A geometrically exact rod model and its Hamiltonian structure

Elena Celledoni joint work with Niklas Säfström

Department of Mathematical Sciences, NTNU, Norway

ICNAAM, Crete, September 21st, 2009
Outline of the talk

1. motivation
2. geometrically exact elastic rod model
3. reformulation in material coordinates
4. reformulation as a Hamiltonian PDE: I and II
5. multi-symplectic formulation
Motivation: risers and pipelines

- Vortex induced vibrations of pipelines
- Control of suspended pipe-lay

Pipe-lay form vessels
ICT and Mathematics

• Center for Ships and Ocean Structures NTNU (Gullik Jensen)
• Department of Engineering Cybernetics (Thor Inge Fossen)
• Marine Cybernetics (Olav Egeland)
• Department of Mathematical Sciences at NTNU (Niklas Säfström, Elena Celledoni)

Publications

• *Modelling and Control of Suspended Pipeline during Pipe-lay Operation Based on a Finite Strain Beam Model*, Jensen, Säfström, Nguyen

• *A Nonlinear PDE Formulation for Offshore Vessel Pipeline Installation*, Jensen, Säfström, Fossen.

• *Hamiltonian and multi-symplectic formulation of a rod model using quaternions*. Celledoni and Säfström.
Rod model: reference frame \( \{E_1, E_2, E_3\} \), body frame \( \{t_1, t_2, t_3\} \).

\[
\tilde{B} = \Phi(B), \quad \mathbf{x} = \Phi(\mathbf{X}),
\]

\[
F_{i,j} = \frac{\partial x_i}{\partial X_j}
\]

**deformation gradient:** \( F \) (Jacobian of \( \Phi \))

**stretch in direction** \( \mathbf{v} \) \( \| F \mathbf{v} \| = \lambda(\mathbf{v}), \quad \| \mathbf{v} \| = 1 \)
Rod model: reference frame \( \{ E_1, E_2, E_3 \} \), body frame \( \{ t_1, t_2, t_3 \} \).

\[ \tilde{B} = \Phi(B), \quad x = \Phi(X), \quad F_{i,j} = \frac{\partial x_i}{\partial X_j} \]

**deformation gradient**: \( F \) (Jacobian of \( \Phi \))

**stretch** in direction \( v \) \( \| Fv \| = \lambda(v), \quad \|v\| = 1 \)

**stress vector**: \( t \) surface force per unit area

\( t = \sigma^T n \quad \sigma = G(F) \) Cauchy stress tensor \( n \) unit normal

When a **strain-energy** function \( U(F) \) is given then

\[ \det(F) F^{-1} \sigma = \frac{\partial U(F)}{\partial F} \]

**Objectivity**:

\[ U(QF) = U(F), \quad \forall Q, \text{ s.t. } Q^T Q = I. \]
Rod model: reference frame \( \{ \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3 \} \), body frame \( \{ \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \} \).

Line of centroids:

\[
\varphi(S, t) \quad \Lambda(S, t) = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3] \quad \mathbf{x} = \Phi(\mathbf{X}, t) = \varphi(S, t) + \Lambda(S, t)\mathbf{X}
\]

Configuration space

\[
C = \left\{ (\varphi, \Lambda) \mid \varphi(S, t) \in \mathbb{R}^3, \Lambda(S, t) \in SO(3), \partial_S \varphi^T \mathbf{t}_3 > 0 \right\}
\]

\( S \in [0, L], \ t > 0. \)
The rotation matrix $\Lambda(S, t)$ satisfies the kinematic equations

\[
\begin{align*}
\partial_S \Lambda &= \hat{\omega} \Lambda = \Lambda \hat{\Omega} \\
\partial_t \Lambda &= \hat{\omega} \Lambda = \Lambda \hat{W}
\end{align*}
\]

$\omega \in \mathbb{R}^3 \rightarrow \hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$

$\Omega$ $\mathbf{W}$ material coordinates

$\omega$ $\mathbf{w}$ spatial coordinates
References

- S. S. Antman, Nonlinear Problems of Elasticity, Springer 1992
- Y. B. Fu and R. W. Ogden, Nonlinear Elasticity, Cambridge University Press
- P.G. Ciarlet, Mathematical Elasticity North Holland, 1988
- T. Hughes and J.E. Marsden, Mathematical Foundations of Elasticity, Dover, 1994
- J. C. Simo (1985)
Static case

**Strain measures** (proposed by Simo et al. 1985):  
\[ \Omega, \quad \Gamma = \Lambda^T (\partial S \varphi - t_3) \]

**Strain-energy function:**

\[ U(\Gamma, \Omega) = \int_0^L \Psi(S, \Gamma, \Omega) \, dS, \quad \Psi(S, \Gamma, \Omega) := \frac{1}{2} (\Gamma^T C_1 \Gamma + \Omega^T C_2 \Omega) \]

\[ C_1 = \text{diag}(GA_1, GA_2, EA), \quad C_2 = \text{diag}(EI_1, EI_2, GJ) \] standard constants in strength of materials.

**Stress measures:**

- stress resultant
- stress couple
- Piola-Kirchhoff tensor

\[ N = \frac{\partial \Psi}{\partial \Gamma}, \quad M = \frac{\partial \Psi}{\partial \Omega}, \quad \left( \frac{\partial U}{\partial F} \right)^T \]

material coordinates spatial coordinates

\[ N = C_1 \Gamma, \quad n = \Lambda N \]
\[ M = C_2 \Omega, \quad m = \Lambda M \]

Minimizing \( U(\Gamma, \Omega) + U_{\text{ext}} \) gives the Euler-Lagrange equations for the static problem. We assume \( U_{\text{ext}} = 0 \).
Dynamic case

Define the momentum densities:

\[ p := A_\rho \partial_t \varphi, \quad \pi := I_\rho \mathbf{w}, \]

with

\[ A_\rho := \int_A \rho_0 dA, \quad I_\rho := \Lambda J_\rho \Lambda^T, \quad J_\rho = \text{diag}(J_1(S), J_2(S), J_3(S)) \]

and the linear and angular momentum

\[ L(t) := \int_0^L p \, dS, \quad J(t) := \int_0^L (\varphi \times p + \pi) \, dS. \]

The kinetic energy

\[ T(p, \pi) = \frac{1}{2} \int_0^L (A_\rho^{-1} \| p \|^2 + \pi^T (I_\rho^{-1} \pi)) \, dS \]

minimizing \( T - U \) one obtains the Euler-Lagrange equations in spatial coordinates:
EL equations: balance of linear and angular momentum

\[ A^{-1}_\rho \partial_t p - \partial_S n = 0, \quad p = A_\rho \partial_t \varphi, \]

\[ \partial_t \pi + (I^{-1}_\rho \pi) \times \pi - \partial_S m - \partial_S \varphi \times n = 0, \]

\( (I^{-1}_\rho \pi) = w, \hat{w} = (\partial_t \Lambda) \Lambda^T, \quad (\partial_S \Lambda) \Lambda^T = C_2 \Lambda^T m. \)
EL equations: balance of linear and angular momentum

\[ A^{-1}_\rho \partial_t p - \partial_S n = 0, \quad p = A_\rho \partial_t \varphi, \]

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\[ (I^{-1}_\rho \pi) = w, \quad \hat{w} = (\partial_t \Lambda)\Lambda^T, \quad (\partial_S \Lambda)\Lambda^T = C_2\Lambda^T m. \]

References: structure preserving methods.

- J. C. Simo and L. Vu Quoc (Newmark exponential integrator: large rotations)
- J. C. Simo, N. Tarnow, M. Doblaré (energy and momentum conserving algorithms)
- O. Gonzalez (energy preservation)
- O. Gonzalez and J.C. Simo (energy-momentum preservation)
- F. Armero and I. Romero (dissipative algorithms)
- P. Betsch and P. Steinmann (constrained Hamiltonian systems)
- M. Dixon (generalized Moser-Veselov algorithms)
Equations and Hamiltonian in material coordinates.

\[
A_\rho^{-1} \partial_t p = \partial_S n, \\
\partial_t \pi + (I_\rho^{-1} \pi) \times \pi = \partial_S m + \partial_S \varphi \times n,
\]
defining \( P := \Lambda^T p = \Lambda^T A_\rho (\partial_t \varphi) \), \( \Pi := J_\rho W = \Lambda^T \pi \) we obtain the equations in material coordinates

\[
\partial_t P + (J_\rho^{-1} \Pi) \times P = \partial_S N + (C_2^{-1} M) \times N, \\
\partial_t \Pi + (J_\rho^{-1} \Pi) \times \Pi = \partial_S M + (C_2^{-1} M) \times M + (C_1^{-1} N + E_3) \times N.
\]
Equations and Hamiltonian in material coordinates.

\[
A^{-1}_\rho \partial_t p = \partial_S n, \\
\partial_t \pi + (I^{-1}_\rho \pi) \times \pi = \partial_S m + \partial_S \varphi \times n,
\]

defining \( P := \Lambda^T p = \Lambda^T A_\rho (\partial_t \varphi), \) \( \Pi := J_\rho W = \Lambda^T \pi \) we obtain the equations in material coordinates

\[
\partial_t P + (J^{-1}_\rho \Pi) \times P = \partial_S N + (C^{-1}_2 M) \times N, \\
\partial_t \Pi + (J^{-1}_\rho \Pi) \times \Pi = \partial_S M + (C^{-1}_2 M) \times M + (C^{-1}_1 N + E_3) \times N.
\]

From the kinematics of the rotations differentiating in turn with respect to \( t \) and \( S \) we also obtain

\[
C^{-1}_1 \partial_t N - A^{-1}_\rho \partial_S P = (C^{-1}_1 N + E_3) \times (J^{-1}_\rho \Pi) + (C^{-1}_2 M) \times (A^{-1}_\rho P), \\
C^{-1}_2 \partial_t M - J^{-1}_\rho \partial_S \Pi = (C^{-1}_2 M) \times (J^{-1}_\rho \Pi).
\]
Equations and Hamiltonian in material coordinates.

\[ A^{-1}_\rho \partial_t p = \partial_S n, \]
\[ \partial_t \pi + (I^{-1}_\rho \pi) \times \pi = \partial_S m + \partial_S \varphi \times n, \]

defining \( P := \Lambda^T p = \Lambda^T A_\rho (\partial_t \varphi), \quad \Pi := J_\rho W = \Lambda^T \pi \) we obtain the equations in material coordinates

\[ \partial_t P + (J^{-1}_\rho \Pi) \times P = \partial_S N + (C^{-1}_2 M) \times N, \]
\[ \partial_t \Pi + (J^{-1}_\rho \Pi) \times \Pi = \partial_S M + (C^{-1}_2 M) \times M + (C^{-1}_1 N + E_3) \times N. \]

From the kinematics of the rotations differentiating in turn with respect to \( t \) and \( S \) we also obtain

\[ C^{-1}_1 \partial_t N - A^{-1}_\rho \partial_S P = (C^{-1}_1 N + E_3) \times (J^{-1}_\rho \Pi) + (C^{-1}_2 M) \times (A^{-1}_\rho P), \]
\[ C^{-1}_2 \partial_t M - J^{-1}_\rho \partial_S \Pi = (C^{-1}_2 M) \times (J^{-1}_\rho \Pi). \]

the Hamiltonian

\[ \mathcal{H} = \frac{1}{2} \int_0^L \left( (A^{-1}_\rho P)^T P + \Pi^T (J^{-1}_\rho \Pi) + (C^{-1}_1 N)^T N + M^T (C^{-1}_2 M) \right) dS \]
Rod model as Hamiltonian PDE I

Assume

\[ \{F, G\}[u] = \int_0^L \frac{\delta F}{\delta u} J(u) \frac{\delta F}{\delta u} \, dS, \]

is a Poisson bracket (Olver 93) (i.e. \(J(z)\) is skew-symmetric ...) and \(\frac{\delta F}{\delta u}\) is the variational derivative:

\[ \left( \frac{d}{d\varepsilon} F(u + \varepsilon v) \right)_{\varepsilon=0} = \int_0^L \frac{\delta F}{\delta u} \cdot v \, dS. \]

If \(H = H(z)\) is an Hamiltonian function then

\[ z_t = \{z, H\} = J(z) \frac{\delta H}{\delta z}, \]

is a Hamiltonian PDE.

For the rod model setting \(z = (\Pi^T, P^T, M^T, N^T)^T\), we have

\[ \frac{\delta H}{\delta z} = D_3 z, \]

with

\[ D_3 = \text{diag}(J^{-1}_\rho, A^{-1} \text{Id}_3, C_1^{-1}, C_2^{-1}). \]
**Theorem** (Simo, Marsden, Krishnaprasad 1986)
The rod model can be written as an Hamiltonian PDE as

$$z_t = J(z) \frac{\delta H}{\delta z}, \quad J(z) = D_1(J_S + J_z)D_1,$$

with $D_1 = \text{diag}(\text{Id}_6, C_1, C_2),$

$$J_S = \left( \begin{array}{cc} 0 & \partial S \\ \partial S & 0 \end{array} \right) \otimes \text{Id}_6$$

$$J_z = \left( \begin{array}{cccccc} \hat{\Pi} & 0 & C_1^{-1}\hat{M} & C_2^{-1}\hat{N} + E_3 \\ 0 & A_{\rho}J_\rho^{-1}\hat{\Pi} & 0 & C_1^{-1}\hat{M} \\ C_1^{-1}\hat{M} & 0 & 0 & 0 \\ C_2^{-1}\hat{N} + E_3 & C_1^{-1}\hat{M} & 0 & 0 \end{array} \right),$$

and $J(z)$ defining a non canonical Poisson bracket.
Geometric numerical integration of this PDE:

- Lie Poisson integrators: semidiscretization in space and splitting in time. *Since the symplectic structure is non-canonical splitting methods are the obvious choice.*
- Symmetric energy preserving methods *are normally a good alternative to symplectic methods.* (Discrete gradients).
- Multi-symplectic formulation and multi-symplectic methods:

\[ Mz_t + Kz_x = \nabla_z S(z), \]

some difficulties arise because \( J(z) \) is not invertible and depends on \( z \) this means that the skew-symmetric matrices \( M \) and \( K \) in the multi-symplectic formulation will possibly depend on \( z \).

1. Derive a canonical Hamiltonian formulation.
2. From there get a multi-symplectic formulation with \( M \) and \( K \) constant.
Formulation of the Hamiltonian in quaternions

\[ H := \{ q = (q_0, \mathbf{q}) \in \mathbb{R} \times \mathbb{R}^3, \mathbf{q} = (q_1, q_2, q_3)^T \} \cong \mathbb{R}^4, \]

addition and multiplication of two quaternions,
\( \mathbf{p} = (p_0, \mathbf{p}), \mathbf{q} = (q_0, \mathbf{q}) \in H, \) are defined by

\[ \mathbf{p} + \mathbf{q} = (p_0 + q_0, \mathbf{p} + \mathbf{q}) \]

and

\[ \mathbf{p}\mathbf{q} = (p_0q_0 - \mathbf{p}^T\mathbf{q}, p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}), \quad (1) \]

respectively. For \( \mathbf{q} \neq (0, \mathbf{0}) \) there exists an inverse

\[ \mathbf{q}^{-1} = \mathbf{q}^c / \| \mathbf{q} \|^2, \quad \| \mathbf{q} \| = \sqrt{q_0^2 + \| \mathbf{q} \|^2_2}. \]
Formulation of the Hamiltonian in quaternions

\[ pq = L(p)q = R(q)p, \]

where

\[
L(p) = \begin{bmatrix} p_0 & -p^T \\ p & (p_0 \mathbb{1} + \hat{p}) \end{bmatrix}, \quad R(q) = \begin{bmatrix} q_0 & -q^T \\ q & (q_0 \mathbb{1} - \hat{q}) \end{bmatrix}. \quad (2)
\]

Euler parameters:

\[ S^3 = \{ q \in \mathbb{H} \mid \|q\| = 1 \}, \]

which is a Lie group, there is a (surjective) group homomorphism \( \mathcal{E} : S^3 \to SO(3) \), defined by

\[ \mathcal{E}(q) = I + 2q_0\hat{q} + 2\hat{q}^2, \]

instead of using \( (\varphi, \Lambda)^T \in \mathbb{R}^3 \times SO(3) \) we use \( u = (\varphi, q)^T \in \mathbb{R}^3 \times \mathbb{H} \) and \( \|q\| = 1 \).
Augmented Lagrangian:

\[
\mathcal{L}(u, u_t, u_S) = T - U - \lambda(\|q\|^2 - 1), \quad u = (\varphi, q)^T,
\]

\[
T = \frac{1}{2} \left[ \langle \dot{\varphi}, \rho_A \dot{\varphi} \rangle + 4 \langle \dot{q}, R(q) \tilde{I}_\rho R(q^c) \dot{q} \rangle \right], \quad (3)
\]
\[
U = \frac{1}{2} \left[ \langle \gamma, D_N \gamma \rangle + 4 \langle q', R(q) \tilde{D}_M R(q^c) q' \rangle \right]. \quad (4)
\]

We now introduce the conjugate variables, \( p_\varphi \) and \( p \), via the Legendre transform

\[
p_\varphi := \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \rho_A \dot{\varphi}, \quad (5)
\]
\[
p := \frac{\partial \mathcal{L}}{\partial \dot{q}} = 4L(q)\tilde{I}_\rho L(q^c) \dot{q} = 4R(q)\tilde{I}_\rho R(q^c) \dot{q}, \quad \in T^*\mathbb{H}, \quad (6)
\]
Hamiltonian formulation II

We obtain the augmented Hamiltonian

\[
\mathcal{H} = \int_0^L h(u, p, u_S) \, dS, \quad p = (p_\varphi, p),
\]

(7)

\[
h(u, p, u_S) = \langle p_\varphi, \dot{\varphi}(p_\varphi) \rangle + \langle p, \dot{q}(q, p) \rangle - L(u, u_t(u, p), u_S)
\]

(8)

\[
\partial_t \mathbf{x} = \mathcal{J} \frac{\delta \mathcal{H}}{\delta \mathbf{x}}, \quad \mathcal{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{14 \times 14},
\]

(9)

\[
g(\mathbf{x}) = 0, \quad g(\mathbf{x}) := \|q\|^2 - 1,
\]

(10)

where \( \mathbb{I} \) is the \( 7 \times 7 \) identity matrix, \( \mathbf{x} = (u, p)^T \), \( u = (\varphi, q)^T \in \mathbb{R}^7 \) and \( p = (p_\varphi, p)^T \in \mathbb{R}^7 \).
\[ Mz_t + Kz_x = \nabla_z S(z), \] (11)

where \( z \in \mathbb{R}^d \), \( M \) and \( K \) are skew-symmetric \( d \times d \)-matrices and \( S : \mathbb{R}^d \rightarrow \mathbb{R} \) is a smooth function. We construct the constrained multi-symplectic formulation in quaternions by defining

\[ S(u, p, v) = \langle p, u_t(p) \rangle + \langle v, u_S(v) \rangle - \mathcal{L}(u, u_t(p), u_S(v)), \] (12)

\( p = (p_\varphi, p) \) \( \mathbb{T} \in \mathbb{R}^7 \) and \( v = (v_\varphi, v) \) \( \mathbb{T} \in \mathbb{R}^7 \) are the second conjugate variables defined by

\[ v_\varphi := \frac{\partial \mathcal{L}}{\partial \varphi'} = -D_N \gamma = -n, \] (13)

\[ v := \frac{\partial \mathcal{L}}{\partial q'} = -4L(q)\tilde{C}_M L(q^c)q' = -4R(q)\tilde{D}_M R(q^c)q' \in T^*H. \] (14)
The equations of motion are

\[ \frac{\partial S}{\partial u} = -\partial_t p - \partial_S v \]  
(15)

\[ \frac{\partial S}{\partial p} = \partial_t u \]  
(16)

\[ \frac{\partial S}{\partial v} = \partial_S u \]  
(17)

\[ 0 = \|q\|^2 - 1, \]  
(18)

Let \( z = (u, p, v, \lambda)^T \in \mathbb{R}^{22} \), then (15)–(18) can be written in the general multi-symplectic form (11)

\[ Mz_t + Kz_S = \nabla_z S(z) \]

where

\[
M = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad K = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \in \mathbb{R}^{22 \times 22}.
\]
We implemented a finite element discretization based on a formulation in Euler angles.

We are working on a discretization of the non canonical Hamiltonian formulation exploiting the similarities with the Euler equations for the free rigid body. Alternatively one can use splitting methods to obtain Lie-Poisson integrators.

The canonical Hamiltonian formulation is easy to discretize into a system of constrained Hamiltonian ODEs and then use symmetric-energy-preserving methods or symplectic methods.

The multi-symplectic formulation has not been used in simulations yet, but we hope that the fact that $M$ and $K$ are constant will lead to well posed multi-symplectic discretizations.

We are also investigating a multi-symplectic formulation with $M$ and $K$ non constant.
Thank you for listening!