cdots < a_k$ in this situation.

the *initial term* $\mathrm{in}(f)$ of f is the largest term of f. That is, $f_a e_a$ for the e_a with $f_a \neq 0$ which ranges highest under the given order. The *initial ideal* $\mathrm{in}(I)$ of an ideal $I \subset E$ is the monomial ideal generated by the $\mathrm{in}(f)$ for all elements f in I.

The analogue of lemma 2.2.4 takes a slightly different form in the exterior algebra:

Lemma 2.2.7. *Given a monomial order on E, if $f, g \in E$, then*

$$\begin{aligned} \mathrm{in}(fg) &\leq \mathrm{in}(f)\,\mathrm{in}(g) \\ \mathrm{in}(f+g) &\leq \max(\mathrm{in}(f), \mathrm{in}(g)) \end{aligned}$$

Proof. Inequality in the first property appears exactly when

$$\mathrm{supp}(\mathrm{in}(f)) \cap \mathrm{supp}(\mathrm{in}(g)) \neq \emptyset.$$

Otherwise, the proof is the same as in the symmetric case. ☐

In the exterior algebra we will prefer the term *graded* ideal for an ideal which is generated by homogeneous elements. Once again we have the following:

Theorem 2.2.8. *Let $I \subset E$ be a graded ideal, and suppose there is given a monomial order. Then I and $\mathrm{in}(I)$ have the same Hilbert function.*

Proof. This is corollary 1.2 of [3]. ☐

We will now consider generic initial ideals. Recall that we have chosen a basis $\{x_1, ..., x_n\}$ for the vector space V. Change of bases for a vector space of is done by an action of the general linear group, in our case we will denote it $GL(V)$. Given an element $g = (g_{ij}) \in GL(V)$, the action is described by what happens to the basis elements:

$$x_j \mapsto \sum_{i=1}^{n} g_{ij} x_i.$$

This action extends in the natural way to the symmetric algebra $S(V)$, so $GL(V)$ is a group of algebra homomorphisms of S.

Let now $\mathcal{B} \subset GL(V)$ be the *Borel subgroup*, consisting of the upper triangular invertible matrices. The following important theorem is due to Galligo:

Theorem 2.2.9. *Given a multiplicative monomial order and a homogeneous ideal $J \subset S$, there is a Zariski open subset $U \subset GL(V)$ such that*

a) there is a monomial ideal $J' \subset S$ such that $J' = \mathrm{in}(\phi(J))$ for all $\phi \in U$.

b) the ideal J' is Borel-fixed, i.e. $\phi(J') = J'$ for all $\phi \in \mathcal{B}$.

Proof. This is theorem 1.27 in [15]. ☐

Definition 2.2.10. The ideal J' in theorem 2.2.9 is called the *generic initial ideal* of J. It will be denoted by $\mathrm{Gin}(J)$.

Borel-fixed ideals admit a very nice combinatorial structure, which is indeed why we are interested in them in this thesis:

Lemma 2.2.11. *Suppose k is of characteristic zero. Let $J \subset S$ be an ideal. Then J is Borel-fixed if and only if J is generated by monomials and*

$$x_1^{a_1} \cdots x_i^{a_i} \cdots x_j^{a_j} \cdots x_n^{a_n} \in J \Rightarrow x_1^{a_1} \cdots x_i^{a_i+q} \cdots x_j^{a_j-q} \cdots x_n^{a_n} \in J$$

whenever $1 \leq i \leq j \leq n$ and $0 \leq q \leq a_j$.

Proof. For the proof, see [8], theorem 15.23. □

Note that it is enough to check this for minimal monomial generators of the ideal.

Example 2.2.12. Let J_1 be the ideal $(x_1, x_2^2, x_2x_3, x_3^3) \subset k[x_1, x_2, x_3]$, and J_2 be the ideal $(x_1^2, x_2^2, x_2x_3, x_3^3)$. Then we see that J_1 is Borel-fixed, while J_2 is not, since $x_2^2 \in J_2$ and $x_1x_2 \notin J_2$.

Now, all this can be transferred to the exterior algebra also. In [3] we find the exact analogues to theorem 2.2.9 and lemma 2.2.11, and the definition of the generic initial ideal $\mathrm{Gin}(I)$ for an ideal $I \subset E$ is obvious. Actually, the analogue of lemma 2.2.11 does not require that k is of characteristic zero, only that it is infinite. It is convenient to have a separate combinatorial description of Borel-fixedness for ideals in E:

Lemma 2.2.13. *Borel-fixed monomial ideals I in E have the following property: If $e_\sigma \in I$ is a monomial, and $j \in \sigma$, then $e_i e_{\sigma \setminus j}$ for all $i < j$.* [3]

Proof. This is proposition 1.7 in [3]. □

2.3 Cohen-Macaulayness and Linear Resolutions

We will now combine the theories of simplicial complexes and generic initial ideals. Suppose we are given a simplicial complex Δ. This corresponds via the Stanley-Reisner correspondence to monomial ideals J_Δ in S and I_Δ in E, as defined in definition 2.1.8. A theorem by Eagon and Reiner gives a criterion for determining whether or not Δ is Cohen-Macaulay. Let us first make a definition.

Definition 2.3.1. A free resolution

$$0 \longrightarrow \oplus_j S(-a_{ij}) \xrightarrow{d_i} \cdots \longrightarrow \oplus_j S(-a_{1j}) \xrightarrow{d_1} \oplus_j S(-a_{oj}) \longrightarrow M \tag{2.2}$$

of an S-module M is said to be *linear* if there is an integer t such that $a_{ij} = i + t$ for all $i, j \geq 0$. The definition for modules over the exterior algebra is similar, but one must be aware that free resolutions over E are generally not finite.

[3]In [3], as well as other places in the literature, an ideal with this property is called *strongly stable*. We will simply call them Borel-fixed.

The Eagon-Reiner theorem is as follows:

Theorem 2.3.2. *A simplicial complex Δ is Cohen-Macaulay over k if and only if the Stanley-Reisner ideal $J_{\Delta^*} \subset S$ corresponding to the Alexander dual complex has linear free resolution.*

Proof. This is [7], theorem 3. Note that you may replace the ideal J_{Δ^*} by the quotient $k[\Delta^*]$. □

By this result by Aramova, Avramov and Herzog, we can transfer this to the exterior algebra as well:

Theorem 2.3.3. *If $I \subset E$ is a monomial ideal, and J is the corresponding square-free monomial ideal in the symmetric algebra S, then I has linear free resolution over E if and only if J has linear free resolution over S.*

Proof. See [1], corollary 2.2 part (2). □

So, combining these theorems, we have that Δ is Cohen-Macaulay if and only if the ideal I_{Δ^*} of the Alexander dual in E has linear resolution as an E-module. We may now use the reverse lexicographic order, and take the generic initial ideal $\text{Gin}(I_{\Delta^*})$ of I_{Δ^*}. Since every monomial ideal in E corresponds to a unique simplicial complex, we must have that $\text{Gin}(I_{\Delta^*}) = I_{(\Delta^*)^s}$ for some simplicial complex $(\Delta^*)^s$. This again has the Alexander dual $((\Delta^*)^s)^*$.

Definition 2.3.4. We will denote the simplicial complex $((\Delta^*)^s)^*$ above by Δ_s.

What can we say about this correspondence? We will need a theorem called the Bayer-Stillman theorem about the (Castelnuovo-Mumford) regularity of ideals:

Definition 2.3.5. Let (2.2) be a minimal resolution of the S-module M. M is said to be *m-regular* if $a_{ij} \leq m + i$ for all i. The *regularity* $\text{reg}(M)$ of M is the smallest number m for which M is m-regular.

And now the Bayer-Stillman theorem:

Theorem 2.3.6. *If J is a homogeneous ideal in the symmetric algebra S, then* $\text{reg}(J) = \text{reg}(\text{Gin}(J))$ *when we use the reverse lexicographic order.*

Proof. This is the essence of [5]. □

Regularity is defined in the analogous way for modules over the exterior algebra. In chapter 4, we will give a proof of the Bayer-Stillman theorem for ideals in the exterior algebra. Note that this theorem does not only deal with monomial ideals, but with graded ideals in general.

The exterior Bayer-Stillman theorem gives us that I_{Δ^*} and $I_{(\Delta^*)^s} = \text{Gin}(I_{\Delta^*})$ have linear resolutions simultaneously. This is a consequence of Gröbner basis theory. Using this we find the following:

Theorem 2.3.7. *If Δ is a simplicial complex, then Δ is Cohen-Macaulay if and only if Δ_s is Cohen-Macaulay.*

Proof. Δ is CM if and only if I_{Δ^*} has linear resolution. This is equivalent to the fact that $\mathrm{Gin}(I_{\Delta^*}) = I_{(\Delta^*)^s}$ has linear resolution, which again happens if and only if $((\Delta^*)^s)^* = \Delta_s$ is CM. □

The other interesting property of this theory is the following:

Theorem 2.3.8. *If Δ is a simplicial complex, then Δ and Δ_s have the same h-vector.*

Proof. The h-vector of Δ uniquely determines the h-vector of Δ^*, which again describes the Hilbert function of I_{Δ^*}. By theorem 2.2.8, this is equal to the Hilbert function of $\mathrm{Gin}(I_{\Delta^*}) = I_{(\Delta^*)^s}$, so the h-vector of $(\Delta^*)^s$ is equal to the h-vector of Δ^*. Therefore, by remark 2.1.6, the h-vector of $((\Delta^*)^s)^* = \Delta_s$ is equal to the h-vector of Δ. □

So, by the discussion in this section, we are led to the study of simplicial complexes which are such that the ideals corresponding to their Alexander duals are Borel-fixed, and this is the theme for chapter 3.

Chapter 3

Borel-fixed Simplicial Complexes

This chapter is devoted to the study of pure simplicial complexes which correspond to Borel-fixed monomial ideals in the exterior algebra. After inspecting the combinatorial structure of such complexes, we show that they are shellable, and consequently Cohen-Macaulay for all characteristics. After that, we give an interpretation of the h-vector of the complexes in terms of the generators of the corresponding ideal. Also, we show that these h-vectors are M-vectors, by constructing a suitable multicomplex.

We then proceed to an investigation of higher Cohen-Macaulayness of Borel-fixed simplicial complexes, and derive a closed criterion for determining when a complex Δ is l-CM and not $(l+1)$-CM. Higher Cohen-Macaulayness puts restrictions on the h-vector, and using our criterion and our description of such h-vectors, we will find that if a Borel-fixed simplicial complex is 2-CM, then the h-vector is a non-decreasing sequence.

Finally, we step back and reduce the requirements we have put on the simplicial complexes, and see that it is then more difficult to say anything about them in general.

It should be remarked that the symmetric group $\mathfrak{S}_n$ acts on simplicial complexes on $[n]$ by permuting the labelling of the vertices. All our results regarding Borel-fixed simplicial complexes can be stated for simplicial complexes which are Borel-fixed after a permutation of the labelling. We will not make this explicit.

3.1 Ideals in E and Simplicial Complexes

Let Δ be a simplicial complex on the vertex set $[n]$, V a k-vector space with basis $e_1, ..., e_n$, and let $E = \bigwedge V$ be the exterior algebra over V.

Definition 3.1.1. Let $I_\Delta \subset E$ be the monomial ideal generated by the monomials e_F such that F is not a face of Δ. Then $k\{\Delta\} = E/I_\Delta$ is the *indicator algebra* of Δ. (It is also known as the exterior face algebra or exterior Stanley-Reisner algebra of Δ.)

This also goes the other way: To a monomial ideal I in the exterior algebra, we can assign a uniquely determined simplicial complex $\Delta = \{F \subset [n] | e_F \notin I\}$.

Example 3.1.2. Let $n = 4$. If $\Delta = \langle\{1,2,3\},\{3,4\}\rangle$ as in example 2.1.7, then $I_\Delta = (e_{\{1,4\}}, e_{\{2,4\}})$. $k\{\Delta\} = E/I_\Delta$ is a vector space over k with basis consisting of the images of the monomials e_F such that F is a face of Δ. This is always the case, so we have the following lemma:

Lemma 3.1.3. *The Hilbert series of* $k\{\Delta\}$ *is given by*

$$H_{k\{\Delta\}}(t) = \sum_{i=0}^{d} f_{i-1}t^i$$

This explains why a simplicial complex Δ^* and the associated simplicial complex $(\Delta^*)^s$ from section 2.3 have the same f- and h-vectors. By theorem 2.2.8, the ideals of the complexes have the same Hilbert function, and by the preceeding lemma, the Hilbert function is given by the f-vector.

Although we will not use it, it might be useful for the understanding of the subject to see how the f-vector of the Alexander dual complex comes into all this:

Remark 3.1.4. Let $f^* = (f^*_{-1}, f^*_0, ..., f^*_{n-1})$ be the f-vector of the simplicial complex Δ^*, and $f = (f_{-1}, f_0, ..., f_{d-1}, ..., f_{n-1})$ the f-vector of Δ, both extended with the necessary 0's. Then the Hilbert series of I_Δ is given by

$$H_{I_\Delta}(t) = \sum_{i=0}^{n} f^*_{n-i-1}t^i$$

Proof. The coefficient of t^i is the dimension of $(I_\Delta)_i$, and this vector space has a basis consisting of one monomial for each i-subset of $[n]$ which is *not* in Δ. Each such subset corresponds via complements to an $(n-i)$-set which is not a face of Δ^*, and vice versa. □

In particular,

$$f_i + f^*_{n-i-2} = \binom{n}{i+1}$$

as we see from lemma 3.1.3 and remark 3.1.4, and the fact that $\dim_k(E_i) = \binom{n}{i}$.

For convenience, we recall the property of Borel-fixedness from lemma 2.2.13 and adopt it as a definition:

Definition 3.1.5. Let $I \subset E$ be a monomial ideal. I is said to be *Borel-fixed* if whenever $e_\sigma \in I$ and $j \in \sigma$, then $e_i e_{\sigma\setminus\{j\}} \in I$ for all $i < j$.

There is a simple connection between a monomial ideal $I_\Delta \subset E$ and the Alexander dual Δ^* of the corresponding simplicial complex Δ: F is a face of Δ^* if and only if F^c is not a face of Δ, which again happens if and only if e_{F^c} is in I_Δ. The condition that I_Δ be Borel-fixed translates into the following: If $\{a_1, ..., a_t\}$ is a face of Δ^*, then $\{a_1, ..., a_{i-1}, a_{i+1}, ..., a_t, a\}$ is also a face of Δ^* whenever $a > a_i$. So if F is a face of Δ^*, then we know that if we replace any vertex of F with a vertex labelled with a higher number, we will have another face of Δ^*.

The connection between the generators of the monomial ideal I_Δ and Δ itself is not as transparent as the connection through Alexander duality. Nevertheless, when the ideal is Borel-fixed, Δ and Δ^* share combinatorial properties.

Lemma 3.1.6. *I_Δ is Borel-fixed if and only if I_{Δ^*} is Borel-fixed. In this case we say that Δ* has the Borel property*, or simply that Δ is Borel-fixed. So in other words, Δ is Borel-fixed if and only if Δ^* is Borel-fixed.*

Proof. The monomials in I_Δ are the monomials such that their sets of indices, which are subsets of $[n]$, are not faces of Δ. Suppose then that $F = \{a_1, ..., a_t\}$ is a face of Δ. Consider the set $F' = \{a_1, ..., a_{i-1}, a_{i+1}, ..., a_t, a\}$, for some $i = 1, ..., t$, where $a > a_i$ and a is not equal to of the other a_j's. Then F' has to be in Δ. For if it was not, $e_{F'}$ would be a monomial in I_Δ, and the Borel-fixedness of I_Δ would require that e_F was in I_Δ, which it is not. Thus Δ has the same combinatorial structure as Δ^* does when I_Δ is Borel-fixed. This carries over to the combinatorial structure of the ideals. □

Example 3.1.7. Let $n = 4$ and consider the monomial ideal $I_\Delta = (e_{12}, e_{13}, e_{14}, e_{23})$ in the exterior algebra E.[1] I_Δ is clearly Borel-fixed. The corresponding simplicial complexes are $\Delta = \langle 34, 24, 1\rangle$ and $\Delta^* = \langle 34, 24, 23, 14\rangle$. We see that both complexes have the Borel property. Geometrically, this means that the complexes are in some sense saturated towards the vertices with the largest labels. See figure 3.1.

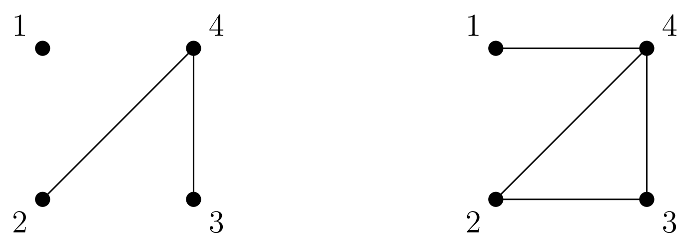

Figure 3.1: The simplicial complexes from example 3.1.7: Δ to the left and Δ^* to the right.

The connection between simplicial complexes and ideals is most easily studied via Alexander duality, so we will use that point of view here.

Recall that a simplicial complex is said to be Cohen-Macaulay if the Stanley-Reisner ring $k[\Delta]$ of the complex is Cohen-Macaulay. We want to show that if I_Δ is a Borel-fixed ideal in the exterior algebra such that the minimal generators of I_Δ are of equal degree, then the Alexander dual Δ^* is Cohen-Macaulay. Cohen-Macaulay complexes are described in [16] and [6], and we will use results from these books.

Lemma 3.1.8. *A Cohen-Macaulay simplicial complex is pure.*

[1] We will employ this short-hand notation when it is clear that the subscript of, say, e_{14} is one-four and not fourteen. The same goes for the sets in the simplicial complexes.

Proof. This is [6], corollary 5.1.5. □

By this lemma it makes sense to look only at ideals I_Δ where the minimal generators are of equal degree. These minimal generators e_F correspond to facets F^c of Δ^*, so if there are minimal generators of different degree, Δ^* will not be pure, and therefore not Cohen-Macaulay.

Definition 3.1.9. A simplicial complex Δ is *shellable* if it is pure, and if the facets can be ordered $F_1, ..., F_s$ such that the following condition holds: Write Δ_i for the subcomplex of Δ generated by $F_1, ..., F_i$. Then for all $1 \leq i \leq s$, the set of faces of Δ_i which are not in Δ_{i-1} has a unique minimal element (with respect to inclusion) $r(F_i)$. (Δ_0 will in this sense be $\emptyset$, and therefore the minimal element of $\Delta_1 \backslash \Delta_0$ is $\emptyset$.) The order on the facets is called a *shelling* of Δ.

There is also a notion of shellability of non-pure simplicial complexes, but we will not consider that here.

Theorem 3.1.10. *A shellable simplicial complex is Cohen-Macaulay for every field k.*

Proof. The proof is given in [6], theorem 5.1.13. □

We will construct a shelling of Δ^* corresponding to our Borel-fixed ideal I_Δ, and thus show that such a Δ^* is Cohen-Macaulay.

Theorem 3.1.11. *Suppose I_Δ is a Borel-fixed ideal in the exterior algebra E with minimal generators of equal degree. Then Δ^* is shellable and therefore Cohen-Macaulay for all fields k.*

Proof. This proof is very technical, and example 3.1.14 will clarify the process.

The generators of I_Δ can be written as $e_{(a_1,...,a_n)}$, where each $a_i \in \{0, 1\}$ indicates whether e_i divides the generator or not (1 for yes, 0 for no). Order the generator-indices after the reverse lexicographic order: $(a_1, ..., a_n)$ before $(b_1, ..., b_n)$ if and only if $a_i < b_i$ for the largest i such that $a_i \neq b_i$. This is the same as ordering the facets of Δ^* such that the facet F is before the facet G if and only if the largest element in the symmetric difference of F and G is in F. Number the facets as $F_1, ..., F_s$ according to this ordering.

Let Δ_j^* be the subcomplex of Δ^* generated by $F_1, ..., F_j$ and $\Delta_0^* = \emptyset$. We will show that when $1 \leq i \leq s$, $\Delta_i^* \backslash \Delta_{i-1}^*$ has a unique minimal element $r(F_i)$ (w.r.t. inclusion). Thus the ordering is a shelling of Δ^*, and Δ^* is Cohen-Macaulay.

Let $F_j = \{i_1, ..., i_k\}$ (where all the i_j's are different from each other) be a facet in Δ^*. Then $F_j^c = \{k_1, ..., k_t\}$ (once again all different) is an index for one of the minimal generators of I_Δ. For $k_t = t, j = 1$, and $r(F_1) = \emptyset$. For $k_t = t + r$ we have $i_1 < i_2 < \cdots < i_r < k_t <$ (all other i's). For such a facet there is an r-set $R = \{i_1, ..., i_r\}$ which is not in any of the facets $F_{j'}, j' < j$. (Since if this set was in one such $F_{j'}, F_{j'}$ would be equal to F_j, since the last $k - r$ numbers also belong to both facets, by the way the ordering was defined.) Also, all subsets of F_j which are not in Δ_{j-1}^* are extensions of this r-set. For suppose there is a set $R' \subset F_j$ which does not contain R. Then there is an i_s, $1 \leq s \leq r$ such that $i_s \notin R$.

Then $R \subset \{i_1, ..., i_{s-1}, i_{s+1}, ..., i_r, k_t, i_{r+1}, ..., i_k\}$, which is a facet of Δ^* with lower index than F_j. So R is the unique minimal element $r(F_j)$ of $\Delta^*_j \backslash \Delta^*_{j-1}$. □

Corollary 3.1.12. *If I_Δ is Borel-fixed and Δ is pure, then Δ is shellable and therefore Cohen-Macaulay.*

Proof. This is theorem 3.1.11 and lemma 3.1.6 combined. □

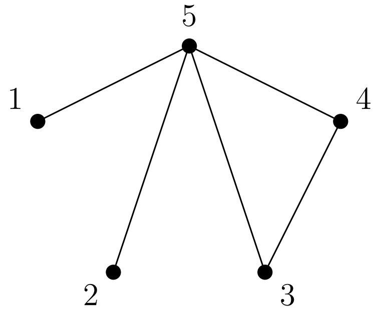

Figure 3.2: Δ^* of example 3.1.13.

Example 3.1.13. An example will hopefully make the theorem more understandable. Let $n = 5$ and $I_\Delta = (e_{123}, e_{124}, e_{125}, e_{134}, e_{234}) \subset E$, which is Borel-fixed. All the minimal generators are of degree 3, and therefore Δ^* is a pure graph (that is, with no isolated vertices). The ordering of the generators and facets is as follows:

$$\begin{array}{rclcl} e_{11100} & = & e_{123} & \leftrightarrow & F_1: \; 45 \\ e_{11010} & = & e_{124} & \leftrightarrow & F_2: \; 35 \\ e_{10110} & = & e_{134} & \leftrightarrow & F_3: \; 25 \\ e_{01110} & = & e_{234} & \leftrightarrow & F_4: \; 15 \\ e_{11001} & = & e_{125} & \leftrightarrow & F_4: \; 34 \end{array}$$

The complex is displayed in figure 3.2. Now $r(F_1)$ is the inclusion-minimal element of $\langle F_1 \rangle$, which is $\emptyset$. $r(F_2)$ is the inclusion-minimal element of $\langle F_1, F_2 \rangle \setminus \langle F_1 \rangle$, so $r(F_2) = \{3\}$. Likewise, $r(F_3) = \{2\}$ and $r(F_4) = \{1\}$. But then F_5 is glued to $\langle F_1, F_2, F_3, F_4 \rangle$ in both ends, so $r(F_5) = \{3, 4\}$. The shelling process is illustrated in figure 3.3.

The table in example 3.1.13 gives us an idea of how to visualize simplicial complexes of higher dimension. We see that the 0's and 1's in the indices of the minimal generators form a 5x5 matrix with a very nice structure. Let us investigate this more generally. To a simplicial complex Δ on $[n]$, we can assign an incidence matrix with n columns, one associated to each vertex of Δ. Vertex 1 is associated to the leftmost column, vertex 2 to the next, and so on up to vertex n associated to the rightmost column. To each facet of Δ we put one row in the matrix, with 0 in columns corresponding to vertices in the facet, and 1 elsewhere. This will shed some light on the proof of theorem 3.1.11, and help us in the next section. We will see it in an example:

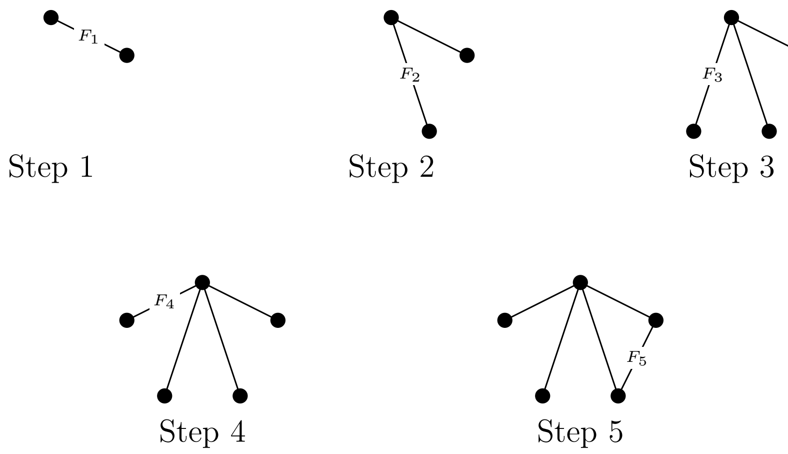

Figure 3.3: The shelling of Δ^*, step by step.

Example 3.1.14. Let $n = 6$ and consider the Borel-fixed ideal I_Δ generated by the degree 3 generators below:

generator-index	facet of Δ^*	**1**	**2**	**3**	**4**	**5**	**6**
123	456	1	1	1	0	0	0
124	356	1	1	0	1	0	0
134	256	1	0	1	1	0	0
234	156	0	1	1	1	0	0
125	346	1	1	0	0	1	0
135	246	1	0	1	0	1	0
235	146	0	1	1	0	1	0
145	236	1	0	0	1	1	0
245	136	0	1	0	1	1	0
126	345	1	1	0	0	0	1
136	245	1	0	1	0	0	1
236	145	0	1	1	0	0	1
146	235	1	0	0	1	0	1
246	135	0	1	0	1	0	1
156	234	1	0	0	0	1	1
256	134	0	1	0	0	1	1

The generators are listed in the reverse lexicographic order, so the ordering of the facets from top to bottom of the table is the shelling we constructed for Borel-fixed complexes. The Borel property is visible in the incidence matrix: A facet is represented by a string of three 1's and three 0's. Replacing a vertex of the facet with a vertex with higher label corresponds to moving a 0 rightward to a position occupied by a 1, and putting the 1 back on the position previously occupied by the 0. (An example: 010110$\mapsto$011010.) The Borel-fixedness says that then the resulting string also represents a facet of Δ^*. The new facet is always earlier in the ordering, so we move upward in the matrix.

How does this relate to the proof of theorem 3.1.11? We note the echelon form of the incidence matrix. The top right corner has only 0's, and this is always true since we use the reverse lexicographic order. In the shelling, each of the facets is glued to the complex generated by the facets above it in the list. The unique minimal new set R is the one represented by all the 0's to the left of the rightmost 1. (An example: If the facet is 010110, then the unique minimal new set is $\{1, 3\}$.) It is new, because if it had been in one of the facets above, the facets would be equal, since the zeros to the right of the rightmost 1 are stable upwards in the matrix. Why are all new sets extensions of this set? Suppose that there is a subset R' of the facet which does not contain R. Then there is one element of R which is not in R'. Move the 0 representing this element to the position of the rightmost 1, and replace the 0 with the 1. ($\underline{0}1011\underline{0} \mapsto \underline{0}1110\underline{0}$.) Then, by the Borel property and the ordering, we get a representation of a facet which is already in the complex. The string representing this facet contains all the 0's representing R', so therefore R' is not a new set. This is exactly what we did in the proof of the theorem.

Remark 3.1.15. As a special case of what we have seen so far, let us consider the class of Borel-fixed pure graphs. Let Δ^* be such a graph. Then I_Δ is generated by monomials of degree $n-2$. It is well known that a graph is Cohen-Macaulay if and only if it is connected. (See [14].) Now let $\{t\}$ be a vertex of Δ^*. Since Δ^* is pure of dimension one, there is some $t' \neq t$ such that $\{t, t'\}$ is an edge of Δ^*. By the Borel property of Δ^*, this implies that $\{t, n\}$ is an edge of Δ^* as well, since $n \geq t'$.

We see that all Borel-fixed pure graphs are connected and therefore Cohen-Macaulay. The importance of pureness is immediate in figure 3.1. In this figure, both complexes are Borel-fixed, but only Δ^* is pure. So Δ^* is Cohen-Macaulay, while Δ is not.

In fact, we will spend some pages on yet another proof of the fact that pure Borel-fixed simplicial complexes are CM. This proof uses simplicial homology and is based on a criterion for Cohen-Macaulayness by Hochster and Reisner. However, the result we reach is slightly weaker than theorem 3.1.11, as we do not prove shellability.

Definition 3.1.16. Given a simplicial complex Δ, let $C_q = C_q(\Delta)$ be the f_q-dimensional k-vector space with basis consisting of the q-dimensional faces of Δ. In particular, C_{-1} is a one-dimensional k-space for non-empty Δ. Let $\partial_q : C_q \to C_{q-1}$, $q \geq 0$, be defined on the basis as

$$\partial_q\{v_0, ..., v_q\} = \sum_{i=0}^{q} (-1)^i \{v_0, ..., \hat{v}_i, ..., v_q\}$$

(where the ˆ symbol denotes that v_i is removed from the set) and linearly extended to the whole vector space.

Lemma 3.1.17. *$\partial_q\partial_{q+1} = 0$, and thus $\{C_q, \partial_q\}$ is a complex. We will call it the* augmented chain complex of Δ.

Proof.

$$\begin{aligned}\partial_q\partial_{q+1}\{v_0,...,v_{q+1}\} &= \partial_q\left[\sum_{i=0}^{q+1}(-1)^i\{v_0,...,\hat{v}_i,...,v_{q+1}\}\right]\\ &= \sum_{j<i}(-1)^i(-1)^j\{v_0,...,\hat{v}_j,...,\hat{v}_i,...,v_{q+1}\}\\ &\quad+\sum_{j>i}(-1)^i(-1)^{j-1}\{v_0,...,\hat{v}_i,...,\hat{v}_j,...,v_{q+1}\}\end{aligned}$$

If we switch i and j in the second sum, we see that the two summations cancel. □

We now have a complex

$$C(\Delta):\cdots\longrightarrow C_{q+1}\xrightarrow{\partial_{q+1}}C_q\xrightarrow{\partial_q}C_{q-1}\longrightarrow\cdots$$

Definition 3.1.18. The q-th reduced homology group $\widetilde{\mathrm{H}}_q(\Delta;k)$ is the homology of the complex $C(\Delta)$ at C_q.

We will apply the following criterion for Cohen-Macaulayness, due to Hochster/Reisner.

Definition 3.1.19. Let Δ be a simplicial complex. The *link* $\operatorname{lk}R$ of R is the complex $\{F\in\Delta_{-R}|F\cup R\in\Delta\}$.

Theorem 3.1.20. *Δ is Cohen-Macaulay over k if and only if for all $F\in\Delta$ and all $i<\dim(\operatorname{lk}F)$, we have $\widetilde{\mathrm{H}}_i(\operatorname{lk}F;k)=0$.*

Proof. This is found in [16], chapter II.4. □

Theorem 3.1.21. *If I_{Δ^*} is Borel-fixed and generated by monomials of degree $n-d$, then Δ is Cohen-Macaulay.*

Proof. We use theorem 3.1.20. If $F=\emptyset$, then $\operatorname{lk}F=\Delta$. We have to check that $\widetilde{\mathrm{H}}_i(\Delta;k)=0$ for $i<\dim\Delta=d-1$.

$$\cdots\longrightarrow C_{i+1}\xrightarrow{\partial_{i+1}}C_i\xrightarrow{\partial_i}C_{i-1}\longrightarrow\cdots$$

$$C_j=\bigoplus_{|F|=j+1}kF\quad,\quad\dim C_j=f_j$$

∂_{i+1} is given by the $f_i\times f_{i+1}$ matrix where there is one column for each face of dimension $i+1$ and one row for each face of dimension i. If we arrange these faces in lexicographic order, we see that the entries in each column is given by the following rule: The entries in the column for F_j are 0 in the rows belonging to faces which are not faces of F_j. The top non-zero entry in each the column is ± 1, depending on whether i is even or odd. Down from this entry the non-zero entries are alternating ± 1.

$$\begin{pmatrix} & 0 & \\ \cdots & \pm 1 & \\ & 0 & \\ & \vdots & \\ & 0 & \\ \cdots & \mp 1 & \\ & \vdots & \end{pmatrix}$$

The matrix has a kind of echelon form, with the zeros in the top right corner. (This is due to the lexicographic ordering of the rows and columns.) Every row that corresponds to an $(i+1)$-tuple $\in \Delta$ which is the bottom part of some $(i+2)$-tuple $\in \Delta$ has one pivot position. The other rows do not, so the rank of ∂_{i+1} is

$$\begin{aligned} \text{rank } \partial_{i+1} &= f_i - \#\{(i+1)\text{-tuples} \in \Delta \\ &\qquad \text{that are not the bottom part of some } (i+2)\text{-tuple} \in \Delta\} \\ &= f_i - \#\{(i+1)\text{-tuples} \in \Delta \\ &\qquad \text{that contain } n\} \end{aligned}$$

where the last equality comes from the facts that Δ is pure of dimension $> i$ and that it has the Borel property.

∂_i has a similar matrix. If we count the pivot positions, the nullity of ∂_i is

$$\begin{aligned} \text{null } \partial_i &= f_i - \text{rank } \partial_i \\ &= f_i - \#\{i\text{-tuples} \in \Delta \\ &\qquad \text{that are the bottom part of some } (i+1)\text{-tuple} \in \Delta\} \\ &= f_i - \#\{(i+1)\text{-tuples} \in \Delta \\ &\qquad \text{that contain } n\} \end{aligned}$$

the last equality again coming from the Borel property. Thus rank $\partial_{i+1} =$ null ∂_i, and $\widetilde{\mathrm{H}}_i(\Delta; k) = 0$.

If $F \neq \emptyset$, $\operatorname{lk} F$ is a pure simplicial complex of smaller dimension than Δ. This also satisfies the Borel property, since Δ does. So we can assume that $\widetilde{\mathrm{H}}_i(\operatorname{lk} F; k) = 0$ for $i < \dim \operatorname{lk} F$ by induction, having already shown that for dimension 1, our Δ is Cohen-Macaulay (remark 3.1.15). ☐

Example 3.1.22. To clarify the preceeding proof, we look at maps in the augmented chain complex of the simplicial complex

$$\Delta = \langle \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\} \rangle$$

which has the Borel property. The 2-simplices are given above in lexicographic order, and the 1-simplices are, in lexicographic order,

$$\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}$$

$\partial_2 : C_2(\Delta) \longrightarrow C_1(\Delta)$ is given by the matrix

$$\begin{array}{r} \{1,3\} \\ \{1,4\} \\ \{1,5\} \\ \{2,3\} \\ \{2,4\} \\ \{2,5\} \\ \{3,4\} \\ \{3,5\} \\ \{4,5\} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

while $\partial_1 : C_1(\Delta) \longrightarrow C_0(\Delta)$ is given by

$$\begin{array}{r} \{1\} \\ \{2\} \\ \{3\} \\ \{4\} \\ \{5\} \end{array} \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We see that ∂_2 has rank 5, and that ∂_1 has rank 4 and nullity 9-4=5.

Although a simplicial complex and its Alexander dual are Borel-fixed at the same time, they do not necessarily share the Cohen-Macaulay property. This happens if and only if both of them are pure. CM complexes that have the property that their dual is also CM are called *bi-Cohen-Macaulay* simplicial complexes. To sum up:

Corollary 3.1.23. *If a simplicial complex Δ is Borel-fixed and pure, and Δ^* is pure, then Δ is bi-Cohen-Macaulay.*

Remark 3.1.24. Bi-CM simplicial complexes have been studied in [13]. Simplicial complexes on $[n]$ give rise to complexes of coherent sheaves on $\mathbb{P}_k^{n-1}$. (This is via the BGG correspondence, which will be introduced in section 4.2, but with a different emphasis.) These complexes reduce to one coherent sheaf if and only if the Alexander dual simplicial complex is CM, and this sheaf is locally CM of pure dimension if and only if the simplicial complex is bi-CM.

3.2 An Interpretation of the h-vector of Δ^*

We will use the shelling of Δ^* which we constructed in the preceeding section to find an interpretation of the h-vector of the simplicial complex Δ^* associated to a Borel-fixed ideal I_Δ with minimal generators of equal degree.

Definition 3.2.1. Let G and F be faces of a simplicial complex Δ such that $G \subseteq F$. Then the (closed) *interval* from G to F is

$$[G, F] = \{H | G \subseteq H \subseteq F\}.^2$$

A pure simplicial complex Δ is *partitionable* if Δ can be written as a disjoint union of intervals

$$\Delta = [G_1, F_1] \cup \cdots \cup [G_s, F_s] \tag{3.1}$$

where the F_i are facets. The right hand side of (3.1) is called a *partitioning* of Δ.

Proposition 3.2.2. *Let* (3.1) *be a partitioning of* Δ. *Let* $(h_0, ..., h_d)$ *denote the h-vector of* Δ. *Then*

$$h_i = \#\{j || G_j| = i\}.$$

Proof. This is [16], proposition III.2.3. □

Proposition 3.2.3. *If* $I_\Delta \subset E$ *is a Borel-fixed ideal generated by monomials of degree* t*, then*

$$h_i(\Delta^*) = \#\{\textit{generators } e_\sigma \textit{ of } I_\Delta | \max(\sigma) = t + i\}.$$

Proof. Let us take another look at the shelling we constructed for Δ^* in the proof of 3.1.11. We see that Δ^* can be written as a disjoint union

$$\Delta^* = [r(F_1), F_1] \cup \cdots \cup [r(F_s), F_s]$$

so this is a partitioning of Δ^*. We make use of proposition 3.2.2. If we look closer at the proof of theorem 3.1.11, we find that if the degree of the generators of I_Δ is t,

$$|r(F_i)| = r = \max(F_i^c) - t.$$

This is also explained in example 3.2.4. □

Example 3.2.4. If we take another look at example 3.1.14, we can easily read the h-vector from the incidence matrix, and this should again shed light on our results. Proposition 3.2.2 asks for the cardinality of the minimal new sets. We recall from example 3.1.14 that the minimal new sets are the ones represented by the 0's to the left of the rightmost 1. So h_i is the number of facets where there are i 0's to the left of the 1. That means that if the minimal generators of I_Δ have degree t, then the rightmost 1 is in position $t + i$. Thus the variable e_j with largest index j which divides the generator is e_{t+i}.

So it is enough to count the height of each step of the echelon form of the matrix, and we see that in the example, $h(\Delta^*) = (1, 3, 5, 7)$.

[2] Note that an interval $[G, F]$ is not a simplicial complex unless $G = \emptyset$.

Example 3.2.5. Let us return to the simplicial complex Δ^* in example 3.1.13. A partitioning of Δ^* is given by

$$\Delta^* = [\emptyset, 45] \cup [3, 35] \cup [2, 25] \cup [1, 15] \cup [34, 34].$$

Now, by proposition 3.2.2, h_0 is the number of "interval-starts" with no elements, which is 1 (in this case and all other non-trivial cases). h_1 is the number of starts with one element, which in this case is 3. h_2 is the number of starts with two elements, which in our example is 1. Thus the h-vector is $h(\Delta^*) = (1, 3, 1)$. Looking at the set of minimal monomial generators $\{e_{123}, e_{124}, e_{125}, e_{134}, e_{234}\}$ of I_Δ, we find that there is $h_0 = 1$ generator with largest index 3+0=3, $h_1 = 3$ generators with largest index 3+1=4 and $h_2 = 1$ generator with largest index 3+2=5.

In the proof of proposition 3.2.3 we only used the Borel property of Δ^*. We can avoid the Alexander duality by a reformulation, but we must require that Δ be pure:

Corollary 3.2.6. *Let Δ be a pure simplicial complex of dimension $d-1$ on $[n]$ such that I_Δ is Borel-fixed, and let $(h_0, ..., h_d)$ be the h-vector of Δ. Then*

$$h_i = \#\{\textit{facets } F \in \Delta \mid \textit{the largest number which is not in } F \textit{ is } n-d+i\}.$$

We will now turn to another aspect of the h-vectors. First we need some definitions.

Definition 3.2.7. A *multicomplex* Γ on $X = \{x_1, ..., x_m\}$ is a collection of (commutative) monomials $x_1^{a_1} \cdots x_m^{a_m}$ which is closed under division, that is, if u divides v and v is a monomial in Γ, then u is also in Γ. For a multicomplex Γ, we define the *h-vector* $(h_0, h_1, ...)$ of Γ by $h_i = \#\{u \in \Gamma | \deg(u) = i\}$. A sequence $h = (h_0, h_1, ...)$ of integers such that there exists a multicomplex with h as its h-vector is called an *M-vector*. [3]

Remark 3.2.8. We see that a simplicial complex is a special case of a multicomplex; it is a *squarefree* multicomplex. Therefore, the term *h-vector* is ambiguous in this case. (The h-vector in the new sense is actually what we call the f-vector, but shifted in index.) However, when we speak of the h-vector of a simplicial complex, we will always mean this in the sense of definition 2.1.11. The terminology is standard.

Two natural questions now are: Are the h-vectors of our simplicial complexes M-vectors, and if so, what do their corresponding multicomplexes look like? We will now see that in the case of pure, Borel-fixed simplicial complexes, the answer to the first question is yes, and the answer to the second question is given in the proof and illustrated in example 3.2.10. The proof is written in an almost narrative way. This is because the proof would otherwise be hard to follow.

Theorem 3.2.9. *If Δ is a pure, Borel-fixed simplicial complex, and $h(\Delta)$ is its h-vector, then $h(\Delta)$ is an M-vector.*

[3] After F. S. Macaulay. Note that in general, an M-vector may be infinite.

Proof. We will construct a multicomplex Γ which has $h(\Delta)$ as its h-vector. By proposition 3.2.3, Γ is supposed to have one monomial of degree i for each minimal monomial generator m of I_{Δ^*} where the variable e_j of maximal j which divides m is e_{t+i}, where t is the degree of m. It will be helpful to visualize this by using the incidence matrix introduced in section 3.1.

Our idea now is to let the number t of elements in the variable set $X = \{x_1, ..., x_t\}$ be equal to the degree of the generators of I_{Δ^*}. We then assign a monomial to each generator by looking at how we can construct the corresponding row in the incidence matrix, starting from the first row and moving 1's to the right and 0's to the left.

For any non-empty simplicial complex, $h_0 = 1$, and this corresponds to the monomial 1 of degree 0 which is in all non-empty multicomplexes. Recalling the description of the h-vector of Δ from example 3.2.4, we understand that h_0 is associated to the generator $e_{1..t}$. In other words, it is associated to the first facet in the shelling order, or to the first row in the incidence matrix.

h_1 counts the number of generators divisible by e_{t+1}, but not divisible by e_j for any $j > t+1$. We will call the corresponding facets h_1 type facets, and similarly for rows in the incidence matrix. Γ is supposed to have one monomial of degree one for each of these generators. So we must understand what it is that distinguishes the generators. Obviously, this is which of the e_j, where $j < t+1$, does *not* divide the generator. Put in terms of the rows in the incidence matrix, in which column is the 0? There are t possible spots, one for each of our varibles. We let the monomial be x_j if the zero is in column number j.

generator of I_{Δ^*}	**1**	**2**		$\mathbf{t-2}$	$\mathbf{t-1}$	$\mathbf{t}$	$\mathbf{t+1}$	$\mathbf{t+2}$		**n**	monomial in Γ
$e_{12...t}$	1	1	..	1	1	1	0	0	..	0	1
$e_{12...(t-1)(t+1)}$	1	1	..	1	1	0	1	0	..	0	x_t
$e_{12..(t-2)t(t+1)}$	1	1	..	1	0	1	1	0	..	0	x_{t-1}

By the Borel property, if x_j is associated to a generator, then all $x_{j'}$, where $j < j' \leq t$ is also associated to a generator, and must be included in Γ. In this way we get h_1 monomials of degree 1 in Γ.

Now let us continue to monomials of degree 2. We are supposed to have h_2 such monomials. Thus we should find one monomial for each of the generators of I_Δ which are divisible by e_{t+2} but not divisible by e_j when $j > t+2$. Also, we can only use the variables which were included in Γ as monomials of degree 1. This last condition will be solved by the Borel-fixedness. Indeed, if m is a generator of this sort, we find that there is only one or two generators of the h_1 type which can be obtained from m by replacing the e_{t+2} by an e_j where $j < t+2$, and by the Borel-fixedness, these are in the ideal, and have been given a degree 1 monomial. Then it is natural to assign the product of the two variables as the monomial for the generator. In the case of only one possible h_1 type generator, we square the variable. This happens exactly when e_{t+1} does not divide the generator m.

What does this look like in the incidence matrix? Consider a row in the matrix with its rightmost 1 in column $t+2$. If we are to make a "Borel-move" and move this to the left (to a place occupied by a 0), there are two possibilities. In the case where there is a

1 in column $t+1$ (i.e. the variable e_{t+1} divides the generator associated to the row), each of these possibilities represent a variable x_i. So in this case we assign the product of these two x_i's as the monomial for the row:

	1	**2**			**i**				**j**			**t + 2**			**n**	monomial in Γ
$e_{12..\hat{i}..\hat{j}..(t+2)}$	1	1	..	1	0	1	..	1	0	1	..	1	0	..	0	x_ix_j

In the other case, when there is a 0 in column $t+1$, one of the two possibilities represents the monomial 1, so in this case we choose to square the variable represented by the other possibility.

	1	**2**			**i**			**t**		**t + 2**			**n**	monomial in Γ
$e_{12..\hat{i}..t(t+2)}$	1	1	..	1	0	1	..	1	0	1	0	..	0	x_i^2

The reason for doing things in this way is the Borel-fixedness. The variables that divide our monomials of degree two are precisely the ones that can be produced from the monomials by a Borel-move. And by the Borel-fixedness, these are included in the multicomplex.

When we come to monomials of degree 3, things are a little different. We may now encounter faces where we are not sure whether to choose $x_i^2x_j$ or $x_ix_j^2$. It is now convenient to introduce another way of looking at this. We may replace the 1's in the incidence matrix by the variables. In each row we find one of each of the t variables. In the top row, x_j is in column number j. For the rows corresponding to h_1 type faces, one of the x_j's is in column $t+1$, while the other variables are in their "initial position" from row 1. The variable which has been moved, is exactly the monomial in Γ which we assigned to the row.

	1	**2**			**i**			**t**	**t + 1**			**n**	monomial in Γ
$e_{12..t}$	x_1	x_2	..	x_{i-1}	x_i	x_{i+1}	..	x_t	0	0	..	0	1
⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	
$e_{12..\hat{i}..(t+1)}$	x_1	x_2	..	x_{i-1}	0	x_{i+1}	..	x_t	x_i	0	..	0	x_i

Continuing, we find that the h_2 type rows are built from the h_1 type rows by moving a variable to column $t+2$. But we must be careful now, because unless $t+1$ is in the corresponding facet, there are two ways to do this. Either

	1			**i**				**j**			**t**					**n**	
$e_{1..\hat{i}..\hat{j}..(t+2)}$	x_1	..	x_{i-1}	0	x_{i+1}	..	x_{j-1}	0	x_{j+1}	..	x_t	x_i	x_j	0	..	0	x_ix_j

or

$e_{1..\hat{i}..\hat{j}..(t+2)}$	x_1	..	x_{i-1}	0	x_{i+1}	..	x_{j-1}	0	x_{j+1}	..	x_t	x_j	x_i	0	..	0	x_ix_j

This difference does not matter to us yet, since the two ways produce the same monomial. But in anticipation of what is ahead, we choose the convention that we will use the first of the two possibilities. So we think of it as follows: If $i<j$, then x_i is moved first to column $t+1$ and then x_j is moved passed it to column $t+2$.

The procedure continues in this way. Suppose R is a row in the incidence matrix. Then there are some zeros in the columns 1 through t. The variables corresponding to these columns are the ones which will be in the monomial. Suppose j_1 is the smallest number such that column j_1 has a zero in R. Then, from the initial row, the one corresponding to the generator $e_{12..t}$, move the variable x_{j_1} to column $t+1$, and then to column $t+2$, and so on until it reaches the next column where R has a 1. Suppose this is column $t+a_1$, so it takes a_1 steps.

$$\begin{array}{rcccccccccccccl} R: & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & x_1 & 0 & x_3 & x_4 & x_5 & x_6 & x_2 & 0 & 0 & 0 & 0 & 0 & 1 \text{ step} \rightarrow x_2 \end{array}$$

Now move the next 0-column variable x_{j_2} to column $t+a_1+1$ and then further on in the same way to the next column with 1 in R. Suppose it takes a_2 steps, so this is column $t+a_1+a_2$.

$$\begin{array}{rcccccccccccccl} R: & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & x_1 & 0 & x_3 & x_4 & x_5 & x_6 & x_2 & 0 & 0 & 0 & 0 & 0 & 1 \text{ step} \rightarrow x_2 \\ & x_1 & 0 & x_3 & x_4 & 0 & x_6 & x_2 & 0 & 0 & x_5 & 0 & 0 & 3 \text{ steps} \rightarrow x_2 x_5^3 \end{array}$$

Continue in this way until all the variables corresponding to the zero-columns among columns $1, ..., t$ have been used.

$$\begin{array}{rcccccccccccccl} R: & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & x_1 & 0 & x_3 & x_4 & x_5 & x_6 & x_2 & 0 & 0 & 0 & 0 & 0 & 1 \text{ step} \rightarrow x_2 \\ & x_1 & 0 & x_3 & x_4 & 0 & x_6 & x_2 & 0 & 0 & x_5 & 0 & 0 & 3 \text{ steps} \rightarrow x_2 x_5^3 \\ & x_1 & 0 & x_3 & x_4 & 0 & 0 & x_2 & 0 & 0 & x_5 & 0 & x_6 & 2 \text{ steps} \rightarrow x_2 x_5^3 x_6^2 \end{array}$$

We now assign to the facet in question the monomial $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_s}^{a_s}$.

We have now assigned a monomial to each row in the incidence matrix, and we see that there are h_i monomials of degree i. It only remains to show that our collection of monomials is closed under division.

It suffices to check that if a monomial m, associated to a row R, is in Γ, then all its divisors of degree $\deg(m) - 1$ are also in Γ. If $m = x_{j_1}^{a_1} \cdots x_{j_s}^{a_s}$, then these divisors are

$$x_{j_j}^{a_1} \cdots x_{j_{i-1}}^{a_{i-1}} x_{j_i}^{a_i - 1} x_{j_{i+1}}^{a_{i+1}} \cdots x_{j_s}^{a_s}$$

and these monomials are exactly the ones associated to the rows in the incidence matrix obtained from R by moving all the $s-i+1$ rightmost 1's in R one column to the left (or, in case $a_i = 1$, this 1 is moved back to column j_i). This is a combination of Borel-moves, and therefore these rows are in the matrix, and the divisors are in the multicomplex.

So we have constructed a multicomplex Γ with $h(\Delta)$ as its h-vector, and may conclude that $h(\Delta)$ is an M-vector. □

Example 3.2.10. Let us once again return to the simplicial complex from example 3.1.14, and see what the multicomplex looks like.

generator-index	**1**	**2**	**3**	**4**	**5**	**6**	monomial in Γ
123	x_1	x_2	x_3	0	0	0	1
124	x_1	x_2	0	x_3	0	0	x_3
134	x_1	0	x_3	x_2	0	0	x_2
234	0	x_2	x_3	x_1	0	0	x_1
125	x_1	x_2	0	0	x_3	0	x_3^2
135	x_1	0	x_3	0	x_2	0	x_2^2
235	0	x_2	x_3	0	x_1	0	x_1^2
145	x_1	0	0	x_2	x_3	0	x_2x_3
245	0	x_2	0	x_1	x_3	0	x_1x_3
126	x_1	x_2	0	0	0	x_3	x_3^3
136	x_1	0	x_3	0	0	x_2	x_2^3
236	0	x_2	x_3	0	0	x_1	x_1^3
146	x_1	0	0	x_2	0	x_3	$x_2x_3^2$
246	0	x_2	0	x_1	0	x_3	$x_1x_3^2$
156	x_1	0	0	0	x_2	x_3	$x_2^2x_3$
256	0	x_2	0	0	x_1	x_3	$x_1^2x_3$

Corollary 3.2.11. *If Δ is a Cohen-Macaulay simplicial complex and $h(\Delta)$ is its h-vector, then $h(\Delta)$ is an M-vector.*

Proof. By the process defined in section 2.3 (and theorems 2.3.7 and 2.3.8), Δ has the same h-vector as some pure Borel-fixed simplicial complex Δ_s. By the preceeding theorem, this h-vector is an M-vector. ☐

Example 3.2.12. The simplicial complex $\Delta = \langle 123, 345 \rangle$ pictured in figure 3.4 has f-vector $(1, 5, 6, 2)$ and, by theorem 2.1.12, h-vector $(1, 2, -1, 0)$. This is obviously not an M-vector, since it has a negative integer for h_2. So, by corollary 3.2.11, we conclude that Δ is not Cohen-Macaulay, even though it is pure.

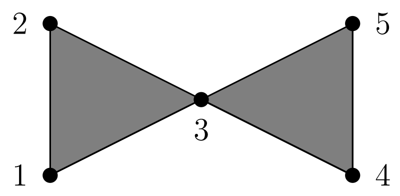

Figure 3.4: The simplicial complex Δ from example 3.2.12.

3.3 l-Cohen-Macaulay Simplicial Complexes

In this section we will study l-Cohen-Macaulay simplicial complexes, introduced by Baclawski in [4]. They are studied in [11]. We will find a procedure for determining when a simplicial complex is l-Cohen-Macaulay, but not $(l+1)$-Cohen-Macaulay. As usual, we will stick to the case when the monomial ideal corresponding to the Alexander dual simplicial complex is Borel-fixed and generated by monomials of equal degree. That is, pure Borel-fixed simplicial complexes.

First a few definitions:

Definition 3.3.1. Let Δ be a simplicial complex on $[n]$, and let $R \subset [n]$. The *restriction* Δ_R of Δ to R is the simplicial complex consisting of the faces of Δ which are contained in R. Δ_{-R} will denote the restriction to the subset $[n]\backslash R$.

Definition 3.3.2. A simplicial complex Δ is said to be *l-Cohen-Macaulay* (or l-CM) if Δ_{-R} is Cohen-Macaulay of the same dimension as Δ for all subsets $R \subset [n]$ of cardinality $\leq l-1$.

Remark 3.3.3. Graphs are CM if and only if they are connected, so if Δ is a graph, then Δ is l-CM if and only if it is l-connected.

Recall the connection between the monomial ideal $I_{\Delta^*} \subset E$ and Δ: A monomial e_F of E is in I_{Δ^*} if and only if F^c is a face of Δ. Thus the minimal generators of I_{Δ^*} are in a one-to-one correspondence with the facets of Δ. Monomial ideals with minimal generators of different degree correspond to simplicial complexes which are not pure, and therefore not Cohen-Macaulay.

Criterion 3.3.4. Let Δ be a Borel-fixed pure graph on $[n]$. Suppose t is the smallest number such that there exists an m where $\{m, n-t\}$ is an edge of Δ, but $\{m, n-t-1\}$ is not an edge of Δ. Then Δ is $(t+1)$-Cohen-Macaulay, but not $(t+2)$-Cohen-Macaulay.

Proof. Suppose Δ is a Borel-fixed graph without isolated vertices. As we know, graphs are Cohen-Macaulay whenever they are connected. Every vertex in Δ is adjacent to the vertex labelled n, by the Borel property. Thus Δ is connected and Cohen-Macaulay.

If $R = \emptyset$, then $\Delta_{-R} = \Delta$. If $R = \{m\}$, where $m \neq n$, then Δ_{-R} is a simplicial complex satisfying the Borel property, except that the vertex m and its adjacent edges are removed. (It corresponds to the monomial ideal which is the same as I_{Δ^*}, except that $e_m = 0$, and this is also Borel-fixed.) Δ_{-R} will therefore be Cohen-Macaulay for the same reasons as Δ.

To check whether Δ is 2-Cohen-Macaulay, we need to see what happens when $R = \{n\}$. If all the edges of Δ are of the kind $\{m, n\}$, Δ_{-R} will just be a collection of isolated vertices. It has dimension $0 < \dim(\Delta)$, and Δ will in this case not be 2-CM. Suppose that there are edges which are not of this kind. If Δ_{-R} is not Cohen-Macaulay, then it must be disconnected. Take any edge $\{m_1, m_2\} \in \Delta_{-R}$. Because of the Borel property, both $\{m_1, n-1\}$ and $\{m_2, n-1\}$ are in Δ_{-R}. So if it is to be disconnected, Δ_{-R} must have an isolated vertex. This happens if and only if there is an m such that $\{m, n\} \in \Delta$, but

$\{m, n-1\} \notin \Delta$. If there is such an m, then Δ is not 2-CM. If there are no such edges, then Δ_{-R} will satisfy the Borel property, and Δ is therefore 2-CM, for the same reason as before.

We can continue this in the obvious way to check for 3-Cohen-Macaulayness, by starting with $\Delta_{-\{n\}}$ and repeat what we did for Δ, with n replaced by $n-1$. We continue until we end up with an isolated vertex. □

The criterion can be stated even simpler for graphs Δ because of the Borel property. Let m be the smallest number of a vertex in the graph. Then vertex m has edges to the vertices $l, l+1, ..., n-1, n$ for some l. Also, every vertex $m' > m$ has edges to these vertices, by the Borel property. That means that if some vertex is cut loose from the graph by removing vertices, then m is cut loose. Thus we can reformulate criterion 3.3.4:

Criterion 3.3.5. In the situation of criterion 3.3.4, let m be the smallest number labelling a vertex of Δ. Let t be the smallest number such that $\{m, n-t\} \in \Delta$. Then Δ is $(t+1)$-CM, but not $(t+2)$-CM.

Example 3.3.6. Consider once again the graph from example 3.1.13 which is displayed in figure 3.2. We see that $\{1, 5\}$ is a face of Δ^* while $\{1, 4\}$ is not. So $t = 1$ in the criterion, and Δ^* is 1-CM, but not 2-CM. See figure 3.5.

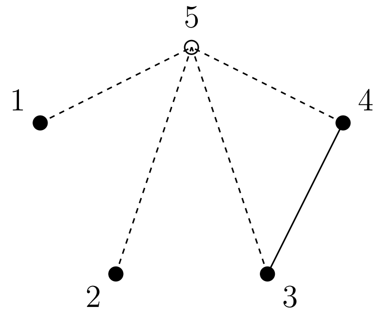

Figure 3.5: Deleting vertex 5 and its adjacent edges in example 3.3.6. The resulting graph is not connected, i.e. not Cohen-Macaulay.

This gives an idea of what we can expect when I_{Δ^*} is generated in other degrees, although the nice description of Cohen-Macaulayness for graphs made the above quite easy.

Theorem 3.3.7. *Let Δ be a $(d-1)$-dimensional pure Borel-fixed simplicial complex. Let t be the smallest number such that there exist different $a_1, ..., a_{d-1} < n-t$ with $F = \{a_1, ..., a_{d-1}, n-t\}$ in Δ, but $\{a_1, ..., a_{d-1}, m\}$ not in Δ for the largest $m < n-t$ not in F. Then Δ is $(t+1)$-CM, but not $(t+2)$-CM.*

Proof. If $R = \{m\}$, then Δ_{-R} is a simplicial complex which still satisfies the Borel property, only skipping m in the definition. If Δ_{-R} has the same dimension as Δ, and if it is pure, then it will be CM for the same reason as Δ. The only way it can be non-CM is that it is non-pure. This happens exactly when there are different $a_1, ..., a_{d-1} \neq m$ such that $\{a_1, ..., a_{d-1}, m\} \in \Delta$, but $\{a_1, ..., a_{d-1}, m'\} \notin \Delta$ for all $m' \neq m, m' \neq a_i$. But since Δ satisfies the Borel property, any $m' > m$ should give a face in Δ. It therefore suffices to check if this happens for $m = n$. If Δ_{-R} is of dimension lower than Δ for some m, then Δ is not 2-CM. This happens when m is in every facet, so by the Borel property, it happens if and only if it happens with $m = n$. Thus we find that the criterion given is correct for 2-Cohen-Macaulay.

To check for 3-CM we only need to check $\Delta_{-\{n\}}$ for 2-CM when Δ is 2-CM. This is done in the exact same way as above, with $n-1$ playing the role of n, since $\Delta_{-\{n\}}$ behaves exactly like Δ, only with fewer facets. Thus Δ is 3-CM if and only if it is 2-CM and there do not exist different $a_1, ..., a_{d-1} < n - 1$ with $\{a_1, ..., a_{d-1}, n - 1\} \in \Delta$ while $\{a_1, ..., a_{d-1}, m\} \notin \Delta$ for the maximal $m < n - 1$, $m \neq a_i, 1 \leq i \leq d - 1$.

We can continue like this until we find that when $n - t$ is removed, the resulting simplicial complex is not Cohen-Macaulay or of lower dimension than Δ. ☐

Example 3.3.8. We will now use our result on higher Cohen-Macaulayness to find out more about our favourite example, the simplicial complex Δ^* from example 3.1.14. The first question is, when we delete vertex 6 and its adjacent faces, will the rest of the complex still be pure of dimension 2? The complex $\Delta^*_{-\{6\}}$ is represented by the matrix:

face	**1**	**2**	**3**	**4**	**5**
45	1	1	1	0	0
35	1	1	0	1	0
25	1	0	1	1	0
15	0	1	1	1	0
34	1	1	0	0	1
24	1	0	1	0	1
14	0	1	1	0	1
23	1	0	0	1	1
13	0	1	0	1	1
345	1	1	0	0	0
245	1	0	1	0	0
145	0	1	1	0	0
235	1	0	0	1	0
135	0	1	0	1	0
234	1	0	0	0	1
134	0	1	0	0	1

This is just the matrix from example 3.1.14 with column 6 removed. The nine first rows represent one-dimensional faces, and our hope is to find that all these one-dimensional faces are subsets of the two-dimensional faces, which we find in the last seven rows.

The answer is yes. Let us take the face $\{3, 5\}$ represented by 11010 as an example. By theorem 3.3.7, it is enough to check if the face $\{3, 4, 5\}$ represented by 11000 is in $\Delta^*_{-\{6\}}$.

And indeed it is. The same can be done for all the other one-dimensional faces. We delete the nine superfluous rows end up with the following incidence matrix for $\Delta^*_{-\{6\}}$:

1	**2**	**3**	**4**	**5**
1	1	0	0	0
1	0	1	0	0
0	1	1	0	0
1	0	0	1	0
0	1	0	1	0
1	0	0	0	1
0	1	0	0	1

Thus Δ^* is 2-CM.

It turns out that Δ^* is 3-CM as well. If we remove vertex five, we find that the remaining one-dimensional faces are subsets of the two-dimensional faces. $\Delta^*_{-\{5,6\}}$ is described by the matrix

1	**2**	**3**	**4**
1	0	0	0
0	1	0	0

From this it is easily seen that if we remove vertex 4, the dimension drops. Thus Δ^* is 3-CM, but not 4-CM.

Our next topic is what higher Cohen-Macaulayness does to the h-vector.

Proposition 3.3.9. *If Δ is a $d-1$-dimensional 2-Cohen-Macaulay Borel-fixed simplicial complex and $(h_0, ..., h_d)$ is its h-vector, then*

$$h_i \leq h_j$$

whenever $1 \leq i \leq j \leq d$.

Proof. Choose some $i > d$. By corollary 3.2.6, h_i is the number of facets which include all the vertices with labels from $n-d+i+1$ to n, but not the vertex labelled $n-d+i$. By theorem 3.3.7, the 2-Cohen-Macaulayness requires that if $\{a_1, ..., a_{d-1}, n\}$, where $a_i \neq n-d+i$, is such a face, then $\{a_1, ..., a_{d-1}, n-d+i\}$ is a face as well. For different facets of the h_i type, we get in this way different facets of the h_d type. Therefore, the number h_i cannot exceed h_d.

We now use the Borel property. The fact that $\{a_1, ..., a_{d-1}, n-d+i\}$ is a face, and that for some j, $a_j = n-1$, implies that $\{a_1, ..., \hat{a_j}, ..., a_{d-1}, n-d+i, n\}$ is a face too. As this face contains the vertex n but not the vertex $n-1$, it is a h_{d-1} type face. All the h_i different faces of h_d type which we showed must exist give different faces of h_{d-1} type. Therefore, h_i cannot exceed h_{d-1} either.

We continue to use the Borel property in this way, finding h_i distinct faces for each step. Thus we may conclude that $h_i \leq h_j$ whenever $i \leq j$. ☐

Example 3.3.10. Let us review the steps in the preceeding proof in terms of the complex Δ^* from example 3.1.14, which is 2-CM, as we saw in example 3.3.8. In this example, $d = 3$. $h_1 = 3$ is the number of faces where vertex 4 is the vertex with highest number which is not in the face. That is, the height of the second step from the top of the incidence matrix. These faces are the ones represented by the rows

facet	**1**	**2**	**3**	**4**	**5**	**6**
356	1	1	0	1	0	0
256	1	0	1	1	0	0
156	0	1	1	1	0	0

All these faces include vertex 6. Theorem 3.3.7 now says that since Δ^* is 2-CM, the set obtained from any of these facets by replacing the vertex 6 with the vertex of highest label which is *not* in the facet (i.e. vertex 4) is also a facet. That is, the sets $\{3, 4, 5\}$, $\{2, 4, 5\}$ and $\{1, 4, 5\}$ must be facets of Δ^*. These are the facets represented by the rows

facet	**1**	**2**	**3**	**4**	**5**	**6**
345	1	1	0	0	0	1
245	1	0	1	0	0	1
145	0	1	1	0	0	1

So, since h_3 counts the number of facets which do not contain vertex 6, we must have that $h_1 \leq h_3$.

Now by the Borel property of Δ^*, we may replace vertex 5 in these facets with vertex 6, and find new facets:

facet	**1**	**2**	**3**	**4**	**5**	**6**
346	1	1	0	0	1	0
246	1	0	1	0	1	0
146	0	1	1	0	1	0

These facets are of h_2 type, and therefore $h_1 \leq h_2$.

Another thing that may be seen from this example is that equality in proposition 3.3.9 may occur. If we remove the facets $\{2, 3, 4\}$ and $\{1, 3, 4\}$ from Δ^*, the resulting complex is still Borel-fixed and 2-CM, but $h_2 = h_3$.

Remark 3.3.11. One may ask, what is the significance of l-Cohen-Macaulayness? As an interesting example of what it describes, we mention that in a recent preprint [12] it is shown that a simplicial complex Δ is l-CM if and only if Δ is CM and the so-called top *enriched cohomology module* can occur as an $l-1$'th syzygy module in a free resolution over the symmetric algebra S.

3.4 Complexes Corresponding to Stable Ideals

Let us now take a short look at what happens if we reduce our requirements, so that the simplicial complexes correspond to ideals which satisfy the weaker condition of being stable (as opposed to strongly stable, i.e. Borel-fixed):

Definition 3.4.1. A monomial ideal I in the exterior algebra E is called *stable* if whenever e_σ is a monomial in I, then $e_i e_{\sigma\setminus\{s\}} \in I$ for all $i < s$ when $s = \max(\sigma)$.

We see that in particular, Borel-fixed ideals are stable.

So what do the involved simplicial complexes look like, combinatorially? We take the approach through Alexander duality. Let I_{Δ^*} be a stable monomial ideal. Suppose $F = \{a_1, ..., a_d\}$ is a face of Δ. Then e_{F^c} is a monomial in I_{Δ^*}. Let $F^c = \{b_1, ..., b_t\}$, $b_1 < \cdots < b_t$. Since I_{Δ^*} is stable, this means that $e_{b_1 b_2 \ldots b_{t-1}} e_{a_i}$ is in I_{Δ^*} for all i such that $a_i < b_t$. In other words, $\{a_1, ..., a_{i-1}, a_{i+1}, ..., a_d, b_t\}$ is a face of Δ.

To sum up: Δ has the property that if you replace a vertex in a face F with the highest labelled vertex which is not in F, then you get a new face of Δ. For graphs, this means that for any edge in Δ, there are edges connecting the endpoints of it to the vertex n. So it consists of rays and triangles wedged together in vertex n, and $\{n-1, n\}$ is always an edge in Δ. See figure 3.6.

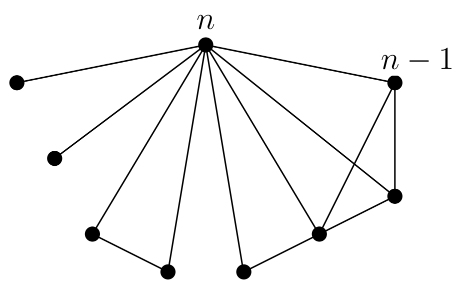

Figure 3.6: A typical graph Δ corresponding to a stable I_{Δ^*}

The equivalence of lemma 3.1.6 does not have a counterpart in this more general situation. We illustrate this in an example:

Example 3.4.2. Consider the simplicial complex

$$\Delta = \langle 12, 15, 25, 34, 35, 45 \rangle$$

which is drawn in figure 3.7. We see that Δ corresponds the stable ideal

$$I_{\Delta^*} = (e_{345}, e_{245}, e_{134}, e_{125}, e_{124}, e_{123}).$$

The Alexander dual of Δ is

$$\Delta^* = \langle 12, 34, 135, 145, 235, 245 \rangle .$$

The topological realization of Δ^* looks like a pyramid-shaped hat with two strings crossing the hole, and vertex 5 as the apex. An attempt to draw it is given in figure 3.8. We see also that $\{1, 3, 5\}$ is a face of Δ^*, while $\{3, 4, 5\}$ is not, so I_Δ is not stable.

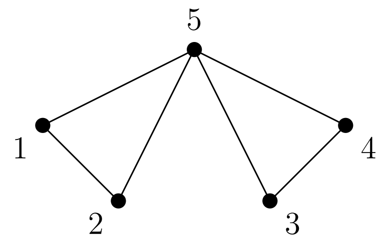

Figure 3.7: Δ of example 3.4.2.

It is obvious that graphs Δ such that I_{Δ^*} is stable are Cohen-Macaulay. Every vertex has an edge to n, so Δ is connected. However, there is nothing about them that can guarantee higher Cohen-Macaulayness. $\Delta_{-\{n\}}$ is CM of dimension 1 if and only if there is a path in Δ between any two given vertices not passing through n. But there does not seem to be a nice way of describing this. Vertex $(n-1)$ does not have a more special position than the other vertices (except that it is always in the complex), as the case is for Borel-fixed graphs. All structure on Δ is lost when we remove vertex n.

With this in mind, we understand that given any graph G, we may construct a new graph $\Delta = \Delta(G)$, containing G as a subgraph, which corresponds to a stable ideal I_{Δ^*}. This is done by adding a vertex $n+1$ and drawing edges from $n+1$ to all the vertices of G. As noted, this graph is CM. If we do the same with the new graph, we will produce a 2-CM graph. And if we continue to do this successively, the CMness rises by one for each step.

Let us move on to two-dimensional simplicial complexes Δ of this sort. Suppose $\{a,b,c\}$ is a facet of Δ. Then, assuming $a,b,c \leq n-2$, the following sets must be in Δ: $\{a,b,n\}, \{a,c,n\}, \{b,c,n\}, \{a,n-1,n\}, \{b,n-1,n\}, \{c,n-1,n\}, \{n-2,n-1,n\}$. Thus the facet $\{a,b,c\}$ generates a complex which looks like the one in figure 3.9.

Now if there is an additional facet $\{a',b',c'\}$ of Δ, then this too will generate such a complex, and these complexes will share the facet $\{n-2,n-1,n\}$ (as well as possibly some other faces, if $\{a,b,c\} \cap \{a',b',c'\} \neq \emptyset$). So in general, two-dimensional complexes Δ corresponding to stable I_{Δ^*} look like complexes like the one in figure 3.9 glued together. The different components of this large complex do not relate to each other in other manners than that they look the same, and that they share some faces.

Proposition 3.4.3. *If Δ is a pure simplicial complex of dimension one or two such that I_{Δ^*} is stable, then Δ is shellable and therefore Cohen-Macaulay for all fields k.*

Proof. All CM graphs are shellable. This is because it does not matter whether it is glued onto zero, one or two vertices. There is always a unique minimal new set anyway.

A pure two-dimensional simplicial complex is shellable exactly when we can glue the facets together sequentially such that each facet (triangle) is connected to the earlier facets along one or more of its edges (except for the first facet, which is glued onto the empty set).

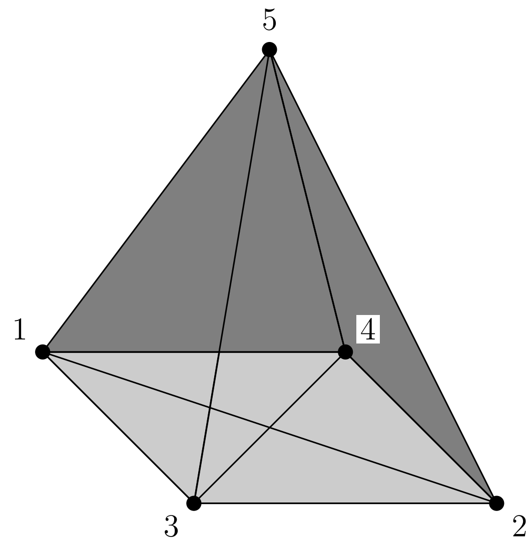

Figure 3.8: The "pyramid-shaped hat" Δ^* of example 3.4.2. It is open in the bottom, with edges crossing the hole.

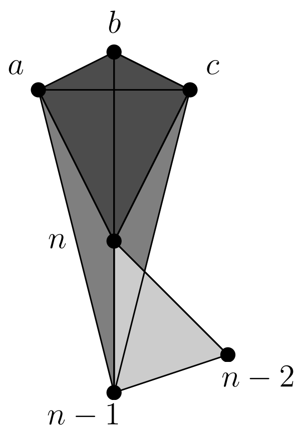

Figure 3.9: The simplicial complex Δ corresponding to a stable I_{Δ^*} generated by $\{a, b, c\}$ such that $a, b, c < n-2$: An empty tetrahedron with three wings meeting in the point $n-1$, and with the face $\{n-2, n-1, n\}$ glued on.

We see that this can be done for the complex in figure 3.9, starting with $\{n-2, n-1, n\}$ as the first facet. No facet has to be glued on in a vertex. Thus it can also be done for the composition of more than one such complexes. □

It would be interesting to study similar problems for higher dimensions, but this is beyond our scope in this thesis.

Chapter 4

Regularity in E

In this chapter we will study (Castelnuovo-Mumford) regularity of ideals in the exterior algebra. In [5], Bayer and Stillman gave a criterion for when a homogeneous ideal I in the symmetric algebra $S = k[x_1, ..., x_n]$ is m-regular. We will prove a similar result for ideals in the exterior algebra. Moreover, Bayer and Stillman used their criterion to prove that if we use the reverse lexicographic order on monomials, the generic initial ideal of I has the same regularity as I. We will arrive at the same statement for ideals in the exterior algebra.

4.1 Definitions and basics

As usual, V will be a vector space over a field k with basis $e_1, ..., e_n$, and $E = \bigwedge V$ will be the exterior algebra over V.

Let M be a module over E. We are considering free resolutions of this module:

$$\cdots \longrightarrow \oplus_j E(-a_{ij}) \xrightarrow{d_i} \cdots \longrightarrow \oplus_j E(-a_{1j}) \xrightarrow{d_1} \oplus_j E(-a_{0j}) \longrightarrow M$$

Such a resolution is said to be *minimal* if all the entries in the matrices describing the d_i have positive degree. Modules over the symmetric algebra always have finite free resolutions. (This is the Hilbert syzygy theorem.) Modules over the exterior algebra, however, do not have finite resolutions unless they are free modules.

Definition 4.1.1. Let

$$\cdots \longrightarrow \oplus_j E(-a_{ij}) \longrightarrow \cdots \longrightarrow \oplus_j E(-a_{1j}) \longrightarrow \oplus_j E(-a_{0j}) \longrightarrow M \tag{4.1}$$

be a minimal resolution of the E-module M. M is said to be *m-regular* if $a_{ij} \leq m + i$ for all i. The *regularity* $\operatorname{reg} M$ of M is the minimal m such that M is m-regular.

We will not be concerned with other modules than ideals and quotients of E and their duals. Even though the resolutions of our modules are infinite, their regularity is bounded:

Proposition 4.1.2. *Let M be a module over E. Then*

$$\operatorname{reg} M \le \max\{d \mid M_d \neq 0\}$$

Proof. This is stated in [3], bottom of page 186. □

There is a description of regularity in terms of the Tor-funktor. Upon tensoring (4.1) with k and taking the homology, we find

$$\operatorname{Tor}_i^E(M,k) = \oplus_j k(-a_{ij})$$

That is, the k-vector space shifted in degree according to the modules in the free resolution. So we can say that M is m-regular if and only if

$$\operatorname{Tor}_i^E(M,k)_d = 0 \text{ for all } d > m+i$$

This enables us to study how the regularities of different modules relate to each other by passing from short exact sequences to long exact Tor-sequences. If

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence, then we get the associated long exact Tor-sequence[1]

$$\cdots \to \operatorname{Tor}_{i+1}(M_3,k) \to \operatorname{Tor}_i(M_1,k) \to \operatorname{Tor}_i(M_2,k) \to \operatorname{Tor}_i(M_3,k) \to \cdots$$

As an example of this, we have the following lemma:

Lemma 4.1.3. *Let m be the regularity of the ideal I. Then* $\operatorname{reg} E/I = m-1$.

Proof. We have a short exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow E/I \longrightarrow 0$$

This provides the long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}(E,k) \to \operatorname{Tor}_{i+1}(E/I,k) \to \operatorname{Tor}_i(I,k) \to \operatorname{Tor}_i(E,k) \to \cdots$$

$\operatorname{Tor}_i(E,k) = 0$ for all $i > 0$, and

$$\operatorname{Tor}_0(E,k) = E \otimes k \simeq k$$

(degree zero), so the exactness of the sequence implies that

$$\operatorname{Tor}_{i+1}(E/I) \simeq \operatorname{Tor}_i(I,k)$$

as graded vector spaces for all $i > 0$. For $i = 0$, they are isomorphic in degrees > 0. So if

$$\operatorname{Tor}_i(I,k)_d = 0 \text{ for all } d > m+i,$$

then

$$\operatorname{Tor}_{i+1}(E/I)_d = 0 \text{ for all } d > (m-1)+(i+1),$$

and similarly the other way. □

This result is of course easy to see from the resolution of E/I, which looks like

$$(\text{resolution of } I) \to E \to E/I \tag{4.2}$$

[1] We will leave out the superscript E when it is clear from the context that we are dealing with E-modules.

4.2 The BGG Correspondence

In this section we will see basic properties of the Bernstein-Gelfand-Gelfand (BGG) correspondence. Our presentation is influenced by the paper [10] and Eisenbud's new textbook [9]. Some relevant material is also from [11].

Let now $W = V^* = \mathrm{Hom}_k(V, k)$, the vector space dual of V, and let $S = S(W)$ be the symmetric algebra on W. If $e_1, ..., e_n$ is a basis for V, we choose a dual basis $x_1, ..., x_n$ of W. That is,

$$x_i(e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.3}$$

So $S = k[x_1, ..., x_n]$. We give the elements of V degree 1, and the elements of W degree -1.

The BGG correspondence assigns to each E-module a complex of S-modules. Suppose $M = \oplus_i M_i$ is a graded E-module. We can form a complex of free S-modules

$$\mathbf{L}(M) : \cdots \longrightarrow S(i) \otimes_k M_i \xrightarrow{d^i} S(i+1) \otimes_k M_{i+1} \longrightarrow \cdots \tag{4.4}$$

where the differential is given by

$$d^i(s \otimes m) = \sum_{\alpha=1}^{n} s x_\alpha \otimes e_\alpha m$$

and extended linearly.

Lemma 4.2.1. *$d^{i+1} \circ d^i = 0$, so (4.4) is indeed a complex.*

Proof.

$$\begin{aligned} d^{i+1} \circ d^i(s \otimes m) &= d^{i+1}\left(\sum_{\alpha=1}^{n} s x_\alpha \otimes e_\alpha m\right) \\ &= \sum_{\beta=1}^{n}\sum_{\alpha=1}^{n} s x_\alpha x_\beta \otimes e_\beta e_\alpha m \\ &= \sum_{\alpha \leq \beta} s x_\alpha x_\beta \otimes (e_\alpha e_\beta + e_\beta e_\alpha) m = 0 \end{aligned}$$

since $e_\alpha e_\beta = -e_\beta e_\alpha$ when $\alpha \neq \beta$ and $e_\alpha^2 = 0$. □

Proposition 4.2.2. $\mathbf{L}$ *is a functor from the category of graded E-modules to the category of linear free complexes of S-modules. In fact, it is an equivalence of categories.*

Proof. See [9], proposition 7.5. □

Proposition 4.2.3. *If M is a graded E-module, then M is m-regular if and only if $\mathrm{H}^p(\mathbf{L}(M^*)) = 0$ for $p < -m$, where $M^* = \mathrm{Hom}_k(M, k)$ denotes the vector space dual of M.*

Proof. This is [10], proposition 2.2 part b), suitably rewritten for our point of view. □

4.3 Generic elements

The key concept in [5] is that of elements of degree 1 in S which are generic for the ideal in question, meaning that they are not zero-divisors on S/I^{sat}, where I^{sat} is the saturation of I. The notion of saturated ideals doesn't make much sense in the exterior algebra, but nevertheless we can find elements which have the desired effect:

Definition 4.3.1. Let I be an ideal in the exterior algebra $E = \bigwedge V$. An element h in $E_1 = V$ is said to be *m-generic* for I if $(I : h)_d = (I, h)_d$ for all $d \geq m$.

Remark 4.3.2. Note that since $h \cdot h = 0$, $(I, h) \subset (I : h)$. Thus the generic elements are the elements h that make $(I : h)$ "smallest possible" in degrees $\geq m$.

We want to show not only that if I is m-regular, then such elements exist, but also that the condition of being m-regular is a Zariski open condition.

First we need some terminology. Lines through zero in V may be viewed as points in $\mathbb{P}(W)$, the projective vector space of W. If $e_1, ..., e_n$ and $x_1, ..., x_n$ are dual vector space bases of V and W respectively (as in (4.3)), and the elements v and w are given in these bases as

$$v = \sum v_i e_i \in V \text{ and } w = \sum w_i x_i \in W,$$

then

$$w(v) = \sum w_i v_i \in k.$$

So, given an element w in W, we consider the set

$$D(w) = \{\text{lines spanned by } v \in V \mid w(v) \neq 0\}$$

which has the structure of an $(n-1)$-dimensional affine k-space.

We will let the coordinate ring of $D(w)$ be denoted by $S_{(w)}$. If v is in $D(w)$, then we may form the residue field at v:

$$k(v) = S_{(w)}/I(v)$$

where $I(v)$ is the maximal ideal of $S_{(w)}$ consisting of polynomials vanishing at the point v.

Suppose now that we have a graded ideal I in the exterior algebra E over V. Suppose also that I is m-regular. This is equivalent to the module E/I being $(m-1)$-regular, by lemma 4.1.3. We want to find an element h in $E_1 = V$ such that $(I : h)_d = (I, h)_d$ for all $d \geq m$. That is, we want the complex

$$(E/I)_{p-1} \xrightarrow{\cdot h} (E/I)_p \xrightarrow{\cdot h} (E/I)_{p+1}$$

to be without homology for $p \geq m$. If we dualize the complex above, this is the same as requiring that the complex

$$(E/I)^*_{p-1} \xrightarrow{\cdot h} (E/I)^*_p \xrightarrow{\cdot h} (E/I)^*_{p+1} \tag{4.5}$$

be without homology for $p \leq -m$. Call the (co)homology of this complex $\mathrm{H}^p((E/I)^*, h)$.

Now we use the BGG correspondence introduced in section 4.2. The complex associated to $(E/I)^*$ is

$$\mathbf{L}((E/I)^*) : \cdots \longrightarrow S(p-1) \otimes_k (E/I)^*_{p-1} \longrightarrow S(p) \otimes_k (E/I)^*_p \longrightarrow \cdots$$

where the differential is given by

$$s \otimes m \mapsto \sum_{\alpha=0}^{n} s x_\alpha \otimes e_\alpha m$$

where m is an element in the quotient E/I. We will make use of the following proposition.

Proposition 4.3.3. *If h is an element of $D(w)$ and $k(h)$ is the residue field at h, then*

$$\mathrm{H}^p((E/I)^*, h) = \mathrm{H}^p(\mathbf{L}((E/I)^*)_{(w)} \otimes_{S_{(w)}} k(h)) \tag{4.6}$$

where $\mathbf{L}((E/I)^*)_{(w)}$ means that we have replaced all the S'es in $\mathbf{L}((E/I)^*)$ with $S_{(w)}$.

Proof. This is stated in [11]. The reason for this equality is that the differential in $\mathbf{L}$ reduces to the differential of (4.5) when we tensor with $k(h)$. The x_α's turn to h_α, if $h = \sum_{\alpha=1}^{n} h_\alpha e_\alpha$. □

Now a general lemma:

Lemma 4.3.4. *Let R be an integral domain and let M be a finitely generated module over R. Then there exists an element f of R such that the localized module M_f is a free R_f-module.*

Proof. Suppose $\{m_1, ..., m_n\}$ is a set of generators for M. Let $K = R_{(0)}$ be the quotient field of R. Then $M_{(0)}$ is a vector space over K. Let us see what happens to the generators m_i when we localize. A generator m_i goes to 0 in $M_{(0)}$ if and only if there exists an element $r_i \in R^*$ such that $r_i m_i = 0$. For each m_i where it is possible, choose such an r_i.

$M_{(0)}$ is generated by the elements of the form $\frac{m_j}{1}$ where no such r_j can be found. We remove the redundant generators to find a vector space basis among these elements. The generators which are removed are expressed in this basis as

$$\frac{m_j}{1} = \sum_k \frac{s_{jk} m_k}{r_{jk}}$$

for suitable $s_{jk}, r_{jk} \in R$. For each of these m_j's, choose such a collection of r_{jk}'s.

Now $M_{(0)}$ has a K-vector space basis consisting of the $\frac{m_k}{1}$'s where there are no relations between the m_k's. Let $f = \Pi r_i \cdot \Pi r_{jk}$ be the product of all the chosen r's. As R is an integral domain, $f \neq 0$. The localized module M_f over R_f is then generated freely by the images of the m_k's which produced the vector space basis of $M_{(0)}$. The other generators are either annihilated by f or such that

$$\frac{m_j}{1} = \sum_k \frac{f_{jk} s_{jk} m_k}{f}, \quad \text{where} \quad f_{jk} = \Pi r_i \cdot \Pi_{a \neq j, b \neq k} r_{ab}.$$

□

Proposition 4.3.5. *If I is a graded ideal in the exterior algebra E, and $\operatorname{reg}(I) = m$, then there exists a Zariski open set $U \subset V$ such that whenever h is an element of U, then h is m-generic for I.*

Proof. Each module $\mathrm{H}^p(\mathbf{L}((E/I)^*)_{(w)})$ is finitely generated over $S_{(w)}$. $S_{(w)}$ is an integral domain, so by lemma 4.3.4 there exists an element f in $S_{(w)}$ such that each $\mathrm{H}^p(\mathbf{L}((E/I)^*)_{(w)})_f$ is a free module over $(S_{(w)})_f$. (This element is obtained by finding f_p's for each p and multiplying them together.)

So, if we choose an element h in the open set

$$D(f) = \{v \in V \mid f(v) \neq 0\},$$

that is, an h such that $f(h) \neq 0$, we have that

$$\mathrm{H}^p\left((\mathbf{L}((E/I)^*)_{(w)})_f \otimes_{(S_{(w)})_f} k(h)\right) = \mathrm{H}^p(\mathbf{L}((E/I)^*)_{(w)})_f \otimes_{S_{(w)}} k(h).$$

since the complex is totally split after we have localized in f. That is, may be decomposed as a direct sum of an acyclic free complex and a complex consisting of the (free) homology and differentials equal to zero.

Since I is m-regular, the first factor on the right is 0 for $p \leq -m$. Thus the left side is also 0 in these degrees, and by proposition 4.3.3, $\mathrm{H}^p((E/I)^*, h) = 0$ for $p \leq -m$, and this is what we wanted. □

Now that we have established the existence of the generic elements, we will see why they are useful to us. The idea is to enlarge the ideal with the generic elements, and we will see that this preserves the regularity.

Lemma 4.3.6.

$$\operatorname{Tor}_i\left(\frac{(I,h)}{I}, k\right) = \operatorname{Tor}_{i-1}((I : h), k)(-1)$$

Proof. There is a resolution of $\frac{(I,h)}{I}$ which looks like

$$(\text{resolution of } (I : h))(-1) \longrightarrow E(-1) \xrightarrow{\phi_0} \frac{(I,h)}{I}$$

where ϕ_0 sends the element $f \in E$ to the residue class of fh modulo I. The kernel of this map is $(I : h)$. Note the shift in degrees. □

Proposition 4.3.7. *Let I be a graded ideal in E. If I is m-regular, and h is m-generic for I, then both (I, h) and $(I : h)$ are m-regular.*

Proof. We have a short exact sequence

$$0 \longrightarrow (I, h) \longrightarrow (I : h) \longrightarrow \frac{(I : h)}{(I, h)} \longrightarrow 0 \tag{4.7}$$

Since $(I:h)_d = (I,h)_d$ for all $d \geq m$,

$$\left(\frac{(I;h)}{(I,h)}\right)_d = 0$$

for $d \geq m$. Then we know ([3], page 186) that

$$\mathrm{Tor}_i\left(\frac{(I:h)}{(I,h)}, k\right)_d = 0$$

for $d > m+i-1$. The sequence (4.7) gives rise to the long exact Tor-sequence

$$\cdots \to \mathrm{Tor}_{i+1}\left(\frac{(I:h)}{(I,h)}, k\right) \to \mathrm{Tor}_i((I,h),k) \to \mathrm{Tor}_i((I:h),k) \to \mathrm{Tor}_i\left(\frac{(I:h)}{(I,h)}, k\right) \to \cdots \tag{4.8}$$

The vector space displayed on the left is zero for degrees $> m+i$, and the one on the right is zero for degrees $> m+i-1$. By the exactness, this means that the spaces in between are isomorphic in degrees $> m+i$. In particular, (I,h) is m-regular if and only if $(I:h)$ is m-regular.

There is another exact sequence

$$0 \longrightarrow \frac{(I,h)}{I} \longrightarrow \frac{E}{I} \longrightarrow \frac{E}{(I,h)} \longrightarrow 0 \tag{4.9}$$

The associated long exact Tor-sequence is

$$\cdots \to \mathrm{Tor}_{i+1}\left(\frac{E}{I}, k\right) \to \mathrm{Tor}_{i+1}\left(\frac{E}{(I,h)}, k\right) \to \mathrm{Tor}_i\left(\frac{(I,h)}{I}, k\right) \to \mathrm{Tor}_i\left(\frac{E}{I}, k\right) \to \cdots$$

By lemma 4.3.6 and resolution (4.2) this transforms into

$$\cdots \to \mathrm{Tor}_i(I,k) \to \mathrm{Tor}_i((I,h),k) \to \mathrm{Tor}_{i-1}((I:h),k)(-1) \to \mathrm{Tor}_{i-1}(I,k) \to \cdots \tag{4.10}$$

As I is assumed to be m-regular, the vector space on the left is zero in degrees $> m+i$, and the space on the right is zero in degrees $> m+i-1$. Thus the two spaces in between are isomorphic in degrees $> m+i$.

We will prove the claim by induction on i, so suppose

$$\mathrm{Tor}_i((I,h),k)_d = 0 \quad \text{for} \quad d > m+i.$$

Then

$$\mathrm{Tor}_i((I:h),k)_d = 0 \quad \text{for} \quad d > m+i$$

by the exactness sequence (4.8). Shifting the degree, we find that

$$\mathrm{Tor}_i((I:h),k)(-1)_d = 0 \quad \text{for} \quad d > m+i+1.$$

For such values of d, these vector spaces are equal to $\mathrm{Tor}_{i+1}((I,h),k)_d$, by sequence (4.10), so we find that

$$\mathrm{Tor}_{i+1}((I,h),k)_d \quad \text{for} \quad d > m + (i+1).$$

We need the start of the induction. But this is obvious, as a start for a resolution of (I,h) is given by

$$\cdots \longrightarrow E(-\deg h) \oplus \Big(\oplus_{j\in J'} E(-a_{0j})\Big) \longrightarrow (I,h)$$

where $\oplus_{j\in J} E(-a_{0j})$ is the first free module in the resolution of I, and J' is the set of indices corresponding to the minimal generators of I which are not divisible by h. (Recall that h has degree 1.) □

We want to show that also the converse of the preceeding proposition is true. However, in order to make the situation easier to handle, we assume that we have already passed to generic coordinates, and that $h = e_n$. This is the situation we will be in when we come to the proof that the regularity of the generic initial ideal is the same as the regularity of the ideal itself.

Proposition 4.3.8. *Let I be a graded ideal in E, and suppose that we are using generic coordinates. If $(I : e_n)_d = (I, e_n)_d$ for all $d \geq m$, and (I, e_n) and $(I : e_n)$ are m-regular, then I is m-regular.*

Proof. Consider the short exact sequence of E-modules

$$0 \longrightarrow I \cap (e_n) \longrightarrow I \longrightarrow \frac{(I,e_n)}{(e_n)} \longrightarrow 0.$$

Dualizing, we get the short exact sequence

$$0 \longrightarrow \left(\frac{(I,e_n)}{(e_n)}\right)^* \longrightarrow I^* \longrightarrow (I \cap (e_n))^* \longrightarrow 0. \tag{4.11}$$

Via the BGG correspondence, sequence (4.11) gives rise to a short exact sequence of complexes of S-modules:

$$0 \longrightarrow \mathbf{L}\left(\left(\frac{(I,e_n)}{(e_n)}\right)^*\right) \longrightarrow \mathbf{L}(I^*) \longrightarrow \mathbf{L}((I\cap(e_n))^*) \longrightarrow 0 \tag{4.12}$$

More explicitly, it looks like this:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(p-1) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{p-1} & \longrightarrow & S(p-1) \otimes I^*_{p-1} & \longrightarrow & S(p-1) \otimes (I \cap (e_n))^*_{p-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(p) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{p} & \longrightarrow & S(p) \otimes I^*_{p} & \longrightarrow & S(p) \otimes (I \cap (e_n))^*_{p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(p+1) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{p+1} & \longrightarrow & S(p+1) \otimes I^*_{p+1} & \longrightarrow & S(p+1) \otimes (I \cap (e_n))^*_{p+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(p+2) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{p+2} & \longrightarrow & S(p+2) \otimes I^*_{p+2} & \longrightarrow & S(p+2) \otimes (I \cap (e_n))^*_{p+2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

There is an associated long exact sequence of cohomology:

$$
\cdots \longrightarrow \mathrm{H}^p(\mathbf{L}(I^*)) \longrightarrow \mathrm{H}^p(\mathbf{L}((I \cap (e_n))^*)) \xrightarrow{\delta^p} \mathrm{H}^{p+1}\left(\mathbf{L}\left(\left(\frac{(I,e_n)}{(e_n)}\right)^*\right)\right) \longrightarrow \mathrm{H}^{p+1}(\mathbf{L}(I^*)) \longrightarrow \cdots
$$

Our goal is to show that the connecting map δ^p is injective for $p < -m$. Then the exactness of the sequence implies that the cohomology module $\mathrm{H}^p(\mathbf{L}(I^*))$ vanishes for $p < -m$, which again is equivalent to I being m-regular, by proposition 4.2.3.

We will describe this map very explicitly. First, we need to look at the vector space bases. Suppose I is generated by elements $m_1, ..., m_r, m_{r+1}e_n, ..., m_s e_n$, where e_n does not divide any of the m_i's. The bases for the different spaces are:

$$
\begin{array}{rcl}
I & : & \text{all multiples of } m_1, ..., m_r, m_{r+1}e_n, ..., m_s e_n \\
(I, e_n) & : & \text{all multiples of } m_1, ..., m_r \text{ and } e_n \\
\dfrac{(I, e_n)}{(e_n)} & : & \text{all multiples of } m_1, ..., m_r \text{ which are not multiples of } e_n \\
(I : e_n) & : & \text{all multiples of } m_1, ..., m_r, ..., m_s \text{ and } e_n \\
\dfrac{(I : e_n)}{(e_n)} & : & \text{all multiples of } m_1, ..., m_s \text{ which are not multiples of } e_n \\
I \cap (e_n) & : & \text{all multiples of } m_1e_n, ..., m_re_n, ..., m_se_n
\end{array}
$$

But we are dealing with the dual vector spaces, so we make a choice of dual bases. Thus, for instance, the basis for the vector space $(I \cap (e_n))^*_d$ consists of elements

$$f^* \in I^*_{-d} = \mathrm{Hom}_k(I_d, k)$$

where f is a multiple of one or more of the $m_i e_n$, $1 \leq i \leq s$, and $f^*(f) = 1$ while $f^*(g) = 0$ for all the other basis elements of I_d.

At this point we must notice the isomorphism

$$(I \cap (e_n)) \simeq \frac{(I : e_n)}{(e_n)}(-1).$$

And in degrees higher than $m+1$, this is also isomorphic to $\frac{(I,e_n)}{(e_n)}(-1)$, by hypothesis. This means that, for $p \leq -(m+1)$, the spaces $(I \cap (e_n))^*_p$ and $\left(\frac{(I,e_n)}{(e_n)}\right)^*_{p+1}$ actually have the same basis, only that in the former space the elements are represented by (duals of) multiples of the m_i which are again multiplied by e_n.

Let us now proceed to find out more about the connecting map. We are interested in the part where cohomological degree is smaller than $-m$. Given an element

$$\xi \in \mathrm{H}^{-(m+1)}(\mathbf{L}((I \cap (e_n))^*)).$$

It is represented by an element

$$z = \sum_i s_i \otimes (a_i m_i e_n)^* \in S(-(m+1)) \otimes (I \cap (e_n))^*_{-(m+1)}$$

where the a_i are elements of E such that the total degree of $a_i m_i e_n$ is $m+1$. z can be regarded as an element of $S(-(m+1)) \otimes I^*_{-(m+1)}$ as well. It is mapped to the element

$$\partial z = \sum_i \left(\sum_{\alpha=1}^{n} s_i x_\alpha \otimes e_\alpha (a_i m_i e_n)^* \right)$$

which is zero in $S(-m) \otimes (I \cap (e_n))^*_{-m}$, but in general non-zero in $S(-m) \otimes I^*_{-m}$. Now we can split this sum into two pieces, corresponding to $\alpha = n$ and $\alpha \neq n$:

$$\partial z = \sum_i s_i x_n \otimes e_n (a_i m_i e_n)^* + \sum_i \left(\sum_{\alpha \neq n} s_i x_\alpha \otimes e_\alpha (a_i m_i e_n)^* \right)$$

Since the second piece equals zero in $S(-m) \otimes (I \cap (e_n))^*_{-m}$, it must equal zero in $S(-m) \otimes I^*_{-m}$ as well. This is simply because $e_\alpha (a_i m_i e_n)^*$ is non-zero in $S(-m) \otimes I^*_{-m}$ if and only if it is non-zero in $S(-m) \otimes (I \cap (e_n))^*_{-m}$ when $\alpha \neq n$.

The first piece automatically equals zero in $S(-m) \otimes (I \cap (e_n))^*_{-m}$, since $e_n (a_i m_i e_n)^*$ maps the product $a_i m_i$ to $1 \in k$ and all other basis elements to zero, and $a_i m_i$ is not a multiple of e_n and thus not in $(I \cap (e_n))_m$. However, it does not necessarily equal zero in $S(-m) \otimes I^*_{-m}$, as the $a_i m_i$ actually are basis elements of this latter vector space.

Thus the image of ξ in $\mathrm{H}^{-m}\left(L\left(\left(\frac{(I,e_n)}{(e_n)}\right)^*\right)\right)$ is the cohomology class $\delta^{-(m+1)}(\xi)$ represented by

$$z' = \sum_i s_i x_n \otimes (a_i m_i)^*.$$

We will show that if z' is the image of some element in $S(-(m+1)) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{-(m+1)}$, then z is also an image, and $\delta^{-(m+1)}$ is injective. The diagram below is an attempt to illustrate the diagram chase.

$$\begin{array}{ccccccc} & & z_2 & \longmapsto & z & \longmapsto & z \rightsquigarrow \xi \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ \delta\xi & \leftsquigarrow & z' & \longmapsto & \partial z & \longmapsto & 0 \end{array}$$

Suppose that there is an element z_2 which maps to z' in this way. It has the form

$$z_2 = \sum_j t_j \otimes g_j^* \in S(-(m+1)) \otimes \left(\frac{(I,e_n)}{(e_n)}\right)^*_{-(m+1)}$$

where the g_j's are multiples of $m_1, ..., m_r$, but not of e_n, of degree $m+1$. The image of z_2 under ∂ is

$$\partial z_2 = \sum_j \left(\sum_{\alpha=1}^{n} t_j x_\alpha \otimes e_\alpha g_j^*\right) = \sum_{j,\alpha} t_j x_\alpha \otimes \left(\frac{g_j}{e_\alpha}\right)^* \tag{4.13}$$

where the starred fraction in the last expression is to be interpreted as zero when e_α does not divide g_j. Recall that we have seen that $(I \cap (e_n))^*_{-(m+1)}$ and $\left(\frac{(I,e_n)}{(e_n)}\right)^*_{-m}$ have "the same" basis, so the elements $(a_i m_i)^*$ appear as basis elements in $\frac{(I,e_n)}{(e_n)}$ also. The expression (4.13) is supposed to equal the expression we already have for z', so

$$\begin{aligned} \sum_i s_i x_n \otimes (a_i m_i)^* &= \sum_{j,\alpha} t_j x_\alpha \otimes \left(\frac{g_j}{e_\alpha}\right)^* \\ &= \sum_i \left(\sum_{\frac{g_j}{e_\alpha} = a_i m_i} t_j x_\alpha \otimes (a_i m_i)^*\right) \\ &= \sum_i \left(\sum_{\frac{g_j}{e_\alpha} = a_i m_i} t_j x_\alpha\right) \otimes (a_i m_i)^* \end{aligned}$$

This implies that we must have

$$\sum_{\frac{g_j}{e_\alpha} = a_i m_i} t_j x_\alpha = s_i x_n$$

Since e_n never divides g_j, this again implies that x_n divides all the t_j. Now replace the t_j by $t'_j = \frac{t_j}{x_n}$. Then

$$s_i = \sum_{\frac{g_j}{e_\alpha} = a_i m_i} t'_j x_\alpha.$$

Let us consider the element

$$\sum_j t'_j \otimes (g_j e_n)^* \in S(-(m+2)) \otimes (I \cap (e_n))^*_{-(m+2)}.$$

Remembering that $\alpha = n$ gives

$$e_\alpha (g_j e_n)^* = g_j^* = 0 \in (I \cap (e_n))^*,$$

since e_n does not divide g_j, we see that ∂ maps this to

$$\begin{aligned}
\sum_j \left(\sum_\alpha t'_j x_\alpha \otimes e_\alpha (g_j e_n)^* \right) &= \sum_{j,\alpha} t'_j x_\alpha \otimes (\frac{g_j}{e_\alpha} e_n)^* \\
&= \sum_i \left(\sum_{\frac{g_j}{e_\alpha} = a_i m_i} t'_j x_\alpha \otimes (a_i m_i e_n)^* \right) \\
&= \sum_i \left(\sum_{\frac{g_j}{e_\alpha} = a_i m_i} t'_j x_\alpha \right) \otimes (a_i m_i e_n)^* \\
&= \sum_i s_i \otimes (a_i m_i e_n)^* \\
&= z
\end{aligned}$$

Thus we have shown that if $\delta^{-(m+1)}$ maps ξ to the zero cohomology class, then $\xi = 0$. In other words, $\delta^{-(m+1)}$ is injective, and

$$\mathrm{H}^{-(m+1)}(\mathbf{L}(I^*)) = 0.$$

The same is true in the exact same way for the cohomology groups of lower cohomological degree. By proposition 4.2.3, this completes the proof. □

The idea now is to choose successive elements such that each element is generic for the ideal generated by I and the preceeding elements.

Definition 4.3.9. For each $j > 0$, let $U_j(I)$ denote the subset

$$U_j^m(I) = \{(h_1, ..., h_j) \in E_1^j \mid h_i \text{ is } m\text{-generic for } (I, h_1, ..., h_{i-1}), 1 \le i \le j\}$$

It should be clear that if I is m-regular, expanding the ideal in this way will finally give us an ideal such that $(I, h_1, ..., h_j)_m = E_m$.

Lemma 4.3.10. *If $I_m = E_m$, then I is m-regular.*

Proof. This is perhaps most easily seen via the **L**-functor. By hypothesis, $(E/I)_p = 0$ for $p \geq m$. That means that $(E/I)^*_p = 0$ for $p \leq -m$, so

$$\mathrm{H}^p(\mathbf{L}((E/I)^*)) = 0 \text{ for } p < -(m-1),$$

and by proposition 4.2.3, E/I is $(m-1)$-regular. By lemma 4.1.3, I is m-regular. □

And now comes the main theorem of this section. We will assume that we have made a generic choice of coordinates, such that e_n is generic for I, e_{n-1} is generic for (I, e_n) and so on, for a generic sequence sufficiently long.

Theorem 4.3.11. *If $I \subset E$ is a graded ideal, then I is m-regular if and only if we can make a generic choice of coordinates such that*

$$(e_n, e_{n-1}, ..., e_j) \in U^m_{n-j+1}(I) \quad \textit{and} \quad (I, e_n, ..., e_j)_m = E_m.$$

Remark 4.3.12. This is the analogue of theorem 1.10 of Bayer and Stillman's paper [5], the "Criterion for m-regularity".

Proof. This is a consequence of the earlier results in this section. By propositions 4.3.7 and 4.3.8, our chosen elements preserve the regularity. By the discussion in the beginning of the section, the elements exist. By lemma 4.3.10, I is m-regular when the entire ring E can be filled in this way. □

4.4 The Generic Initial Ideal

We will use the results we found in the previous section to show that the regularity of the generic initial ideal of an ideal $I \subset E$ is equal to the regularity of the ideal itself when we use the reverse lexicographic order on monomials.

For convenience, we repeat the definitions of the reverse lexicographic order and initial ideals:

Definition 4.4.1. *(The reverse lexicographic order)* Given two monomials $m_1 = e_{\sigma_1}$ and $m_2 = e_{\sigma_2}$, we arrange them $m_1 > m_2$ if and only if either $\deg(m_1) > \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and the largest element in the symmetric difference of σ_1 and σ_2 is in σ_2.

Definition 4.4.2. Let f be an element of E. The *initial term* $\mathrm{in}(f)$ of f is the greatest term of f under the reverse lexicographic order. If $I \subset E$ is an ideal, then the *initial ideal* $\mathrm{in}(I)$ is the ideal generated by all the initial terms of elements of I.

Throughout this section we will deal only with the reverse monomial order, so the symbol "$>$" will always mean "$>_{\text{revlex}}$", and "in" will mean "$\mathrm{in}_{\text{revlex}}$". Let us now investigate some properties of this particular order.

Lemma 4.4.3. *Suppose f is a homogeneous element in E, and that we use the reverse lexicographic order. Then*

$$\operatorname{in}(f) \in (e_j, e_{j+1}, ..., e_n) \Rightarrow f \in (e_j, e_{j+1}, ..., e_n)$$

Proof. Suppose

$$f = \sum_a c_a e_{\sigma_a} \notin (e_j, ..., e_n) \quad , \quad c_a \in k$$

and that $\operatorname{in}(f)$ is in $(e_j, ..., e_n)$. Then there must be an e_{σ_a} with non-zero coefficient which is not divisible by any of the $e_j, ..., e_n$. But if there were such a σ_a, then e_{σ_a} would be greater than $\operatorname{in}(f)$, since $\operatorname{in}(f)$ is divided by some $e_{j'}$, $j' \geq j$. By the definition of the initial term, this is a contradiction. □

So in particular, if e_n divides $\operatorname{in}(f)$, then e_n divides f. Now a couple of technical lemmas:

Lemma 4.4.4. *Let I be an ideal in the exterior algebra E and let $>$ be the reverse lexicographic order. Choose i in the range $1 \leq i \leq n$.*

a) $\operatorname{in}(I, e_n, ..., e_i) = (\operatorname{in}(I), e_n, ..., e_i)$.

b) Let $e_n, ..., e_{i+1} \in I$, and let $m \geq 0$. Then

$$(I : e_i)_{\geq m} = (I, e_i)_{\geq m} \Leftrightarrow (\operatorname{in}(I) : e_i)_{\geq m} = (\operatorname{in}(I), e_i)_{\geq m}$$

Proof. a) Let f be an element in the ideal $(\operatorname{in}(I), e_n, ..., e_i)$. Then we can write f as

$$f = f_1 g + f_n e_n + \cdots f_i e_i,$$

where g is some element of $\operatorname{in}(I)$, so

$$f = f_1 g_1 \operatorname{in}(h_1) + \cdots + f_1 g_t \operatorname{in}(h_t) + f_n e_n + \cdots f_i e_i,$$

where the h_j's are in I. Thus f is in $\operatorname{in}(I, e_n, ..., e_i)$.

To show the opposite inclusion, let f be some element of $(I, e_n, ..., e_i)$. Then $\operatorname{in}(f)$ is a generator of $\operatorname{in}(I, e_n, ..., e_i)$. If e_j divides $\operatorname{in}(f)$ for some j, $i \leq j \leq n$, then

$$\operatorname{in}(f) \in (e_j) \subset (\operatorname{in}(I), e_n, ..., e_i).$$

So assume there is no such e_j. Then we can write f as

$$f = g + h_n e_n + \cdots + h_i e_i,$$

where g is in I, and $h_n, ..., h_i$ are suitable elements in E. As none of the e_j divide $\operatorname{in}(f)$ when $j \geq i$,

$$\operatorname{in}(f) > \operatorname{in}(h_n e_n + \cdots + h_i e_i).$$

So

$$\text{in}(f) = \text{in}(g) \in \text{in}(I) \subset (\text{in}(I), e_n, ..., e_i).$$

Since all the generators of $\text{in}(I, e_n, ..., e_i)$ are in $(\text{in}(I), e_n, ..., e_i)$, the entire ideal must also be.

b) Suppose $(I : e_i)$ and (I, e_i) are equal in degrees $\geq m$, and let f be an element in $(\text{in}\, I : e_i)_{\geq m}$. That is, f has degree at least m and fe_i is in $\text{in}(I)$. We want to show that f is in $(\text{in}(I), e_i)$ as well. We can express fe_i as

$$fe_i = a_1\, \text{in}(f_1) + \cdots + a_t\, \text{in}(f_t),$$

with all f_j elements of I. By allowing repetitions of the $\text{in}(f_j)$'s, we may choose the a_j's as monomials. We may also assume that none of the terms are multiples of $e_{i+1}, ..., e_n$, as these are already in the ideal $\text{in}(I)$. For all $j = 1, ..., t$, we either have that e_i divides a_j or that e_i divides $\text{in}(f_j)$. In the former case,

$$a'_j\, \text{in}(f_j) \in \text{in}(I) \quad \text{where} \quad a'_j = \frac{a_j}{e_i}$$

so this is OK. In the latter case, e_i divides all terms in f_j which are not divisible by $e_{i+1}, ..., e_n$ by lemma 4.4.3, so we may replace f_j by the parts of it wich are divisible by e_i, as the rest is also in I, and does not contribute to $\text{in}(f)$. So we find that

$$a_j f'_j \in (I : e_i)_{\geq m} = (I, e_i)_{\geq m} \quad \text{where} \quad f'_j = \frac{f_j}{e_i}$$

Hence

$$a_j f'_j = g_j + b_j e_i \quad \text{with} \quad g_j \in I.$$

$a_j f_j$ equals $g_j e_i$, and

$$a_j\, \text{in}(f_j) = \text{in}(a_j f_j) = \text{in}(g_j) e_i \in \text{in}(I)$$

So we see that f consists of parts which are in $(\text{in}(I), e_i)$, the multiples of e_i naturally being annihilated upon mulitiplication by e_i. The inclusion $(\text{in}(I), e_i) \subset (\text{in}(I) : e_i)$ is obvious, since $e_i^2 = 0$.

We now turn to the converse. Suppose $(\text{in}(I) : e_i)$ and $(\text{in}(I), e_i)$ are equal in degrees $\geq m$. Let f be an element in $(I : e_i)_{\geq m}$. That is, fe_i is in I and f has degree at least m. Assume by induction that for all elements g of E of the same degree as f such that $\text{in}(g) < \text{in}(f)$ and ge_i is in I, we also have that g is in (I, e_i). Now

$$\text{in}(f)e_i = \text{in}(fe_i) \in \text{in}(I),$$

so

$$\text{in}(f) \in (\text{in}(I) : e_i)_{\geq m} = (\text{in}(I), e_i)_{\geq m}.$$

$\text{in}(f)$ is a monomial, so either e_i divides $\text{in}(f)$ or $\text{in}(f) = \text{in}(g)$ for some g in I. In the former case, e_i divides all terms in f which are not divisible by $e_{i+1}, ..., e_n$ (which are members of I), and therefore f must be in (I, e_i). In the latter case, $(f - g)e_i$ will be in I, and $(f - g)$ has initial term smaller than $\text{in}(f)$, so by induction, $(f - g)$ is in (I, e_i) as well. f is therefore in $(I, e_i)_{\geq m}$. The induction starts nicely, since e_n is in the ideal (I, e_i). The other inclusion is again obvious. □

Lemma 4.4.5. *Let $r \geq 0$, $m \geq 0$, and $>$ be the reverse lexicographic order. The following conditions are equivalent:*

a) $((I, e_n, ..., e_{i+1}) : e_i)_m = (I, e_n, ..., e_i)_m$ *for* $i = n, n-1, ..., n-r+1$, *and* $(I, e_n, ..., e_{n-r+1})_m = E_m$.

b) $((\mathrm{in}(I), e_n, ..., e_{i+1}) : e_i)_m = (\mathrm{in}(I), e_n, ..., e_i)_m$ *for* $i = n, n-1, ..., n-r+1$, *and* $(\mathrm{in}(I), e_n, ..., e_{n-r+1})_m = E_m$.

Proof. This follows from lemma 4.4.4 parts a) and b). For the "$= E_m$"-parts, note that if all the homogeneous elements of degree m in E are in $(I, e_n, ..., e_{n-r+1})$, then in particular the monomials are, and these generate $\mathrm{in}(I, e_n, ..., e_{n-r+1})_m$, which is equal to $(\mathrm{in}(I), e_n, ..., e_{n-r+1})_m$. On the other hand, if all monomials of degree m are in this initial ideal, then every one of these monomials is the initial part of some homogeneous element of degree m in $(I, e_n, ..., e_{n-r+1})$. The only way the reverse-lexicographically *last* monomial can be an initial part of an element, is that the element is the monomial itself. Thus it must be in the ideal. And since it is in the ideal, the *second last* monomial must be in the ideal, as it is also initial for some element. By induction, all the monomials of E_m must be in $(I, e_n, ..., e_{n-r+1})$. □

And now the punchline:

Theorem 4.4.6. *Let I be an ideal in the exterior algebra E, and let* $\mathrm{Gin}(I)$ *be the generic initial ideal under the reverse lexicographic order. Then* $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{Gin}(I))$.

Remark 4.4.7. This is the Bayer-Stillman theorem for the exterior algebra.

Proof. This is a consequence of theorem 4.3.11 of the previous section, and the lemmas in this section. Suppose we have made a generic choice of coordinates. By lemma 4.4.5, $(e_n, ..., e_j) \in U^m_{n-j+1}(I)$ and $(I, e_n, ..., e_j)_m = E_m$ exactly when $(e_n, ..., e_j) \in U^m_{n-j+1}(\mathrm{in}(I))$ and $(\mathrm{in}(I), e_n, ..., e_j)_m = E_m$. By theorem 4.3.11, this happens exactly when the ideals are m-regular. Thus the regularities are equal. □

At this point, we should remark that we could have cheated, and avoided the rather complicated proof of proposition 4.3.8. In the rest of this section we will discuss this briefly. We could have appealed to this result:

Lemma 4.4.8. *If $I \subset E$ is a Borel-fixed monomial ideal, then the regularity of I is the maximal degree of a minimal generator for I.*

Proof. This is corollary 3.1 of [3]. □

Remark 4.4.9. The regularity of an ideal does not depend on the labelling of the variables. Therefore, the lemma above also applies to ideals which are Borel-fixed after a relabelling of the variables.

Lemma 4.4.10. *If I is a Borel-fixed monomial ideal, e_n is m-generic for I, and (I, e_n) is m-regular, then I is m-regular.*

Proof. Let $\{m_1, ..., m_r, m_{r+1}e_n, ..., m_s e_n\}$ be the set of minimal monomial generators for I, where e_n does not divide any of the monomials $m_1, ..., m_r$. Then the set of generators for (I, e_n) is $\{m_1, ..., m_r, e_n\}$ and for $(I : e_n)$ $\{m_1, ..., m_s, e_n\}$.

Note that (I, e_n) and $(I : e_n)$ also have the property that the regularity is the highest degree of a minimal generator, since a permutation of the variables sending e_t to e_{t+1}, $t = 1, ..., n-1$, and e_n to e_1, gives a Borel-fixed ideal with the same structure as the original ones (cf. remark 4.4.9).

So the question is whether or not I can have minimal generators of degree $> m$. As $m_1, ..., m_r$ are minimal generators of (I, e_n) as well, and (I, e_n) is m-regular, these cannot be of degree $> m$. Therefore, a minimal generator for I of degree $> m$ must be among the $m_{r+1}e_n, ..., m_s e_n$. We must then have that there is an m_t, $r + 1 \leq t \leq s$ of degree exactly m, since $(I : e_n)$ is m-regular. But then we will have someting of degree m which is in $(I : e_n)$, but not in (I, e_n), contradicting the hypothesis that e_n is m-generic for I. Thus there cannot be minimal generators for I of degree $> m$. □

The point now is that we already know that the generic initial ideal is Borel-fixed, and therefore lemma 4.4.10 applies in this situation. We could have chosen elements successively as in definition 4.3.9, and produce enlargements of I such that the regularities of the enlargements would not exceed the regularity of I.

Lemmas 4.4.3, 4.4.4 and 4.4.5 do not depend on regularity, so they would still be accessible to us. After enlarging I, the enlargement would meet the enlargement of $\text{in}(I)$ in E_m for some m. By 4.4.10, we could have pulled this enlargement back to $\text{in}(I)$ in a regularity-preserving fashion, thus showing that the regularity of the generic initial ideal is $\leq \text{reg}(I)$.

The fact that $\text{reg}(I) \leq \text{reg}(\text{Gin}(I))$ is a general fact, which can be deduced from Gröbner basis theory. A minimal free resolution of I is obtained from a minimal free resolution of $\text{Gin}(I)$ by cancelling some terms, the regularity of I cannot exceed the regularity of $\text{Gin}(I)$. This is briefly explained in [15], and stated (in a somewhat different language) as proposition 1.8 in [3].

So we would have reached the statement $\text{reg}(I) = \text{reg}(\text{Gin}(I))$, at the cost of sacrificing the "criterion for exterior m-regularity", theorem 4.3.11.

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