The Stochastic Wave Equation

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Abstract

In this thesis the stochastic wave equation is studied in the setting of [HØUZ96]. It is proved that the 1D and 3D initial value problems for the inhomogeneous equation have unique, C^2 solutions given by formulas. Furthermore the formulas and the smoothness assumptions are similar to the ones in the classical, non-stochastic initial value problems. It is also explained why the 2D initial value problem can not be solved using the method from the 1D and 3D problems. But the 2D homogeneous problem is solved using another method.

Preface

This is a diploma thesis in industrial mathematics at the Norwegian University of Science and Technology, Faculty of Physics and Mathematics, Department of Mathematical Sciences. The duration of this work has been 5 months, of which one month is extra time mainly due to my position as teaching assistant in the spring of 1996. The thesis counts as 48 bt (belastningstimer), that is one semesters work.

The work on this thesis has to a large extent consisted of reading and understanding the book by Holden et.al. [HØUZ96]. At least to understand enough to solve the problem at hand: The stochastic wave equation. In order to understand [HØUZ96], I studied several books on topics like distributions and stochastic analysis. Even so I got a picture of how I should proceed to solve this equation quit early. But there were still a lot details to fill in. Some of the calculations and proofs are quite long. I can only hope there are now free of errors.

Finally I want to thank my supervisor Professor Helge Holden for finding a problem that has been challenging and very interesting. I also want to thank him for help and advise during the work.

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Trondheim 26.8.96

Contents

Chapter 1

Introduction

1.1 Mathematical Background

In every mathematical work you have to start developing the theory at some point. This section will describe the starting point of this thesis.

We require some familiarity with point set topology, especially with the notions of a continuous function and a compact set. Results like "a linear combination of continuous functions is continuous", "the product of two continuous functions is continuous", "a closed and bounded set in $\mathbb{R}^d$ is compact", and other basic results about continuous functions and compact sets will sometimes be taken for granted and not be stated in this text. I will also use some basic concepts from the theory of PDE's without explanation. Here follows a list of results that we will use frequently throughout the text. Each result will be followed by a reference.

- The Cauchy-Schwartz inequality. See Rudin [Rud76], or Chernoff [Che93].
- The mean value theorem. In one and more dimensions. See Rudin [Rud76] for the 1D case, and Kreyszig [Kre88] for the 2D and 3D cases.
- The fundamental theorem of calculus. See Rudin [Rud76] or Royden [Roy88].
- The bounded convergence theorem. See Royden [Roy88].
- The Fubini Theorem. See Royden [Roy88].

In addition we will need two other results of a general kind, these I have proved in Appendix A. The first result is about continuity, and the second result is about differentiation of an integral.

Lemma 1.1 *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a function and $\alpha \in \mathbb{N}^d$ a multi index. Assume $D^\alpha u(x)$ exists and is continuous for all α such that $|\alpha| = k$. Then $D^\beta u(x)$ is continuous for all multi indexes β such that $|\beta| < k$.*

Lemma 1.2 *Let $f : V \times \mathbb{R} \to \mathbb{R}$, $u : V \to \mathbb{R}$, and $l : V \to \mathbb{R}$ be functions, and let $V \subset \mathbb{R}^d$ be an open set. If $u(x)$, $l(x)$, and $f(x,s)$ are C^1, then*

$$\frac{\partial}{\partial x_i}\int_{l(x)}^{u(x)} f(x,s)ds = u_{x_i}(x)f(x,u(x)) - l_{x_i}(x)g(x,l(x)) + \int_{l(x)}^{u(x)} f_{x_i}(x,s)ds.$$

1.2 Notation

I have tried to use standard notation throughout this text. But I should explain my differential notation. Let us define the operators ∂ and D:

$$\partial_a \equiv \tfrac{\partial}{\partial a} \text{ and } D^k u(x) = \{\text{the collection of all partial derivatives of } u(x) \text{ of order k}\}.$$

A lower index on my differential symbols denotes which variables the differentiation is with respect to. Note that $u_t(x) = (\partial_t u)(x) = \frac{\partial u}{\partial t}(x)$, but $\partial_t u(x) = u_t(x)x_t$. Also note that $Du = \nabla u$, the gradient of u, and $D^2 u$ is the Hessian matrix.

I will frequently use numbers as lower indices on ∂ and D. This is supposed to mean the following. Let $g(f_1(x), ..., f_i(x), ..., f_n(x))$ be a function.

- If $f_j(x)$ is 1-dimensional, then $\partial_i g(f_1(x), ..., f_i(x), ..., f_n(x)) = \frac{\partial g}{\partial f_i}(f_1(x), ..., f_n(x))$.
- If $f_j(x) = (f_j^1(x), ..., f_j^m(x))$, then
$$D_i g(f_1(x), ..., f_i(x), ..., f_n(x)) = (D_{f_i^1,...,f_i^m} g)(f_1(x), ..., f_i(x), ..., f_n(x)).$$

In this thesis all $(\mathcal{S})_{-1}$-processes will be denoted by capital letters. If F is a $(\mathcal{S})_{-1}$-processes, we let a lowercase f denote the Hermite transform of F, $f \equiv \mathcal{H}F$. See Chapter 2 for the definitions of a $(\mathcal{S})_{-1}$-process and the Hermite transformation.

Let the function space $C^2_{loc}(U \times V, (\mathcal{S})_{-1})$ be such that $u(t,x) \in C^2_{loc}(U \times V, (\mathcal{S})_{-1})$ if and only if $u(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$ for all compact sets $K_1 \subset U$ and $K_2 \subset V$.

Finally, $|\cdot|$ is the 2-norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

1.3 Outline

Chapter 2 contains a brief description of the setting in which we will work in this thesis. That is we will define what we mean by a stochastic PDE and state some results from this theory for later use. A lot of the notation used in this thesis will be explained here.

In Chapter 3 we will state and prove a lot of results that will be used in the proofs of later chapters. Most of the results are of a general kind, and will save repeating similar arguments several times in the following chapters.

Chapter 4 and 5 contains the main results of this work. Namely the existence, uniqueness, formula and smoothness results for the stochastic wave equation. Chapter 4 deals with the 1D problem, and Chapter 5 deals with the 3D problem.

Then there is a discussion in Chapter 5. We will discuss some of the assumptions made and why the 2D problem was not solved. And we will use the 3D results to partially solve the 2D problem. I will also list some of unanswered and closely related problems I did not have time to do.

Finally there are two appendices. In the first one we prove the two lemmas from section 1.1. In the second one we will calculate the derivatives of the classical solutions of the 1D and 3D problems. These calculations are long and tedious, but the results will be needed in Chapters 4 and 5.

Chapter 2

Basic stochastic PDE Theory

Now we will describe the setting in which we will work. This chapter is going to be very brief, and practically only results that will be of direct use later are stated. We give no proofs, they can be found in the books by Holden et.al. [HØUZ96] or Hida et.al. [HKPS93]. This chapter is mainly based on [HØUZ96].

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space, then the dual $\mathcal{S}'(\mathbb{R}^d)$ is the space of Schwartz distributions. Then define $\mathcal{S} \equiv \Pi_{i=1}^m \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}' \equiv \Pi_{i=1}^m \mathcal{S}'(\mathbb{R}^d)$. Also let $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$ and $\mathcal{B}(\mathcal{S}')$ be the Borel σ-algebras generated by the weak topologies on $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}'$ respectively. Then by the Bochner-Minlos theorem there exists a unique Gaussian probability measure μ on $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$. Let $\mu_m = \Pi_{k=1}^m \mu$, then μ_m is a Gaussian probability measure on $\mathcal{S}'$.

Definition 2.1 *The triplet $(\mathcal{B}(\mathcal{S}'), \mathcal{S}', \mu_m)$ is called the m-dimensional white noise probability space.*

Now we are going to define a basis for the space $L^2(\mu_m)$. Let $h_n(x)$ denote the *Hermite polynomials*, $h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})$. Then we define the *Hermite functions* $\xi_n(x)$ as follows, $\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{n-1}(\sqrt{2}x)$. Note that $\{\xi_n\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$. Let $\delta^{(i)} \in \mathbb{N}^d$ with an ordering such that $i < j$ implies $\delta_1^{(i)} + ... + \delta_d^{(i)} \leq \delta_1^{(j)} + ... + \delta_d^{(j)}$. Define $\eta_j \equiv \xi_{\delta^{(j)}} \equiv \xi_{\delta_1^{(j)}} \otimes \xi_{\delta_2^{(j)}} \otimes ... \otimes \xi_{\delta_d^{(j)}}$. When $\delta \in \mathbb{N}^d$, $\{\eta_j\}_{j=1}^\infty$ constitutes an orthonormal basis for $L^2(\mathbb{R}^d)$. Now define

$$e^{(km)} = (\eta_k, 0, ..., 0)\ ,\ e^{(km+1)} = (0, \eta_k, ..., 0)\ ,\ ...\ ,\ e^{((k+1)m)} = (0, 0, ..., \eta_k),\ m, k = 0, 1, 2, ...$$

The set $\{e^{(k)}\}_k$ is an orthonormal basis for $\oplus_{k=1}^m L^2(\mathbb{R}^d)$.

Definition 2.2 *Let $\mathcal{I} \equiv (\mathbb{N}_0^{\mathbb{N}})_c$, i.e. the space of $\mathbb{N}_0$-valued sequences with only a finite number entries different from 0 (compact support).*

Definition 2.3 (Definition 2.2.1 in [HØUZ96]) *Let $\alpha \in \mathcal{J}$. Then we define*

$$H_\alpha^{(m)}(\omega) = H_\alpha(\omega) \equiv \Pi_{k=1}^\infty h_{\alpha_k}(<\omega, e^{(k)}>) \text{ for } \omega \in \mathcal{S}',$$

where $<\omega, e^{(k)}>=<\omega_1, e_1^{(k)}> + ... + <\omega_1, e_m^{(k)}>$. $<\cdot,\cdot>$ *is the pairing between* $\mathcal{S}$ *and* $\mathcal{S}'$.

The set $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ constitutes an orthogonal basis for $L(\mu_m)$ and $\|H_\alpha\|^2_{L^2(\mu_m)} = \alpha! \equiv \Pi_{i=1}^\infty \alpha_i!$. This means that every $f(\omega) \in L^2(\mu_m)$ can be written as $f(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha(\omega)$, where $c_\alpha \in \mathbb{R}$.

Now we know enough to define the spaces we will be working with in this thesis. Let $\alpha \in \mathcal{J}$ then define $(2\mathbb{N})^\alpha \equiv \Pi_j (2j)^{\alpha_j}$.

Definition 2.4 (Definition 2.3.2 in [HØUZ96])

a) Let the spaces $(\mathcal{S})_\rho$, $0 \le \rho \le 1$, consist of those $f = \sum_\alpha c_\alpha H_\alpha$ with $c_\alpha \in \mathbb{R}$ such that $\|f\|^2_{\rho,k} \equiv \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} < \infty$ for all $k \in \mathbb{N}$.

b) Let the spaces $(\mathcal{S})_{-\rho}$, $0 \le \rho \le 1$, consist of all formal expansions $F = \sum_\alpha b_\alpha H_\alpha$ with $b_\alpha \in \mathbb{R}$ such that $\|f\|^2_{-\rho,-q} \equiv \sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$.

Note that the space $(\mathcal{S})_\rho$ and $(\mathcal{S})_{-\rho}$ are denoted by $(\mathcal{S})_\rho^{m;1}$ and $(\mathcal{S})_{-\rho}^{m;1}$ in [HØUZ96], and are called the Kondratiev spaces of stochastic test functions and distributions respectively. The spaces $(\mathcal{S})_0$ and $(\mathcal{S})_{-0}$ corresponds to the Hida spaces $(\mathcal{S})$ and $(\mathcal{S})^*$.

The family of semi norms $\|\cdot\|_{\rho,k}$ give rise to a Frechet topology on $(\mathcal{S})_\rho$. Furthermore we can regard $(\mathcal{S})_{-\rho}$ as the dual of $(\mathcal{S})_\rho$ by the action $<F, f>= \sum_\alpha b_\alpha c_\alpha$ for $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})_{-\rho}$ and $f = \sum_\alpha c_\alpha H_\alpha \in (\mathcal{S})_\rho$. Note that for $\rho \in [0,1]$ we have $(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S})_0 \subset L^2(\mu) \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1}$.

Definition 2.5 (Definition 2.6.1 in [HØUZ96]) *Let* $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})_{-1}$ *with* $b_\alpha \in \mathbb{R}$. *Then the Hermite transform of* F*, denoted* $\mathcal{H}F$ *or* $\tilde{F}$*, is defined by*

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C} \text{ (when convergent)}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^\mathbb{N}$ *and* $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{J}$ *with the convention that* $z_j^0 = 1$.

Definition 2.6 (Definition 2.6.4 in [HØUZ96]) *For* $R > 0$, $q < \infty$ *let* $\mathbb{K}_q(R) \subset \mathbb{C}^\mathbb{N}$ *be the infinite-dimensional neighborhood of* 0 *defined by*

$$\mathbb{K}_q(R) \equiv \{(\zeta_1, \zeta_2, ...) \in \mathbb{C}^\mathbb{N} : \sum_{\alpha\neq 0} |\zeta^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2\}$$

Note that

$$q \leq Q, r \leq R \Rightarrow \mathbb{K}_Q(r) \subset \mathbb{K}_q(R). \tag{2.1}$$

The following theorem describes the properties of $\mathcal{H}f$ when $f \in (\mathcal{S})_{-1}$. It also describes what kind of f satisfy $\mathcal{H}^{-1}f \in (\mathcal{S})_{-1}$.

Theorem 2.7 (Theorem 2.6.11 in [HØUZ96])

a) If $F(\omega) = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})_{-1}$, $a_\alpha \in \mathbb{R}$ then there exists q,$M_q < \infty$ such that

$$|\tilde{F}(z)| \leq \sum_\alpha |a_\alpha||z^\alpha| \leq M_q \left(\sum_\alpha (2\mathbb{N})^{q\alpha} |z^\alpha|^2 \right)^{\frac{1}{2}} \text{ for all } z \in (\mathbb{C}^{\mathbb{N}})_c,$$

i.e. $\tilde{F}$ is bounded analytic on $\mathbb{K}_q(R)$ $\forall R < \infty$.

b) If $g(z) = \sum_\alpha b_\alpha z^\alpha$, $z \in (\mathbb{C}^{\mathbb{N}})_c$, $b_\alpha \in \mathbb{R}$, and there exists $q < \infty$, $\delta > 0$ such that $g(z)$ is absolutely convergent when $z \in \mathbb{K}_q(\delta)$ and $\sup_{z \in \mathbb{K}_q(\delta)} |g(z)| < \infty$. Then there exists a unique $G \in (\mathcal{S})_{-1}$ such that $\tilde{G} = g$, namely $G = \sum_\alpha b_\alpha H_\alpha$.

Now we can relate convergence in $(\mathcal{S})_{-1}$ to convergence of the Hermite transformed functions in $\mathbb{K}_q(\delta)$.

Theorem 2.8 (Theorem 2.8.1 in [HØUZ96]) *The following statements are equivalent:*

i) $X_n \to X$ in $(\mathcal{S})_{-1}$.

ii) There exist $\delta > 0$, $q < \infty$ such that $\tilde{X}_n \to \tilde{X}$ uniformly in $\mathbb{K}_q(\delta)$.

iii) There exist $\delta > 0$, $q < \infty$ such that $\tilde{X}_n \to \tilde{X}$ point-wise boundedly in $\mathbb{K}_q(\delta)$.

Definition 2.9 (Definition 2.8.3 in [HØUZ96])

a) A measurable function $u : \mathbb{R}^d \to (\mathcal{S})_{-1}$ is called a stochastic distribution process or a $(\mathcal{S})_{-1}$-process.

b) u is continuous, differentiable, C^k respectively if u has these properties as a $(\mathcal{S})_{-1}$-process (this means that all limits has to exist in $(\mathcal{S})_{-1}$).

We will need some results concerning integration and differentiation in $(\mathcal{S})_{-1}$. The following two lemmas give sufficient conditions for differentiation and integration of $(\mathcal{S})_{-1}$-process to be well defined.

Lemma 2.10 (Lemma 2.8.4 in [HØUZ96]) *Suppose $X(t,\omega)$ and $F(t,\omega)$ are $(\mathcal{S})_{-1}$-processes satisfying:*

1) $\frac{d\tilde{X}}{dt}(t,z) = \tilde{F}(t,z)$ *for* $(t,z) \in (a,b) \times \mathbb{K}_q(\delta)$.

2) $\tilde{F}(t,z)$ *is bounded for* $(t,z) \in (a,b) \times \mathbb{K}_q(\delta)$ *and for each* $z \in \mathbb{K}_q(\delta)$ *continuous for* $t \in (a,b)$.

Then $X(t,\omega)$ *is a differentiable* $(\mathcal{S})_{-1}$*-process and* $\frac{dX}{dt}(t,\omega) = F(t,\omega)$.

Lemma 2.11 (Lemma 2.8.5 in [HØUZ96]) *Let* $X(t)$ *be an* $(\mathcal{S})_{-1}$*-process. Assume there exist* $q < \infty, \delta > 0$ *such that:*

a) $\sup\{\tilde{X}(t,z) \mid (t,z) \in [a,b] \times \mathbb{K}_q(\delta)\} < \infty$.

b) *For each* $z \in \mathbb{K}_q(\delta)$ $\tilde{X}(t,z)$ *is continuous for* $t \in [a,b]$.

Then $X(t)$ *is strongly integrable and* $\mathcal{H}(\int_a^b X(t)dt) = \int_a^b \tilde{X}(t)dt$.

This means that $\int_a^b X(t)dt \in (\mathcal{S})_{-1}$.

Now it is time for main theorem in this chapter. It asserts that under certain conditions a stochastic PDE has a solution. Let $f(t,x,z)$ such that $f : G \times \mathbb{K}_q(\delta) \to \mathbb{C}$, for some q and δ and $G \subset \mathbb{R}^{d+1}$. Then we define properties (P1), (P2), and (P3) as follows:

(P1) f is bounded for $(t,x,z) \in G \times \mathbb{K}_q(\delta)$.

(P2) f is continuous with respect to (t,x) for each $z \in \mathbb{K}_q(\delta)$.

(P3) f is analytic with respect to $z \in \mathbb{K}_q(\delta)$ for each $(t,x) \in G$.

Theorem 2.12 (Theorem 4.1.1 in [HØUZ96]) *Assume* $u(t,x,z)$ *is a solution (in the strong point-wise sense) of the equation*

$$\tilde{A}(t,x,\partial_t,\nabla_x,u,z) = 0 \tag{2.2}$$

for $(t,x,z) \in G \times \mathbb{K}_q(\delta)$ *for some* q, δ *where* $G \subset \mathbb{R} \times \mathbb{R}^d$ *is some bounded, open set. Also assume that* $u(t,x,z)$ *and all its partial derivatives involved in the above equation satisfies (P1), (P2), and (P3).*

Then there exists $U(t,x)$ *lying* $(\mathcal{S})_{-1}$ *for fixed* t *and* x*, such that* $u(t,x,z) = (\mathcal{H}U)(t,x,z)$ *for all* $(t,x,z) \in G \times \mathbb{K}_q(\delta)$ *and* $U(t,x)$ *solves (in the strong sense in* $(\mathcal{S})_{-1}$*) the equation*

$$A^{\diamond}(t,x,\partial_t,\nabla_x,U,\omega) = 0 \text{ in } (\mathcal{S})_{-1} \tag{2.3}$$

Note: $A^{\diamond}(t, x, \partial_t, \nabla_x, U, \omega)$ is the Wick version of the function A. This is based on the Wick product, which is a way to multiply $(\mathcal{S})_{-1}$-processes. We will not need it. In the case A represents the stochastic wave equation, $A^{\diamond}(t, x, \partial_t, \nabla_x, U, \omega) = A(t, x, \partial_t, \nabla_x, U, \omega)$.

We now have the following procedure for solving a stochastic PDE:

1) Take the Hermite transform of the equation. We then get an equation in $(t, x, z) \in G \times \mathbb{K}_q(\delta)$ for some q, δ.

2) Solve this complex PDE. Call the solution u.

3) Show that u and all its derivatives that appears in the equation satisfy (P1) - (P3) on some domain.

4) Then Apply Theorem 2.12 which says that you can take the inverse Hermite transformation $\mathcal{H}^{-1}u$, and that this is the solution of the stochastic PDE.

Chapter 3

Intermediate Results

In the procedure for solving a stochastic PDE (Chapter 2), the third task was to show that all derivatives appearing in the Hermite transformed equation satisfy (P1) - (P3) on some domain. This chapter contains results that will help us do that when we solve the stochastic wave equation. Most of the results are of a general kind, and will save repeating similar arguments several times later.

3.1 Continuous $(\mathcal{S})_{-1}$-processes

In this section we will answer the following question: If F is a continuous $(\mathcal{S})_{-1}$-process, what properties does $f \equiv \mathcal{H}F$ have? Let us start by showing what continuity in $(\mathcal{S})_{-1}$ translates to when we take the Hermite transform.

Lemma 3.1 *The following two statements are equivalent:*

1) $X(x,\omega)$ is a continuous $(\mathcal{S})_{-1}$-process.

2) For every $\epsilon > 0$ there exist $\gamma > 0$ such that

$$|x - x_0| < \gamma \Rightarrow \sup_{z \in \mathbb{K}_q(\delta)} |\tilde{X}(x,z) - \tilde{X}(x_0,z)| < \epsilon \text{ for some } q \text{ and } \delta.$$

Proof:

The result follows from the definition of continuity and theorem 2.8.

$$
\begin{aligned}
X(x,\omega) \text{ is continuous} \quad &\Leftrightarrow \quad \lim_{x\to x_0} X(x,\omega) = X(x_0,\omega)\ \forall\omega \\
&\Leftrightarrow \quad \exists\delta, q \text{ such that } \lim_{x\to x_0} \tilde{X}(x,z) = \tilde{X}(x_0,z) \text{ uniformly in } \mathbb{K}_q(\delta) \\
&\Leftrightarrow \quad \exists\delta, q\ \ \forall\epsilon > 0\ \ \exists\gamma > 0 \text{ such that} \\
&\qquad |x - x_0| < \gamma \Rightarrow \sup_{z\in\mathbb{K}_q(\delta)} |\tilde{X}(x,z) - \tilde{X}(x_0,z)| < \epsilon
\end{aligned}
$$

□

The second statement in the lemma implies that $\tilde{X}(x,z)$ is continuous in x. The next lemma shows that under certain conditions a function is bounded. This result will be used in the proof of Theorem 3.3 to show that a Hermite transformed continuous $(\mathcal{S})_{-1}$-process is bounded on compact sets.

Lemma 3.2 *If* $u(x,z)$,$u : \mathbb{R}^d \times Z \to \mathbb{C}$, *satisfies for every* $\epsilon > 0$ *there exists* $\delta = \delta(x_0,\epsilon)$ *such that*

1) $|x - x_0| < \delta \Rightarrow \sup_{z\in Z} |f(x,z) - f(x_0,z)| < \epsilon$ *and*

2) $f(x,z)$ *is bounded for fixed* x,

then $\sup_{\substack{x \in K \\ z \in Z}} |u(x,z)| < \infty$ *for all compact* $K \subset \mathbb{R}^d$.

Proof:

Let $K \subset \mathbb{R}^d$ be compact. By assumption 1) for every x there exists $\delta_x > 0$ such that

$$
\text{if } y \in B(x,\delta_x) \text{ then } \sup_z |f(x,z) - f(y,z)| < 1.
$$

$\{B(x,\delta_x)\}_{x\in K}$ is a covering for K. Since K is compact, this cover has a finite sub-cover $\{B(x_i,\delta_i)\}_{i=1}^n$, i.e. $K \subset \bigcup_{i=1}^n B(x_i,\delta_i)$. Now fix $x_0 \in K$.

$$
\begin{aligned}
&\sup\nolimits_{z;x\in K} |f(x,z) - f(x_0,z)| \ \leq\ \sup\nolimits_{z;\, x,\, y\, \in\, K} |f(x,z) - f(y,z)| \\
&\leq\ \textstyle\sum_{i,j=1}^n \sup\nolimits_{z;\, x\, \in\, B(x_i,\,\delta_i),\, y\, \in\, B(x_j,\,\delta_j)} |f(x,z) - f(y,z)| \ \leq\ n^2\ <\ \infty
\end{aligned}
$$

From this and boundedness for fixed x we have

$$
\sup_{z;x\in K} |f(x,z)| \leq \sup_{z;x\in K} |f(x,z) - f(x_0,z)| + \sup_{z;x\in K} |f(x_0,z)| < \infty.
$$

□

Now we can get to the main result. That is characterizing the properties of continuous $(\mathcal{S})_{-1}$-processes.

Theorem 3.3 *If $X(x,\omega)$ is a continuous $(\mathcal{S})_{-1}$-process, $x \in U \subset \mathbb{R}^d$ open, then there exists $q < \infty, \delta > 0$ such that the following properties hold:*

(E1) *$\tilde{X}(x,z)$ is bounded on $K \times \mathbb{K}_q(\delta)$ for all compact $K \subset U$.*

(E2) *For each $z \in \mathbb{K}_q(\delta)$ $\tilde{X}(x,z)$ is continuous in x.*

(E3) *For fixed x $\tilde{X}(x,z)$ is analytic in $\mathbb{K}_q(\delta)$.*

Proof:

(E1) From lemma 3.1 and theorem 2.7 we see that the assumptions in lemma 3.2 are satisfied, and we thus have the desired result for some δ_a and q_a.

(E2) Follows from Lemma 3.1 for some δ_b and q_b.

(E3) Follows from theorem 2.7 for some δ_c and q_c.

If we choose $q = \max\{q_a, q_b, q_c\}$ and $\delta = \min\{\delta_a, \delta_b, \delta_c\}$ we are done by (2.1). □

Note that (E1) - (E3) are closely related to (P1) - (P3). Let us state the following corollary without proof.

Corollary 3.4 *If $X(x,\omega)$ is a continuous $(\mathcal{S})_{-1}$-process, $x \in U \subset \mathbb{R}^d$ open, then there exists $q < \infty, \delta > 0$ such that (P1) - (P3) hold on every compact subset $K \subset U$.*

3.2 Functions satisfying (E1) - (E3) or (P1) - (P3)

3.2.1 General Results

Let f and g satisfy (E1) - (E3) on some domain. In this section we will study questions like "Does $f + g$ satisfy (E1) - (E3)?" and "Does $\int f$ satisfy (E1) - (E3)?". And we will ask similar questions about functions satisfying (P1) - (P3). The contents of the lemmas will be described in their headings.

In the next lemma let U be a subset of $\mathbb{R}^d$.

Lemma 3.5 (Linear combination)

a) A linear combination $f(x,z) = a_1 f_1(x,z) + a_2 f_2(x,z)$ of functions $f_1(x,z)$, $f_2(x,z)$ defined for $(x,z) \in U \times \mathbb{K}_q(\delta)$ and satisfying (P1) - (P3), also satisfies these properties on the same domain.

b) A linear combination $f(x,z) = a_1 f_1(x,z) + a_2 f_2(x,z)$ of functions $f_1(x,z)$, $f_2(x,z)$ defined for $(x,z) \in U \times \mathbb{K}_q(\delta)$ and satisfying (E1) - (E3), also satisfies these properties on the same domain.

Proof:

Let us prove b). The proof of a) is similar.

(E1) On every compact subset of U f is bounded since it is the finite sum of bounded functions.

(E2) Fix $z \in \mathbb{K}_q(\delta)$. Since $f_1(x,z)$ and $f_2(x,z)$ are continuous in x there exists $\gamma > 0$ such that if $|\Delta x| < \gamma$ then $|f_1(x+\Delta x, z) - f_1(x,z)| < \frac{\epsilon}{2|a_1|}$ and $|f_2(x+\Delta x,z) - f_2(x,z)| < \frac{\epsilon}{2|a_2|}$. Now we have

$$\begin{aligned} |f(x+\Delta x, z) - f(x,z)| &\leq |a_1||f_1(x+\Delta x,z) - f_1(x,z)| + |a_2||f_2(x+\Delta x,z) - f_2(x,z)| \\ &< |a_1|\tfrac{\epsilon}{2|a_1|} + |a_2|\tfrac{\epsilon}{2|a_2|} = \epsilon. \end{aligned}$$

This means that $f(x,z)$ is continuous in x.

(E3) f is analytic, this is a consequence of the fact that the space of analytic functions is vector space under addition and scalar multiplication. (By the triangle inequality $\sum_\alpha |a_\alpha + b_\alpha||z^\alpha| \leq \sum_\alpha |a_\alpha||z^\alpha| + \sum_\alpha |b_\alpha||z^\alpha| < \infty$). □

In the following two lemmas we let $f : U \to \mathcal{H}[(\mathcal{S})_{-1}]$ and $g : \mathbb{R}^d \to U$ be functions, and let $U, K \subset \mathbb{R}^n$ be sets such that K is compact.

Lemma 3.6 (Composition with a continuous function)

a) Assume that $g(x)$ is continuous and that there exist q, δ such that $f(x;z)$ satisfies (P1) - (P3) for $(x,z) \in U \times \mathbb{K}_q(\delta)$. Then $f(g(y);z)$ satisfies these properties for $(y,z) \in g^{-1}(U) \times \mathbb{K}_q(\delta)$.

b) Assume that the following conditions hold:

   1) $g(x)$ is continuous in $\mathbb{R}^n$, and for every bounded $B \subset \mathbb{R}^n$, $g(B) \subset \mathbb{R}^d$ is also bounded.

   2) There exist q, δ such that $f(x;z)$ satisfies (E1) - (E3) for $(x,z) \in U \times \mathbb{K}_q(\delta)$.

   Then $f(g(y);z)$ satisfies (E1) - (E3) for $(y,z) \in g^{-1}(U) \times \mathbb{K}_q(\delta)$.

Proof:

Proof of a)

(P1) Since f satisfies (P1), $\sup_{y \in g^{-1}(K), z} |f(g(y);z)| = \sup_{x \in K, z} |f(x;z)| < \infty$.

(P2) Fix $z \in \mathbb{K}_q(\delta)$. Let $O \subset U$ be open. $f^{-1}(O;z) \equiv V$ is open since $f(\cdot;z)$ is continuous by (P2). $g^{-1}(V) \equiv W$ is open since $g(\cdot)$ is continuous. We then have $[f \circ g]^{-1}(O;z) = g^{-1}(f^{-1}(O;z)) = W$, so $f(g(y);z)$ is continuous in y.

(P3) Fix $y \in g^{-1}(U)$. Then $g(y) \in U$, and since f satisfies (P3) on this domain, $f(g(y);z)$ is analytic in z.

Proof of b)

(E1) Let $K \subset g^{-1}(U)$ be compact (relative the subspace topology of $g^{-1}(U)$), then by assumption 1) $\overline{g(K)}$ is compact (bounded and closed). Then $f(x;z)$ is bounded on $\overline{g(K)}$ by (E1). $\sup_{y \in K, z} |f(g(y);z)| = \sup_{x \in g(K), z} |f(x;z)| \leq \sup_{x \in \overline{g(K)}, z} |f(x;z)| < \infty$.

(E2) Fix $z \in \mathbb{K}_q(\delta)$. Let $O \subset U$ be open. $f^{-1}(O;z) \equiv V$ is open since $f(\cdot;z)$ is continuous by (E2). $g^{-1}(V) \equiv W$ is open since $g(\cdot)$ is continuous. We then have $[f \circ g]^{-1}(O;z) = g^{-1}(f^{-1}(O;z)) = W$, so $f(g(y);z)$ is continuous in y.

(E3) Fix $y \in g^{-1}(U)$. Then $g(y) \in U$, and since f satisfies (P3) on this domain, $f(g(y);z)$ is analytic in z.

□

Lemma 3.7 (Product with a continuous function)

a) Assume that $g(x)$ is continuous, and that there exist q, δ such that $f(x;z)$ satisfies (P1) - (P3) for $(x,z) \in K \times \mathbb{K}_q(\delta)$. Then $f(x;z)g(x)$ satisfies these properties for $(y,z) \in K \times \mathbb{K}_q(\delta)$.

b) Assume that $g(x)$ is continuous, and that there exist q, δ such that $f(x;z)$ satisfies (E1) - (E3) for $(x,z) \in U \times \mathbb{K}_q(\delta)$. Then $f(x;z)g(x)$ satisfies these properties for $(x,z) \in U \times \mathbb{K}_q(\delta)$.

Proof:

We will only prove a), since the proof of b) is similar.

(P1) Since f satisfies (P1) and since a continuous function is bounded on a compact domain $\sup_{x,z} |f(x;z)g(x)| \leq \sup_{x,z} |f(x;z)| \sup_x |g(x)| < \infty$.

(P2) Fix $z \in \mathbb{K}_q(\delta)$, then $f(\cdot;z) : \mathbb{R}^d \to \mathbb{R}$. Multiplication $m(x,y) = xy$, $x, y \in \mathbb{R}$ is continuous. Write $f(x,z)g(x) = m(f(x,z), g(x))$, this function is continuous since the composition of continuous functions is continuous.

(P3) Fix $x \in K$. Then $g(x)$ is just a number, so since $f(x;z)$ is analytic in z by (P3), $g(x)f(x;z)$ is still analytic.

□

In the next two lemmas let $f : K \times \mathbb{R} \to \mathcal{H}[(\mathcal{S})_{-1}]$, $u : K \to \mathbb{R}$, and $l : K \to \mathbb{R}$ be functions, and let $K \subset \mathbb{R}^d$ be a compact set. Assume that $l(x) \leq u(x)$ and that these functions are bounded in K. Then we define $a \equiv \inf_K |l(x)|$ and $b \equiv \sup_K |u(x)|$, where $a, b < \infty$ since l, u are bounded.

Lemma 3.8 (Hermite transform of an integral) *Assume that there exist* q, δ *such that* $f(x, s; z)$ *satisfies (P1) - (P3) for* $(x, s, z) \in K \times [a, b] \times \mathbb{K}_q(\delta)$. *Then* $\int_{l(x)}^{u(x)} F(x, s; \omega) ds \in (\mathcal{S})_{-1}$ *and*

$$\mathcal{H} \int_{l(x)}^{u(x)} F(x, s; \omega) ds = \int_{l(x)}^{u(x)} f(x, s; z) ds.$$

Proof:

The fact that f satisfies (P1) and (P3) and Theorem 2.7 b) asserts that there exists a $(\mathcal{S})_{-1}$ process $F(x, s; \omega)$ such that $\mathcal{H}[F(x, s; \omega)] = f(x, s; z)$. By (P1) and (P2), $f(x, s; z)$ satisfies conditions a) and b) in Lemma 2.11. By this lemma $\int_{l(x)}^{u(x)} F(x, s; z) ds \in (\mathcal{S})_{-1}$ and $\mathcal{H}[\int_{l(x)}^{u(x)} F(x, s; \omega) ds] = \int_{l(x)}^{u(x)} f(x, s; z) ds$. □

Lemma 3.9 (Integration) *Assume that the following conditions hold:*

1) *$l(x)$ and $u(x)$ are continuous functions.*

2) *There exist q' and δ' such that $f(x, s; z)$ satisfies (P1) - (P3) for $(x, s, z) \in K \times [a, b] \times \mathbb{K}_{q'}(\delta')$.*

3) *For fixed z, $D_x f(x, s; z)$ is continuous for $(x, s) \in K \times [a, b]$.*

Then there exists q, δ *such that* $\int_{l(x)}^{u(x)} f(x, s; z) ds$ *satisfies (P1) - (P3) for* $(t, x) \in K \times \mathbb{K}_q(\delta)$.

Proof:

Fix q', δ'. Let $M_1 \equiv \sup_{x,s,z} |f(x, s; z)|$ and $M_2 \equiv \sup_{x,s} |D_x f(x, s; z)|$. $M_1 < \infty$ by (P1), and $M_2 < \infty$ for every z by conditions 1) and 3) and the fact that K is compact.

(P1) $|\int_{l(x)}^{u(x)} f(x, s; z) ds| \leq \int_{l(x)}^{u(x)} |f(x, s; z)| ds \leq \sup_x |u(x) - l(x)| M_1 \leq |b - a| M_1 < \infty$.

(P2) Fix $z \in \mathbb{K}_{q'}(\delta')$.

$$|\int_{l(x+\Delta x)}^{u(x+\Delta x)} f(x + \Delta x, s; z) ds - \int_{l(x)}^{u(x)} f(x, s; z) ds|$$
$$\leq |(\int_{l(x+\Delta x)}^{l(x)} + \int_{u(x)}^{u(x+\Delta x)}) f(x + \Delta x, s; z) ds| + |\int_{l(x)}^{u(x)} [f(x + \Delta x, s; z) - f(x, s; z)] ds|$$
$$\leq (|l(x) - l(x + \Delta x)| + |u(x + \Delta x) - u(x)|) M_1 + |\Delta x| M_2 |b - a|$$

Here we used the mean value theorem and Cauchy-Schwartz inequality to get

$$|f(x + \Delta x, s; z) - f(x, s; z)| \leq |D_x f(\xi, s; z) \Delta x| \leq \sup_s |D_x f(\xi, s; z)| |\Delta x| \leq M_2 |\Delta x|.$$

Fix $\epsilon > 0$. Since $u(x)$ and $l(x)$ are continuous, there exist a δ'_x such that if $|\Delta x| < \delta'_x$ then $|u(x+\Delta x) - u(x)| < \frac{\epsilon}{4M_1}$ and $|l(x+\Delta x) - l(x)| < \frac{\epsilon}{4M_1}$. Now choose $\delta_x = \min(\delta'_x, \frac{\epsilon}{2M_2|b-a|})$, then

$$|\Delta x| < \delta_x \quad \Rightarrow \quad |\int_{l(x+\Delta x)}^{u(x+\Delta x)} f(x+\Delta x, s; z)ds - \int_{l(x)}^{u(x)} f(x, s; z)ds| < \epsilon.$$

(P3) By Lemma 3.8 and Theorem 2.7 b) there exist q'', δ'' such that $\int_{l(x)}^{u(x)} f(x, s; z)ds$ is analytic for $(x, z) \in K \times \mathbb{K}_{q''}(\delta'')$.

Choose $q = \max(q', q'')$ and $\delta = \min(\delta', \delta'')$, then by (2.1) $\int_{l(x)}^{u(x)} f(x, s; z)ds$ satisfies (P1) - (P3) for $(x, z) \in K \times \mathbb{K}_q(\delta)$. □

3.2.2 Results concerning the 3D Problem

In Chapter 5 we will need the following four lemmas. The first two lemmas are concerned with area integrals. The first lemma shows that the Hermite transformed $(\mathcal{S})_{-1}$ area integral equals the area integral of the Hermite transformed integrand. The second lemma shows that the area integral of functions satisfying (E1) - (E3), satisfy (P1) - (P3) on some domain. Let $f : K \to \mathcal{H}[(\mathcal{S})_{-1}]$ be a function, and let $U, K \subset \mathbb{R}^3$ and $K_t \subset \mathbb{R}^+$ be sets such that K and K_t are compact.

Lemma 3.10 *Assume that there exist q and δ such that $f(x; z)$ satisfies (E1) - (E3) for $(x, z) \in U \times \mathbb{K}_q(\delta)$, and that $D_x f(x; z)$ is continuous in x for fixed z. Then $\int_{\partial B(x,r)} F(y)dS(y) \in (\mathcal{S})_{-1}$ and $\mathcal{H} \int_{\partial B(x,r)} F(y)dS(y) = \int_{\partial B(x,r)} f(y)dS(y)$ when $\partial B(x, r) \subset U$.*

Proof:

Let us change to spherical coordinates,

$$\int_{\partial B(x,r)} f(y)dS(y) = r^2 \int_{\partial B(0,1)} f(x+ry)dS(y) = r^2 \int_0^{2\pi} \int_0^{\pi} f(h(x, r, \phi, \theta)) \sin\theta d\theta d\phi,$$

where $h(x, r, \phi, \theta) = x + r(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. h is C^∞. This follows from the fact that a linear combination of products of C^∞ functions is C^∞, and that a vector consisting of C^∞ components also is C^∞. Also note that h takes bounded sets to bounded sets.

Since $f(y)$ satisfies (E1) - (E3) for $y \in U$, $f(h(x, r, \phi, \theta))$ satisfies the conditions in Lemma 3.6. Then by Lemmas 3.6 b) and 3.7 b) there exist q', δ' such that $f(h(x, r, \phi, \theta)) \sin\theta$ satisfies (E1) - (E3) for $(x, r, \phi, \theta, z) \in h^{-1}(U) \times \mathbb{K}_{q'}(\delta')$. By (E2) $f(h(x, r, \phi, \theta)) \sin\theta$ is continuous and thus integrable in $h^{-1}(U)$.

Fix x, r such that $\partial B(x, r) \subset U$. Then the set $B \equiv \{x\} \times \{r\} \times [0, 2\pi] \times [0, \pi] \subset h^{-1}(U)$, and it is compact. Then since $f(h(x, r, \phi, \theta)) \sin\theta$ satisfies (E1) - (E3), it satisfies (P1) - (P3) on B. By (P1)

$\sup_{(r,\phi,\theta)\in B,z} |f(h(x,r,\phi,\theta))\sin\theta| \equiv M < \infty$. $\int_0^{2\pi}\int_0^{\pi}|f((h(x,r,\phi,\theta))\sin\theta|d\theta d\phi \leq 2\pi^2 r^2 M < \infty$.

By Fubinis theorem we can evaluate this integral as an iterated integral. Define $i(x,r,\phi) = \int_0^{\pi} f(h(x,r,\phi,\theta))\sin\theta d\theta$. Since $f(h(x,r,\phi,\theta))\sin\theta$ satisfies (P1) - (P3) on B, Lemma 3.8 yields $I(x,r,\phi) = \mathcal{H}^{-1}i(x,r,\phi) \in (\mathcal{S})_{-1}$ and

$$\mathcal{H}I(x,r,\phi) = \int_0^{\pi} f(h(x,r,\phi,\theta))\sin\theta d\theta. \tag{3.1}$$

We differentiate with respect to (ϕ,θ),

$$D[f(h(x,r,\phi,\theta))\sin\theta] = (D_x f)(h(x,r,\phi,\theta))\cdot Dh(x,r,\phi,\theta)\sin\theta + f(h(x,r,\phi,\theta))\cos\theta.$$

Since $D_x f$ is continuous and h is C^∞, we see that the composition $Df(h)$ is continuous. Since $f(h(x,r,\phi,\theta))\sin\theta$ also satisfies (P1) - (P3) on B, the conditions in Lemma 3.9 are satisfied. Then by Lemma 3.9 $i(x,r,\phi)$ satisfies (P1) - (P3), so Lemma 3.8 yields $\int_0^{2\pi} I(x,r,\phi)d\phi \in (\mathcal{S})_{-1}$ and

$$\mathcal{H}\int_0^{2\pi} I(x,r,\phi)d\phi = \int_0^{2\pi} \mathcal{H}I(x,r,\phi)d\phi. \tag{3.2}$$

From (3.1) and (3.2) we now get

$$\begin{aligned}\mathcal{H}\int_{\partial B(x,t)} F(y)dS(y) &= r^2\int_0^{2\pi}\mathcal{H}I(x,r,\phi)d\phi = r^2\int_0^{2\pi}\int_0^{\pi} f(h(x,r,\phi,\theta))\sin\theta d\theta d\phi \\ &= \int_{\partial B(x,t)} f(y)dS(y).\end{aligned}$$

□

Lemma 3.11 *Assume that there exist q' and δ' such that $f(x;z)$ satisfies (E1) - (E3) for $(x,z)\in\mathbb{R}^3\times\mathbb{K}_{q'}(\delta')$, and that $D_x f(x;z)$ is continuous in x for fixed z. Then there exist q,δ such that $\frac{1}{t^2}\int_{\partial B(x,t)} f(y)dS(y)$ satisfies (P1) - (P3) for $(t,x,z)\in K_t\times K\times\mathbb{K}_q(\delta)$.*

Proof:

(P1) Define $B \equiv \{y : y\in\partial B(x,t) \text{ for } (t,x)\in K_t\times K\}$. Since f satisfies (E1) and $\overline{B}$ is compact, $|\frac{1}{t^2}\int_{\partial B(x,t)} f(y;z)dS(y)| \leq \frac{1}{t^2}4\pi t^2 \sup_{y\in\overline{B}}|f(y;z)| < \infty$.

(P2) Note that $\frac{1}{t^2}\int_{\partial B(x,t)} f(y)dS(y) = \int_{\partial B(0,1)} f(x+ty)dS(y)$. Let $\tilde{f}(x,t) = f(x+ty)$. Then $D_{t,x}\tilde{f}(t,x,y) = (\sum_i y_i\partial_{x_i}f(x+ty), \partial_{x_1}f(x+ty),...,\partial_{x_d}f(x+ty))$. $D\tilde{f}$ is continuous since Df is continuous. So $\sup_{t,x,y}|D\tilde{f}| \equiv M < \infty$, since $(t,x,y)\in K_t\times K\times\partial B(0,1)$ and this set is compact. Now we use the mean value theorem and the Cauchy-Schwartz inequality.

$|\int_{\partial B(0,1)} f(x+\Delta x+(t+\Delta t)y)dS(y) - \int_{\partial B(0,1)} f(x+ty)dS(y)|$
$\leq \int_{\partial B(0,1)} |f(x+\Delta x+(t+\Delta t)y) - f(x+ty)|dS(y)$
$\leq 4\pi|(\Delta t, \Delta x)| \sup_y |D_{t,x}\tilde{f}| \leq 4\pi|(\Delta t, \Delta x)|M$

This means that the function is Lipschitz continuous and therefore continuous.

(P3) By Lemma 3.10 $\frac{1}{t^2}\int_{\partial B(x,t)} f(y)dS(y) \in (\mathcal{S})_{-1}$ for fixed t, x. Theorem 2.7 b) then yields that $\mathcal{H}\frac{1}{t^2}\int_{\partial B(x,t)} f(y)dS(y)$ is analytic in z for some q'' and δ''.

Choose $q = \max(q', q'')$ and $\delta = \min(\delta', \delta'')$, then (P1) - (P3) is satisfied on $K_t \times \mathbb{K} \times \mathbb{K}_q(\delta)$.

□

The following two lemmas are concerned with two particular volume integrals. The first lemma shows that the Hermite transformed $(\mathcal{S})_{-1}$ volume integrals equals the volume integrals of the Hermite transformed integrands. The second lemma shows that two particular volume integrals of functions satisfying (E1) - (E3) multiplied by some function, satisfy (P1) - (P3) on some domain. Let $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathcal{H}[(\mathcal{S})_{-1}]$ and $g : \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be functions, and let $U, K \subset \mathbb{R}^3$ and $K_t \subset \mathbb{R}^+$ be sets such that K and K_t are compact.

Lemma 3.12 *Assume that there exist q and δ such that $f(t,x;z)$ satisfies (E1) - (E3) for $(t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{K}_q(\delta)$, that $D_{t,x}f(t,x;z)$ is continuous in (t,x) for fixed z, and the function $g(x,y)$ is continuous. Then the following statements hold:*

a) $\int_{B(x,r)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|}dy \in (\mathcal{S})_{-1}$ *and*
$\mathcal{H}\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy$ *when* $B(x,t) \subset K$.

b) $\int_{B(x,r)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy \in (\mathcal{S})_{-1}$ *and*
$\mathcal{H}\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy = \int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy$ *when* $B(x,t) \subset K$.

Proof:

Fix x, t such that $B(x,t) \subset K$.

$$\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_{B(0,t)} \frac{f(t-|y|,x+y;z)g(x,x+y)}{|y|}dy$$
$$= \int_0^t \int_0^{2\pi} \int_0^{\pi} \frac{1}{r} f(t-r, x+rh(\phi,\theta);z)g(x,x+rh(\phi,\theta))r^2 \sin\theta d\theta d\phi dr$$

$$\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy = \int_{B(0,t)} \frac{f(t-|y|,x+y;z)g(x,x+y)}{|y|^2}dy$$
$$= \int_0^t \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} f(t-r, x+rh(\phi,\theta);z)g(x,x+rh(\phi,\theta))r^2 \sin\theta d\theta d\phi dr$$

Here $h(\phi,\theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The sets $K \times \overline{B}(x,t)$ and $[0,t] \times K$ are compact. This means that $M_g \equiv \sup_{K\times\overline{B}(x,t)} |g(x,y)| < \infty$, and since f satisfies (E1), $M_f \equiv \sup_{[0,t]\times K\times\mathbb{K}_q(\delta)} |f(s,y;z)| < \infty$.

$\int_0^t \int_0^{2\pi} \int_0^\pi |\frac{1}{r} f(t-r, x+rh(\phi,\theta); z) g(x, x+rh(\phi,\theta))| r^2 \sin\theta d\theta d\phi dr$
$\leq M_f M_g \int_0^t \int_0^{2\pi} \int_0^\pi r \sin\theta d\theta d\phi dr = M_f M_g \pi t^2 < \infty$

$\int_0^t \int_0^{2\pi} \int_0^\pi |\frac{1}{r^2} f(t-r, x+rh(\phi,\theta); z) g(x, x+rh(\phi,\theta))| r^2 \sin\theta d\theta d\phi dr$
$\leq M_f M_g \int_0^t \int_0^{2\pi} \int_0^\pi \sin\theta d\theta d\phi dr = M_f M_g 2\pi t < \infty$

By Fubinis theorem we can evaluate these integrals as iterated integrals. Note that $|y| = r$ when $y \in \partial B(0, r)$.

$\int_{B(x,t)} \frac{f(t-|y-x|, y; z) g(x,y)}{|y-x|} dy = \int_{B(0,t)} \frac{f(t-|y|, x+y; z) g(x, x+y)}{|y|} dy$
$= \int_0^t \int_{\partial B(0,r)} \frac{f(t-|y|, x+y; z) g(x, x+y)}{|y|} dS(y) dr = \int_0^t \frac{1}{r} r^2 \int_{\partial B(0,1)} f(t-r, x+ry; z) g(x, x+ry) dS(y) dr$

$\int_{B(x,t)} \frac{f(t-|y-x|, y; z) g(x,y)}{|y-x|^2} dy = \int_0^t \frac{1}{r^2} r^2 \int_{\partial B(0,1)} f(t-r, x+ry; z) g(x, x+ry) dS(y) dr$

Define $\bar{f}(y) \equiv f(t-r, x-ry) g(x, x+ry)$. This function satisfies (P1) - (P3) for $y \in \overline{B(x,t)}$ by Lemma 3.7 a) since $f(t,x)$ satisfies (E1) - (E3) on $\mathbb{R}^+ \times \mathbb{R}^3$. Now $\bar{f}(y)$ satisfies the conditions of Lemma 3.10 b), so

(A) $\int_{\partial B(0,1)} F(t-r, x+ry; \omega) g(x, x+ry) dS(y) \in (\mathcal{S})_{-1}$, and

(B) $\mathcal{H} \int_{\partial B(0,1)} F(t-r, x+ry; \omega) g(x, x+ry) dS(y) = \int_{\partial B(0,1)} f(t-r, x+ry; z) g(x, x+ry) dS(y)$.

Define $j(r) \equiv \int_{\partial B(0,1)} f(t-r, x+ry; z) g(x, x+ry) dS(y)$. Now we will show that $j(r)$ satisfies (P1) - (P3) for $r \in [0, t]$.

(P1) $|\int_{\partial B(0,1)} f(t-r, x+ry; z) g(x, x+ry) dS(y)| \leq M_f M_g \int_{\partial B(0,1)} dS(y) = M_f M_g 4\pi$. This expression is independent of z.

(P2) Define $\tilde{f}(r) \equiv f(t-r, x+ry; z)$. Then $\partial_r \tilde{f}(r) = (D_2 f)(t-r, x+ry; z) \cdot y - (\partial_1 f)(t-r, x+ry; z)$. This function is continuous since Df is continuous. Then $M_D \equiv \sup_{[0,t]} |\partial_r \tilde{f}(r)| < \infty$. Now we use the mean value theorem:

$$\begin{aligned} |j(r+\Delta r) - j(r)| &\leq \int_{\partial B(0,1)} |\tilde{f}(r+\Delta r) g(x, x+(r+\Delta r)y) - \tilde{f}(r) g(x, x+ry)| dS(y) \\ &\leq |\Delta r| M_f M_g 4\pi. \end{aligned}$$

This is Lipschitz continuity, which implies continuity.

(P3) This property follows from (A) and Theorem 2.7 b) for some q', δ'.

Since $j(r)$ satisfies (P1) - (P3) for $r \in [0, t]$ and $q'' = \max(q, q')$ and $\delta'' = \min(\delta, \delta')$, $rj(r)$ satisfies (P1) - (P3) for $r \in [0, t]$ and q'', δ'' by Lemma 3.7 a). By Lemma 3.8

(C) $\int_0^t J(r) dr \in (\mathcal{S})_{-1}$ and $\int_0^t rJ(r) dr \in (\mathcal{S})_{-1}$, and

(D) $\mathcal{H}\int_0^t J(r)dr = \int_0^t j(r)dr$ and $\mathcal{H}\int_0^t rJ(r)dr = \int_0^t rj(r)dr$.

Since $\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_0^t rJ(r)dr$ and $\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy = \int_0^t J(r)dr$ they both lie in $(\mathcal{S})_{-1}$ by (C). Using (D) and (B) and keeping in mind the definition of $j(r)$ we get

$$\mathcal{H}\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_0^t r\mathcal{H}\int_{\partial B(0,1)} F(t-r,x+ry;\omega)g(x,x+ry)dS(y)dr$$

$$= \int_0^t r\int_{\partial B(0,1)} f(t-r,x+ry;z)g(x,x+ry)dS(y)dr = \int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy$$

$$\mathcal{H}\int_{B(x,t)} \frac{F(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy$$ □

Lemma 3.13 *Assume that there exist q' and δ' such that $f(t,x;z)$ satisfies (E1) - (E3) for $(t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{K}_{q'}(\delta')$, that $D_{t,x}f(t,x;z)$ is continuous in (t,x) for fixed z, and the function $g(x,y)$ is continuous. Then there exist q,δ such that the following statements hold:*

a) $\frac{1}{t^2}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy$ *satisfies (P1) - (P3) for* $(t,x,z) \in K_t \times K \times \mathbb{K}_q(\delta)$.

b) $\frac{1}{t}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy$ *satisfies (P1) - (P3) for* $(t,x,z) \in K_t \times K \times \mathbb{K}_q(\delta)$.

Proof:

Note that $\int_{B(0,1)} \frac{1}{|y|}dy = \int_0^1\int_0^{2\pi}\int_0^{\pi} \frac{1}{r}r^2\sin\theta d\theta d\phi dr = \pi$, and similarly $\int_{B(0,1)} \frac{1}{|y|^2}dy = 2\pi$.

Define $B \equiv \{y : y \in B(x,t) \text{ for } (t,x) \in K_t \times K\}$. Note that the set $S \equiv K_t \times \overline{B}$ is compact. Then since g is continuous $M_g \equiv \sup_{K\times\overline{B}} |g(x,y)| < \infty$, and since f satisfies (E1) $M_f \equiv \sup_{K_t\times\overline{B}\times\mathbb{K}_{q'}(\delta')} |f(s,y;z)| < \infty$. Let $T = \sup_{K_t} |t| < \infty$.

(P1) $|\frac{1}{t^2}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy| \leq M_f M_g \frac{1}{t^2}\int_{B(x,t)} \frac{1}{|y-x|}dy = M_f M_g \int_{B(0,1)} \frac{1}{|y|}dy = M_f M_g \pi$

$|\frac{1}{t}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy| \leq M_f M_g \frac{1}{t}\int_{B(x,t)} \frac{1}{|y-x|^2}dy = M_f M_g \int_{B(0,1)} \frac{1}{|y|^2}dy = M_f M_g 2\pi$

(P2) $\frac{1}{t^2}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|}dy = \int_{B(0,1)} \frac{f(t-t|y|,x+ty;z)g(x,x+ty)}{|y|}dy$

$\frac{1}{t}\int_{B(x,t)} \frac{f(t-|y-x|,y;z)g(x,y)}{|y-x|^2}dy = \int_{B(0,1)} \frac{f(t-t|y|,x+ty;z)g(x,x+ty)}{|y|^2}dy$

Let $\tilde{f}(t,x,y) \equiv f(t-t|y|,x+ty;z)$, then $D_{t,x}\tilde{f}(t,x,y) =$ $((\partial_1 f)(t-t|y|,x+ty;z)(1+|y|) + (D_2 f)(t-t|y|,x+ty;z)\cdot y, (D_2 f)(t-t|y|,x+ty;z))$.

This derivative is continuous since Df is continuous, so

$$M_D \equiv \sup_{K_t\times\overline{B}\times\overline{B}(0,1)\times\mathbb{K}_{q'}(\delta')} |D\tilde{f}(t,x,y;z)| < \infty.$$

Now we use the mean value theorem and the Cauchy-Schwartz inequality:

$|\int_{B(0,1)} \frac{\tilde{f}(t+\Delta t,x+\Delta x,y)g(x+\Delta x,x+\Delta x+(t+\Delta t)y)-\tilde{f}(t,x,y)g(x,x+ty)}{|y|}dy|$

$\leq M_D M_g |(\Delta t, \Delta x)| \int_{B(0,1)} \frac{1}{|y|} dy = M_D M_g |(\Delta t, \Delta x)| \pi$

$| \int_{B(0,1)} \frac{\tilde{f}(t+\Delta t, x+\Delta x, y) g(x+\Delta x, x+\Delta x+(t+\Delta t)y) - \tilde{f}(t,x,y) g(x, x+ty)}{|y|^2} dy|$
$\leq M_D M_g |(\Delta t, \Delta x)| \int_{B(0,1)} \frac{1}{|y|^2} dy = M_D M_g |(\Delta t, \Delta x)| 2\pi$

This is Lipschitz continuity, which implies continuity.

(P3) By Lemma 3.12 $\frac{1}{t^2} \int_{B(x,t)} \frac{F(t-|y-x|,y)}{|y-x|} dy$ and $\frac{1}{t} \int_{B(x,t)} \frac{F(t-|y-x|,y)}{|y-x|^2} dy$ are in $(\mathcal{S})_{-1}$ for fixed t, x. Theorem 2.7 b) then yields that $\frac{1}{t^2} \int_{B(x,t)} \frac{f(t-|y-x|,y)}{|y-x|} dy$ and $\frac{1}{t} \int_{B(x,t)} \frac{f(t-|y-x|,y)}{|y-x|^2} dy$ are analytic in z for some q'' and δ''.

Choose $q = \max(q', q'')$ and $\delta = \min(\delta', \delta'')$, then (P1) - (P3) is satisfied for $z \in \mathbb{K}_q(\delta)$. □

Chapter 4

1D Wave Equation

In this chapter we will consider the stochastic wave equation in one space dimension. That is the wave equation with stochastic distributions as initial values and/or forcing term. In the first section we will study the homogeneous wave equation with stochastic distributions as initial values. Then we turn to the stochastic inhomogeneous wave equation in the following section. First we set the initial values to zero and let the forcing term be a stochastic distribution process. Then we consider the general initial value problem for the inhomogeneous wave equation. This problem can be viewed as a homogeneous problem plus an inhomogeneous problem. These problems are the two problems we just considered, and the sum of the solutions of these problems is a solution for the general problem.

In solving these problems we will use the general procedure outlined in the Chapter 2. We will be looking at strong $(\mathcal{S})_{-1}$-solutions only, and derive a formula for the solution in each case. The formulas for the solutions are the same as in the deterministic case. And in the homogeneous problem the smoothness conditions on the initial values are completely analogous to those in the deterministic case. In the inhomogeneous problem, it seems that you need a smoother forcing term than is needed in the deterministic case. In all cases uniqueness in the classical problem yields uniqueness in the stochastic problem.

Notation. If u is a complex variable, let $u^r \equiv \mathrm{Re}(u)$ and $u^i \equiv \mathrm{Im}(u)$. We will also use the space C^2_{loc} in this chapter. This space is defined in Section 1.2.

4.1 The Homogeneous Wave Equation

In this section we will study an initial value problem for the homogeneous wave equation where the initial values are $(\mathcal{S})_{-1}$-processes. To be precise, we will consider the following problem:

$$\boxed{\begin{array}{rcl} U_{tt}(t,x) - U_{xx}(t,x) &=& 0 \\ U(0,x) &=& G(x) \\ U_t(0,x) &=& H(x) \\ \\ \multicolumn{3}{c}{G(x) \in C^2(\mathbb{R}, (\mathcal{S})_{-1})} \\ \multicolumn{3}{c}{H(x) \in C^1(\mathbb{R}, (\mathcal{S})_{-1})} \end{array}} \tag{4.1}$$

We want to find a formula for a solution in this problem. First let us take the Hermite transform of (4.1).

$$\begin{aligned} u_{tt}(t,x) - u_{xx}(t,x) &= 0 \\ u(0,x) &= g(x) \\ u_t(0,x) &= h(x) \end{aligned} \tag{4.2}$$

By Theorem 3.3 there exist q and δ such that g and h are C^2 and C^1 respectively in (t,x) for $z \in \mathbb{K}_q(\delta)$. Fix z and let us look at the real part of (4.2),

$$\begin{aligned} u^r_{tt}(t,x) - u^r_{xx}(t,x) &= 0 \\ u^r(0,x) &= g^r(x) \\ u^r_t(0,x) &= h^r(x) \end{aligned}$$

where $g^r \in C^2$, $h^r \in C^1$. This is the classical 1D wave equation. It is solved in [Eva94] and [Joh82], and the solution is

$$u^r(t,x) = \frac{1}{2}(g^r(x+t) + g^r(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h^r(s)ds.$$

The imaginary case is similar, $u^i(t,x) = \frac{1}{2}(g^i(x+t) + g^i(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h^i(s)ds$. The solution of (4.2) is then:

$$\begin{aligned} u(t,x) &= u^r(t,x) + i\,u^i(t,x) \\ &= \frac{1}{2}\{(g^r + i\,g^i)(x+t) + (g^r + i\,g^i)(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} (h^r + i\,h^i)(s)ds \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds \end{aligned} \tag{4.3}$$

Now we need to show that this solution $u(t,x)$ satisfies the conditions in Theorem 2.12 so that it can be inversely transformed to give a solution of the initial value problem (4.1). We will also show that this solution is a C^2 $(\mathcal{S})_{-1}$-process.

Theorem 4.1 *The initial value problem (4.1) has a unique solution in $C^2_{loc}(\mathbb{R}^+ \times \mathbb{R}, (\mathcal{S})_{-1})$ which takes the form*

$$U(t,x) = \frac{1}{2}(G(x+t) + G(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} H(s)ds.$$

Proof:

Let $K_1 \subset \mathbb{R}^+$ and $K_2 \subset \mathbb{R}$ be compact sets, and let $(t,x) \in K_1 \times K_2$.

Uniqueness of the solution

The classical boundary value problem has a unique solution, see [Eva94] or [Joh82]. This implies that the solution of (4.2) is unique. Fix (t,x), then by Theorem 2.7 b) there is an unique element $U(t,x)$ in $(\mathcal{S})_{-1}$ such that $\mathcal{H}U(t,x) = u(t,x)$. This function $U(t,x)$ is therefore uniquely defined for t, x, and $\omega \in \mathcal{S}'$. So if (4.1) has a solution, it is unique.

Existence and Formula for the solution

By Theorem 2.12 we need to show that $u(t,x)$ and its derivatives up to second order satisfies (P1) - (P3) for $z \in K_q(\delta)$ where $q, \delta < \infty$. From Appendix B.1 we know that there exist q_d, δ_d such that for $z \in \mathbb{K}_{q_d}(\delta_d)$ these derivatives takes the following form:

$$\begin{aligned}
u_t(t,x) &= \frac{1}{2}(g'(x+t) - g'(x-t)) + \frac{1}{2}(h(x+t) + h(x-t)) \\
u_x(t,x) &= \frac{1}{2}(g'(x+t) + g'(x-t)) + \frac{1}{2}(h(x+t) - h(x-t)) \\
u_{tt}(t,x) &= u_{xx}(t,x) = \frac{1}{2}(g''(x+t) + g''(x-t)) + \frac{1}{2}(h'(x+t) - h'(x-t)) \\
u_{xt}(t,x) &= u_{tx}(t,x) = \frac{1}{2}(g''(x+t) - g''(x-t)) + \frac{1}{2}(h'(x+t) + h'(x-t))
\end{aligned}$$

Since $G \in C^2(\mathbb{R}, (\mathcal{S})_{-1})$ and $H \in C^1(\mathbb{R}, (\mathcal{S})_{-1})$, by definition $G'', G', G, H', H \in C(\mathbb{R}, (\mathcal{S})_{-1})$. Then by Corollary 3.4, we know that there exist some q_1, δ_1 such that for all compact sets $K \subset \mathbb{R}$ $g''(y)$, $g'(y)$, $g(y)$, $h'(y)$, and $h(y)$ satisfies (P1) - (P3) for $z \in \mathbb{K}_{q_1}(\delta_1)$ and $y \in K$.

Note that a common q' and δ' can always be found by letting q be the biggest of the q's and δ' the smallest of the δ's. This follows from (2.1).

Let $f(t,x) = x+t$, then $f(K_1, K_2) = K_1 + K_2$. The set $K \equiv \overline{K_1 + K_2}$ is compact. Then $g(y)$ satisfies (P1) - (P3) for $y \in K_1 + K_2 \subset K$. By Lemma 3.6 a) $g(x+t)$ satisfies (P1) - (P3) for $(t,x) \in K_1 \times K_2 = f^{-1}(K_1 + K_2)$, and $z \in \mathbb{K}_{q_1}(\delta_1)$. Similar for $g''(x+t)$, $g''(x-t)$, $g'(x+t)$, $g'(x-t)$, $g(x-t)$, $h'(x+t)$, $h'(x-t)$, $h(x+t)$, and $h(x-t)$.

Let $l(x) = x - t$ and $u(x) = x + t$. l, u are continuous. Since $H \in C^1(\mathbb{R}, (\mathcal{S})_{-1})$, by Corollary 3.4 there exist q', δ' such that h and Dh satisfies (P1) - (P3) on every compact subset of $\mathbb{R}$. The closures of the sets $K_1 \pm K_2$ are compact, and by (P2) Dh is continuous. This means that the conditions of Lemma 3.9 hold for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_{q'}(\delta')$. Thus there exist q_2, δ_2 such that $\frac{1}{2}\int_{x-t}^{x+t} h(s)ds$ satisfies (P1) - (P3) for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_{q_2}(\delta_2)$.

Let $q = \max(q_d, q_1, q_2)$ and $\delta = \min(\delta_d, \delta_1, \delta_2)$. Then every term in the expressions for $u(t,x)$ and its derivatives satisfy (P1) - (P3) for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_q(\delta)$. Since a linear combination of functions satisfying (P1) - (P3) still satisfies these properties on the same domain (Lemma 3.5 a)), it follows that $u(t,x)$ and all its derivatives up to order 2 satisfies (P1) - (P3) for the above mentioned domain. Thus by Theorem 2.12 there exists a $(\mathcal{S})_{-1}$-process:

$$\begin{aligned} U(t,x) &= \mathcal{H}^{-1}u(t,x) \\ &= \frac{1}{2}\mathcal{H}^{-1}(g(x+t)+g(x-t)) + \frac{1}{2}\mathcal{H}^{-1}\int_{x-t}^{x+t} h(s)ds \\ &= \frac{1}{2}(G(x+t)+G(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} H(s)ds \end{aligned}$$

The fact that $\mathcal{H}^{-1}\int_{x-t}^{x+t} h(s)ds = \int_{x-t}^{x+t} H(s)ds$ follows from Lemma 3.8 since $h(s)$ satisfies (P1) - (P3) for $(s,z) \in [K_2 + K_1] \cup [K_2 - K_1] \times \mathbb{K}_q(\delta)$.

Smoothness of the solution

We are now going to show that $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. Let us start by showing that $D^2U(t,x)$ is continuous. Fix (t,x) and $\epsilon > 0$, then by Lemma 3.1 there exist q', δ' and $\gamma > 0$ such that for $z \in \mathbb{K}_{q'}(\delta')$ and $|\Delta t| + |\Delta x| < \gamma$ we have:

$$\begin{aligned} \sup_z |g''(x+t+\Delta x+\Delta t) - g''(x+t)| &< \frac{\epsilon}{2} \\ \sup_z |g''(x-t+\Delta x-\Delta t) - g''(x-t)| &< \frac{\epsilon}{2} \\ \sup_z |h'(x+t+\Delta x+\Delta t) - h'(x+t)| &< \frac{\epsilon}{2} \\ \sup_z |h'(x-t+\Delta x-\Delta t) - h'(x-t)| &< \frac{\epsilon}{2} \end{aligned}$$

$\sup_z |u_{tt}(x+\Delta t, x+\Delta t) - u_{tt}(t,x)| = \sup_z |\frac{1}{2}[g''(x+t+\Delta x+\Delta t) + g''(x-t+\Delta x-\Delta t)]$

$+\frac{1}{2}[h'(x+t+\Delta x+\Delta t) - h'(x-t+\Delta x-\Delta t)] - \{\frac{1}{2}[g''(x+t)+g''(x-t)] + \frac{1}{2}[h'(x+t)-h'(x-t)]\}|$

$\leq \frac{1}{2}\sup_z |g''(x+t+\Delta x+\Delta t) - g''(x+t)| + \frac{1}{2}\sup_z |g''(x-t+\Delta x-\Delta t) - g''(x-t)|$

$+\frac{1}{2}\sup_z |h'(x+t+\Delta t+\Delta x) - h'(x+t)| + \frac{1}{2}\sup_z |h'(x-t+\Delta x-\Delta t) - h'(x-t)|$

$< \frac{1}{2}(\frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2}) \ = \ \epsilon$

By Lemma 3.1 this implies that U_{tt} $(= U_{xx})$ is continuous. A similar argument will show that U_{tx} $(= U_{xt})$ is continuous. By Theorem 1.1 $U(t,x)$ and $DU(t,x)$ are also continuous, so $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. □

4.2 The Inhomogeneous Wave Equation

4.2.1 A particular Initial Value Problem

In this section we are going to study the following initial value problem for the inhomogeneous wave equation.

$$\boxed{\begin{array}{rcl} U_{tt}(t,x) - U_{xx}(t,x) &=& F(x,t) \\ U(0,x) &=& 0 \\ U_t(0,x) &=& 0 \\ \\ \multicolumn{3}{c}{F(x,t) \in C^2(\mathbb{R}^2, (\mathcal{S})_{-1})} \end{array}} \tag{4.4}$$

As before we take the Hermite transform of (4.4), and try to solve the transformed equation.

$$\begin{array}{rcl} u_{tt}(t,x) - u_{xx}(t,x) &=& f(x,t) \\ u(0,x) &=& 0 \\ u_t(0,x) &=& 0 \end{array} \tag{4.5}$$

By Theorem 3.3 there exist q and δ such that f is C^1 in (t,x) for $z \in \mathbb{K}_q(\delta)$. Let us look at the real part of (4.2),

$$\begin{array}{rcl} u^r_{tt}(t,x) - u^r_{xx}(t,x) &=& f^r(x,t) \\ u^r(0,x) &=& 0 \\ u^r_t(0,x) &=& 0 \end{array}$$

where f^r is C^1. This initial value problem is solved in [Eva94] and [Joh82]. The solution is

$$u^r(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f^r(y,s)dyds.$$

Replacing ^{r} with ^{i} gives a solution for the imaginary part of (4.5). Combining these two results, we get a solution of (4.5).

$$\begin{aligned} u(t,x) &= u^r(t,x) + i\, u^i(t,x) \\ &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} [f^r(y,s) + i\, f^i(y,s)]dyds \\ &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds \end{aligned} \tag{4.6}$$

Theorem 4.2 *The initial value problem (4.4) has a unique solution in* $C^2_{loc}(\mathbb{R}^+ \times \mathbb{R}, (\mathcal{S})_{-1})$ *which takes the form*

$$U(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(y,s)dyds.$$

Proof:

Let $K_1 \subset \mathbb{R}^+$ and $K_2 \subset \mathbb{R}$ be compact sets, and let $(t,x) \in K_1 \times K_2$.

Uniqueness of the solution

The classical boundary value problem has a unique solution, see [Eva94] or [Joh82]. This implies that the solution of (4.5) is unique. Fix (t,x), then by Theorem 2.7 b) there is an unique element $U(t,x)$ in $(\mathcal{S})_{-1}$ such that $\mathcal{H}U(t,x) = u(t,x)$. This function $U(t,x)$ is therefore uniquely defined for t, x, and $\omega \in \mathcal{S}'$. So if (4.4) has a solution, it is unique.

Existence and Formula for the solution

By Theorem 2.12 we need to show that $u(t,x)$ and it is derivatives up to second order satisfies (P1) - (P3) for some q and δ. By Appendix B.2 there exist q_d, δ_d such that for $z \in \mathbb{K}_{q_d}(\delta_d)$ these derivatives have the following form:

$$\begin{aligned}
u_t(t,x) &= \frac{1}{2}\int_0^t [f(x+(t-s),s) + f(x-(t-s),s)]ds \\
u_x(t,x) &= \frac{1}{2}\int_0^t [f(x+(t-s),s) - f(x-(t-s),s)]ds \\
u_{tt}(t,x) &= f(t,x) + \frac{1}{2}\int_0^t [f_t(x+(t-s),s) + f_t(x-(t-s),s)]ds \\
u_{xx}(t,x) &= \frac{1}{2}\int_0^t [f_x(x+(t-s),s) - f_x(x-(t-s),s)]ds \\
u_{xt}(t,x) &= u_{tx}(t,x) = \frac{1}{2}\int_0^t [f_x(x+(t-s),s) + f_x(x-(t-s),s)]ds
\end{aligned}$$

Since $F(t,x)$ is a C^2 $(\mathcal{S})_{-1}$-process, Theorem 3.3 yields that $f(t,x)$ and its derivatives up to second order satisfies (E1) - (E3) for $(t,x,z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{K}_{q_1}(\delta_1)$. This holds for some q_1 and δ_1.

Since $g_1(t,x,s) = x + (t-s)$ and $g_2(t,x,s) = x - (t-s)$ are continuous, and the closure of $g_1(K_1,K_2,K_1)$ and $g_2(K_1,K_2,K_1)$ are compact, the conditions in Lemma 3.9 are satisfied for $f(t,x)$. This means that there exist q', δ' such that $\int_{x-(t-s)}^{x+(t-s)} f(y,s)dy$ satisfies (P1) - (P3) for $(t,x,s,z) \in K_1 \times K_2 \times K_1 \times \mathbb{K}_{q'}(\delta')$.

Let $g(t,x,s) \equiv \int_{x-(t-s)}^{x+(t+s)} f(y,s)dy$. By Lemma B.2 in Appendix B.2 $Dg(t,x,s)$ is continuous for $z \in \mathbb{K}_{q_a}(\delta_a)$, and as showed above $g(t,x,s)$ satisfies (P1) - (P3) for q' and δ'. Let $q_2 = \max(q_a, q')$ and $\delta_2 = \min(\delta_a, \delta')$. By Lemma 3.9 $\int_0^t g(t,x,s)ds$ then satisfies (P1) - (P3) with q_2 and δ_2.

Since $g_1(t,x,s) = x + (t-s)$ and $g_2(t,x,s) = x - (t-s)$ are continuous, and the closure of $g_1(K_1,K_2,K_1)$ and $g_2(K_1,K_2,K_1)$ are compact, the conditions in Lemma 3.6 a) are satisfied for $f(t,x)$ and its derivatives up to second order. This means that $f(x+(t-s),s)$ and $f(x-(t-s),s)$ and their derivatives up to second order satisfies (P1) - (P3) for $(t,x,s,z) \in$

$K_1 \times K_2 \times K_1 \times \mathbb{K}_{q_1}(\delta_1)$.

$f(x+(t-s),s)$ and its first order derivatives satisfies the conditions of Lemma 3.9 on the above mentioned domain. By this lemma there exist q', δ' such that $\int_0^t f(x+(t-s),s)ds$ and $\int_0^t \partial_1 f(x+(t-s),s)ds$ satisfy (P1) - (P3) for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_{q_3}(\delta_3)$. Similar for $\int_0^t f(x+(t-s),s)ds$ and $\int_0^t \partial_1 f(x+(t-s),s)ds$ for some q'', δ''. Let $q_3 = \max(q', q'')$ and $\delta_3 = \min(\delta', \delta'')$.

Choose $q = \max(q_1, q_2, q_3, q_d)$ and $\delta = \min(\delta_1, \delta_2, \delta_3, \delta_d)$. Then since a linear combination of terms satisfying (P1) - (P3) also satisfies (P1) - (P3) on the original domain (Lemma 3.5 a)), $u(t,x)$ and all its derivative up to second order satisfies (P1) - (P3) for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_q(\delta)$.

Thus by Theorem 2.12 there exists a $U(t,x) = \mathcal{H}^{-1}u(t,x)$ satisfying the initial value problem (4.4), where $u(t,x)$ is given by equation (4.6). $U(t,x)$ takes the following form:

$$
\begin{aligned}
U(t,x) = \mathcal{H}^{-1}u(t,x) &= \frac{1}{2}\mathcal{H}^{-1}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds \\
&= \frac{1}{2}\int_0^t \mathcal{H}^{-1}\int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds \\
&= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(y,s)dyds.
\end{aligned}
$$

The third and fourth equality follow form Lemma 3.8 since f and the previously defined g satisfy (P1) - (P3) for q, δ and the integral limits are continuous functions.

Smoothness of the solution

We are now going to show that $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. Let us start by showing that $D^2U(t,x)$ is continuous. Let $h^{\pm}(t,x,s) = x \pm (t-s)$, $\Delta h^{\pm}(t,x,s) = \Delta x \pm (\Delta t - \Delta s)$, $y = (t,x,s)$, and $\Delta y = (\Delta t, \Delta x, \Delta s)$.

$F(t,x) \in C^2(\mathbb{R}^2, (\mathcal{S})_{-1})$. By Corollary 3.4 there exist q' and δ' such that for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_{q'}(\delta')$ $f(t,x)$, $Df(t,x)$, and $D^2f(t,x)$ satisfies (P1) - (P3). By (P1) there exist $M_1, M_2 < \infty$ such that $\sup_{r,s,z}|f_t(r,s)| = M_1$ and $\sup_{r,s,z}|Df_t(r,s)| = M_2$. We also know that $M_3 \equiv \sup_{t \in K_1}|t| < \infty$ since K_1 is compact. We will now use the mean value theorem, the Cauchy-Schwartz inequality, and the fact that $|\Delta h^{\pm}(t,x,s)| \leq \|\Delta y\|_1 \leq k|\Delta y|$, where $0 < k < \infty$.

$|\int_0^{t+\Delta t} f_t(h^+(t,x,s) + \Delta h^+(t,x,s),s)ds - \int_0^t f_t(h^+(t,x,s),s)ds|$

$\leq |\int_t^{t+\Delta t} f_t(h^+(t,x,s) + \Delta h^+(t,x,s),s)ds| + |\int_0^t [f_t(h^+(t,x,s) + \Delta h^+(t,x,s),s) - f(t,x,s)]ds|$

$\leq |\Delta t| \sup_{r,s,z}|f_t(r,s)| + |t||\Delta h^+| \sup_{r,s,z}|Df_t(r,s)| \;\leq\; |\Delta t|M_1 + |\Delta y|kM_2M_3$

Similarly we have $|\int_0^{t+\Delta t} f_t(h^-(t,x,s) + \Delta h^-(t,x,s),s)ds - \int_0^t f_t(h^-(t,x,s),s)ds| \leq |\Delta t|M_1 +$

$|\Delta y|kM_2M_3$. All in all we have

$$\sup_z |u_{xx}(t+\Delta t, x+\Delta x) - u_{xx}(t,x)| \;\leq\; \frac{1}{2}(2|\Delta t|M_1 + 2|\Delta y|kM_2M_3).$$

Fix $\epsilon > 0$ and choose $\delta = \min(\frac{\epsilon}{2M_1}, \frac{\epsilon}{2kM_2M_3})$ then

$$|\Delta y| < \delta \quad \Rightarrow \quad \sup_z |u_{xx}(t+\Delta t, x+\Delta x) - u_{xx}(t,x)| < \epsilon.$$

By Lemma 3.1 this implies that U_{xx} is continuous. A similar argument will show that U_{tt} and U_{tx} $(= U_{xt})$ are continuous. By Lemma 1.1 $U(t,x)$ and $DU(t,x)$ are also continuous, so $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. □

Remark: We have assumed one extra degree of differentiability, compared to the classical case. See the comment on this in Chapter 6.

4.2.2 The general Initial Value Problem

Now we turn to the general initial value problem for the inhomogeneous wave equation. In this section both the initial values and the forcing term will be stochastic distributions.

$$\boxed{\begin{array}{rcl} U_{tt}(t,x) - U_{xx}(t,x) &=& F(x,t) \\ U(0,x) &=& G(x) \\ U_t(0,x) &=& H(x) \\ \\ F(x,t) &\in& C^2(\mathbb{R}^2, (\mathcal{S})_{-1}) \\ G(x) &\in& C^2(\mathbb{R}, (\mathcal{S})_{-1}) \\ H(x) &\in& C^1(\mathbb{R}, (\mathcal{S})_{-1}) \end{array}} \tag{4.7}$$

Theorem 4.3 *The initial value problem (4.7) has a unique solution in $C^2_{loc}(\mathbb{R}^+ \times \mathbb{R}, (\mathcal{S})_{-1})$ which takes the form*

$$U(t,x) = \frac{1}{2}(G(x+t) + G(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} H(s)ds + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(y,s)dyds.$$

Proof:

Existence, Formula and Smoothness

Let $U^h(t,x)$ be be a solution of (4.1) and $U^p(t,x)$ be a solution of (4.4). We see that $U(t,x) = U^h(t,x) + U^p(t,x)$, and that this function therefore is C^2. We need to show that this function satisfies the initial value problem (4.7).

$$\begin{array}{rcl} U_{tt}(t,x) - U_{xx}(t,x) &=& [U^h_{tt}(t,x) + U^p_{tt}(t,x)] - [U^h_{xx}(t,x) + U^p_{xx}(t,x)] \\ &=& [U^h_{tt}(t,x) - U^h_{xx}(t,x)] + [U^p_{tt}(t,x) - U^p_{xx}(t,x)] \\ &=& 0 + F(x,t) \;=\; F(x,t) \end{array}$$

$$
\begin{aligned}
U(0,x) &= U^h(0,x) + U^p(0,x) = G(x) + 0 = G(x) \\
U_t(0,x) &= U_t^h(0,x) + U_t^p(0,x) = H(x) + 0 = H(x)
\end{aligned}
$$

Uniqueness

Let $U(t,x)$ and $V(t,x)$ be two solutions of (4.7). Then the difference $U(t,x) - V(t,x)$ satisfies the homogeneous initial value problem (4.1) with $G(x) = 0$ and $H(x) = 0$. By Theorem 4.1 this problem has a unique solution taking form $U(t,x) - V(t,x) = 0$. So $U(t,x) = V(t,x)$ and this solution is unique. □

Chapter 5

3D Wave Equation

In this chapter we will consider the stochastic wave equation in three space dimension. This chapter will be very much like Chapter 4.

Notation. If u is a complex variable, let $u^r \equiv \mathrm{Re}(u)$ and $u^i \equiv \mathrm{Im}(u)$. We will also use the space C^2_{loc} in this chapter. This space is defined in Section 1.2.

5.1 The Homogeneous Wave Equation

Let us consider an initial value problem for the homogeneous wave equation where the initial values are $(\mathcal{S})_{-1}$-processes. To be precise, we will consider the following problem:

$$\boxed{\begin{array}{c} \begin{aligned} U_{tt}(t,x) - \nabla^2 U(t,x) &= 0 \\ U(0,x) &= G(x) \\ U_t(0,x) &= H(x) \end{aligned} \\ \\ G(x) \in C^4(\mathbb{R}^3, (\mathcal{S})_{-1}) \\ H(x) \in C^3(\mathbb{R}^3, (\mathcal{S})_{-1}) \end{array}} \tag{5.1}$$

We want to find a formula for a solution in this problem. First let us take the Hermite transform of (5.1).

$$\begin{aligned} u_{tt}(t,x) - \nabla^2 u(t,x) &= 0 \\ u(0,x) &= g(x) \\ u_t(0,x) &= h(x) \end{aligned} \tag{5.2}$$

By Theorem 3.3 there exist q and δ such that g and h are C^3 and C^2 respectively in (t,x) for

$z \in \mathbb{K}_q(\delta)$. Fix z and let us look at the real part of (5.2),

$$\begin{aligned} u_{tt}^r(t,x) - \nabla^2 u^r(t,x) &= 0 \\ u^r(0,x) &= g^r(x) \\ u_t^r(0,x) &= h^r(x) \end{aligned}$$

where $g^r \in C^3$, $h^r \in C^2$. This is the classical 3D wave equation. It is solved in [Eva94] and [Joh82], and the solution is:

$$u^r(t,x) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [th^r(y) + g^r(y) + \nabla g^r(y) \cdot (y-x)] dS(y)$$

Replacing ^{r} with ^{i} gives a solution for the imaginary part of (5.2). The solution is

$$\begin{aligned} u(t,x) &= u^r(t,x) + i\, u^i(t,x) \\ &= \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [t(h^r + i\, h^i)(y) + (g^r + i\, g^i)(y) + (\nabla g^r + i\, \nabla g^i)(y) \cdot (y-x)] dS(y) \\ &= \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [th(y) + g(y) + \nabla g(y) \cdot (y-x)] dS(y). \end{aligned} \tag{5.3}$$

Now we need to show that the solution $u(t,x)$ satisfies the conditions in Theorem 2.12 so that it can be inversely transformed to give a solution of the initial value problem (5.1). We will also show that this solution is a C^2 $(\mathcal{S})_{-1}$-process.

Theorem 5.1 *The initial value problem (5.1) has a unique solution in* $C^2(\mathbb{R}^+ \times \mathbb{R}^3, (\mathcal{S})_{-1})$ *which takes the form*

$$U(t,x) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [tH(y) + G(y) + \nabla G(y) \cdot (y-x)] dS(y).$$

Proof:

Let $K_1 \subset \mathbb{R}^+$ and $K_2 \subset \mathbb{R}^3$ be compact sets, and let $(t,x) \in K_1 \times K_2$.

Uniqueness of the solution

The classical boundary value problem has a unique solution, see [Eva94] or [Joh82]. This implies that the solution of (5.2) is unique. Fix (t,x), then by Theorem 2.7 b) there is an unique element $U(t,x)$ in $(\mathcal{S})_{-1}$ such that $\mathcal{H}U(t,x) = u(t,x)$. This function $U(t,x)$ is therefore uniquely defined for t, x, and $\omega \in \mathcal{S}'$. So if (5.1) has a solution, it is unique.

Existence and Formula for the solution

By Theorem 2.12 we need to show that $u(t,x)$ and its derivatives up to second order satisfies (P1) - (P3) for $z \in K_q(\delta)$ where $q, \delta < \infty$. From Appendix B.3 we know that there exist q_d, δ_d

such that for $z \in \mathbb{K}_{q_d}(\delta_d)$ $u(t,x)$ and its derivatives takes the following form:

$$\begin{aligned}
u(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th(y)+g(y)+\nabla g(y)\cdot(y-x)]dS(y)\\
u_{x_i}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th_{x_i}(y)+g_{x_i}(y)+\nabla g_{x_i}(y)\cdot(y-x)]dS(y)\\
u_t(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[h(y)+\nabla h(y)\cdot(y-x)+t\nabla^2 g(y)]dS(y)\\
u_{x_ix_j}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th_{x_ix_j}(y)+g_{x_ix_j}(y)+\nabla g_{x_ix_j}(y)\cdot(y-x)]dS(y)\\
u_{tx_i}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[h_{x_i}(y)+\nabla h_{x_i}(y)\cdot(y-x)+t\nabla^2 g_{x_i}(y)]dS(y)\\
u_{tt}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[t\nabla^2 h(y)+\nabla^2 g(y)+\nabla(\nabla^2 g)(y)\cdot(y-x)]dS(y)
\end{aligned}$$

The idea is now to show that all the integrands above satisfies (E1) - (E3) on $K_1\times\mathbb{R}$ for some q_1,δ_1, and that the gradients of the integrands are continuous. Then by Lemma 3.11 there exist q_2,δ_2 such that the integrals above satisfy (P1) - (P3) for $(t,x)\in K_1\times K_2\times\mathbb{K}_{q_2}(\delta_2)$.

1) Let us show that the integrands satisfies (E1) - (E3). G and H are C^4 and C^3 $(\mathcal{S})_{-1}$-processes on $\mathbb{R}^3$ respectively. Then by Theorem 3.3 there exist q_1,δ_1 such that g, h and all their derivatives up to fourth and third order respectively, satisfies (E1) - (E3) on $\mathbb{R}^3$. Note that all the integrands are linear combinations of products of such derivatives and continuous functions. By Lemmas 3.7 b) and 3.5 b) the integrands satisfies (E1) - (E3) on $K_1\times\mathbb{R}$ for q_1,δ_1.
2) If these integrands were differentiated once, they would still be linear combinations of products of derivatives of g and h and continuous functions. Furthermore the order of the highest derivative would be 3 and 4 for h and g respectively. So by Lemmas 3.7 b) and 3.5 b) the gradients of the integrands satisfies (E1) - (E3) on $K_1\times K_2$ for q_1,δ_1.

Choose $q=\max(q_d,q_1,q_2)$ and $\delta=\min(\delta_q,\delta_1,\delta_2)$. For this choice of q,δ the conditions in Theorem 2.12 are satisfied for $(t,x,z)\in K_1\times K_2\times\mathbb{K}_q(\delta)$. This means that there exists a $(\mathcal{S})_{-1}$-process:

$$\begin{aligned}
U(t,x) &= \mathcal{H}^{-1}u(t,x)\\
&= \frac{1}{4\pi t^2}\mathcal{H}^{-1}\int_{\partial B(x,t)}[th(y)+g(y)+\nabla g(y)\cdot(y-x)]dS(y)\\
&= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[tH(y)+G(y)+\nabla G(y)\cdot(y-x)]dS(y),
\end{aligned}$$

where the last equality follows from Lemma 3.10.

Smoothness of the solution

We are now going to show that $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. Let us show that $U_{tt}(t,x)$ is continuous.

Claim 1

Assume that $f(x)$ is such that $Df(x)$ satisfies (E1) - (E3) for $(x,z) \in \mathbb{R}^3 \times \mathbb{K}_q(\delta)$. Let $g(t,x,y) = x + ty$ and $(t,x,y) \subset K_1 \times K_2 \times K_3$, where K_1, K_2 and K_3 are compact sets. Define $\tilde{f}(t,x,y) = f(g(t,x,y))$, then

$$M \equiv \sup_{(t,x,y)\in K_1\times K_2\times K_3,z} |D_{t,x}\tilde{f}(t,x,y)| < \infty.$$

Proof:

$$D_{t,x}\tilde{f}(t,x,y) = (Df)(x+ty) \cdot D_{t,x}g(t,x,y)$$

Note that $g(t,x,y)$ is C^∞ and takes bounded sets to bounded sets. Since $Df(x)$ satisfies (E1) - (E3) on $\mathbb{R}^3 \times \mathbb{K}_q(\delta)$, Lemma 3.6 b) yields that $(Df)(x+ty)$ satisfies (E1) - (E3) on $g^{-1}(\mathbb{R}^3) \times \mathbb{K}_q(\delta)$. Furthermore since $D_{t,x}g(t,x,y)$ is continuous, $(Df)(x+ty) \cdot D_{t,x}g(t,x,y)$ satisfies (E1) - (E3) on $g^{-1}(\mathbb{R}^3) \times \mathbb{K}_q(\delta)$ by Lemmas 3.7 b) and 3.5 b). Since $K_1 \times K_2 \times K_3 \subset g^{-1}(\mathbb{R}^2)$ is compact, we get by (E1)

$$\begin{aligned} M &\equiv \sup_{(t,x,y)\in K_1\times K_2\times K_3,z} |D_{t,x}\tilde{f}(t,x,y)| \\ &= \sup_{(t,x,y)\in K_1\times K_2\times K_3,z} |(Df)(x+ty)\cdot D_{t,x}g(t,x,y)| < \infty. \end{aligned}$$

QED

Let us now change coordinates from y to $x + ty$.

$$\begin{aligned} u_{tt}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)} [t\nabla^2 h(y) + \nabla^2 g(y) + \nabla(\nabla^2 g)(y)\cdot(y-x)]dS(y) \\ &= \frac{1}{4\pi}\int_{\partial B(0,1)} [t\nabla^2 h(x+ty) + \nabla^2 g(x+ty) + \nabla(\nabla^2 g)(x+ty)\cdot ty]dS(y) \end{aligned}$$

Since H, G are C^3 and C^4 $(\mathcal{S})_{-1}$-processes on $\mathbb{R}^3$ respectively, by Theorem 3.3 $D^3h(x)$, $D^3g(x)$ and $D^4g(x)$ satisfies (E1) - (E3) on $\mathbb{R}^3 \times \mathbb{K}_q(\delta)$, for some q, δ. Let $B \equiv \{y : y \in B(x,t) \text{ for } (t,x) \in K_1 \times K_2\}$, this set is bounded since K_1 and K_2 are compact. The sets K_1, K_2, and $\overline{B}$ are compact, so by Claim 1 we have

$$\begin{aligned} M_1 &\equiv \sup_{t,x,y,z} |\widetilde{D^3h}(t,x,y)| < \infty, \\ M_2 &\equiv \sup_{t,x,y,z} |\widetilde{D^3g}(t,x,y)| < \infty, \text{ and} \\ M_3 &\equiv \sup_{t,x,y,z} |\widetilde{D^4g}(t,x,y)| < \infty. \end{aligned}$$

The $\tilde{}$ notation is used as in Claim 1, that is $\tilde{f}(x) = f(x+ty)$. Let us also define $T \equiv \sup_{K_1} |t|$ and $Y = \sup_{\overline{B}} |y|$. By the mean value theorem and the Cauchy-Schwartz inequality we get:

$$|\widetilde{\nabla^2 h}(t+\Delta t, x+\Delta x, y) - \widetilde{\nabla^2 h}(t,x,y)| = |(\Delta t, \Delta x) \cdot (D_{t,x}\widetilde{\nabla^2 h})(\xi_t, \xi_x, y)| \leq |(\Delta t, \Delta x)| M_1,$$

where $t < \xi_t < t + \Delta t$ and $x < \xi_x < x + \Delta x$. Similarly

$$|\widetilde{\nabla^2 g}(t+\Delta t, x+\Delta x, y) - \widetilde{\nabla^2 g}(t,x,y)| \leq |(\Delta t, \Delta x)| M_2, \text{ and}$$

$$|\widetilde{\nabla(\nabla^2 g)}(t+\Delta t, x+\Delta x, y)\cdot y - \widetilde{\nabla(\nabla^2 g)}(t,x,y)\cdot y| \leq |(\Delta t, \Delta x)| M_3 Y.$$

$$\begin{aligned}
&|u_{tt}(t+\Delta t, x+\Delta x) - u_{tt}(t,x)| \\
&\leq \tfrac{1}{4\pi}\int_{\partial B(0,1)} |(t+\Delta t)\widetilde{\nabla^2 h}(t+\Delta t, x+\Delta x, y) + \widetilde{\nabla^2 g}(t+\Delta t, x+\Delta x, y) \\
&\quad + (t+\Delta t)\widetilde{\nabla(\nabla^2 g)}(t+\Delta t, x+\Delta x, y)\cdot y - t\widetilde{\nabla^2 h}(t,x,y) - \widetilde{\nabla^2 g}(t,x,y) - t\widetilde{\nabla(\nabla^2 g)}(t,x,y)\cdot y|\, dS(y) \\
&\leq \tfrac{1}{4\pi}4\pi \sup_y |\Delta t[\widetilde{\nabla^2 h}(t+\Delta t, x+\Delta x, y) + \widetilde{\nabla(\nabla^2 g)}(t+\Delta t, x+\Delta x, y)\cdot y] \\
&\quad + t[\widetilde{\nabla^2 h}(t+\Delta t, x+\Delta x, y) - \widetilde{\nabla^2 h}(t,x,y)] + [\widetilde{\nabla^2 g}(t+\Delta t, x+\Delta x, y) - \widetilde{\nabla^2 g}(t,x,y)] \\
&\quad + t[\widetilde{\nabla(\nabla^2 g)}(t+\Delta t, x+\Delta x, y)\cdot y - \widetilde{\nabla(\nabla^2 g)}(t,x,y)\cdot y]| \\
&\leq |\Delta t|[M_1 + M_3 Y] + |(\Delta t, \Delta x)|[TM_1 + M_2 + TM_3 Y]
\end{aligned}$$

Note that this holds for for every $z \in \mathbb{K}_q(\delta)$. For any $\epsilon > 0$ choose

$$\delta = \min\left(\tfrac{\epsilon}{2M_1 + 2M_3 Y}, \tfrac{\epsilon}{2TM_1 + 2M_2 + 2TM_3 Y}\right).$$

This choice of δ leads to:

$$|(\Delta t, \Delta x)| < \delta \quad \Rightarrow \quad \sup_z |u_{tt}(t+\Delta t, x+\Delta x) - u_{tt}(t,x)| < \epsilon$$

By Lemma 3.1 $U_{tt}(t,x)$ is continuous. In a similar fashion we could show that $U_{x_i x_j}$ and U_{tx_i} are continuous. Thus $D^2U(t,x)$ is continuous. By Lemma 1.1 $U(t,x)$ and $DU(t,x)$ is also continuous, so $U(t,x)$ is C^2. □

Remark: We have assumed one extra degree of differentiability, compared to the classical case. See the comment on this in Chapter 6.

5.2 The Inhomogeneous Wave Equation

5.2.1 A particular Initial Value Problem

In this section we are going to study the following initial value problem for the inhomogeneous wave equation.

$$\boxed{\begin{array}{rcl} U_{tt}(t,x) - \nabla^2 U(t,x) &=& F(x,t) \\ U(0,x) &=& 0 \\ U_t(0,x) &=& 0 \\ \\ \multicolumn{3}{c}{F(x,t) \in C^3(\mathbb{R} \times \mathbb{R}^3, (\mathcal{S})_{-1})} \end{array}} \tag{5.4}$$

As before we take the Hermite transform of (5.4), and try to solve the transformed equation.

$$\begin{aligned} u_{tt}(t,x) - \nabla^2 u(t,x) &= f(x,t) \\ u(0,x) &= 0 \\ u_t(0,x) &= 0 \end{aligned} \tag{5.5}$$

By Theorem 3.3 there exist q and δ such that f is C^3 in (t,x) for $z \in \mathbb{K}_q(\delta)$. Let us look at the real part of (5.5),

$$\begin{aligned} u^r_{tt}(t,x) - \nabla^2 u^r(t,x) &= f^r(x,t) \\ u^r(0,x) &= 0 \\ u^r_t(0,x) &= 0 \end{aligned}$$

where f^r is C^3. This initial value problem is solved in [Eva94] and [Joh82]. The solution is

$$u^r(t,x) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f^r(y, t - |y-x|)}{|y-x|} dy$$

Replacing ^{r} with ^{i} gives a solution for the imaginary part of (5.5). Combining these two results, we get a solution of (5.5).

$$\begin{aligned} u(t,x) &= u^r(t,x) + i\, u^i(t,x) \\ & \frac{1}{4\pi} \int_{B(x,t)} \frac{(f^r + i\, f^i)(y, t - |y-x|)}{|y-x|} dy \\ &= \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y, t - |y-x|)}{|y-x|} dy \end{aligned} \tag{5.6}$$

Theorem 5.2 *The initial value problem (5.4) has a unique solution in* $C^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^3, (\mathcal{S})_{-1})$ *which takes the form*

$$U(t,x) = \frac{1}{4\pi} \int_{B(x,t)} \frac{F(y, t - |y-x|)}{|y-x|} dy.$$

Proof:

Let $K_1 \subset \mathbb{R}^+$ and $K_2 \subset \mathbb{R}^3$ be compact sets, and let $(t,x) \in K_1 \times K_2$.

Uniqueness of the solution

The classical boundary value problem has a unique solution, see [Eva94] or [Joh82]. This implies that the solution of (5.5) is unique. Fix (t,x), then by Theorem 2.7 b) there is an unique element $U(t,x)$ in $(\mathcal{S})_{-1}$ such that $\mathcal{H}U(t,x) = u(t,x)$. This function $U(t,x)$ is therefore uniquely defined for t, x, and $\omega \in \mathcal{S}'$. So if (5.4) has a solution, it is unique.

Existence and Formula for the solution

By Theorem 2.12 we need to show that $u(t,x)$ and its derivatives up to second order satisfies (P1) - (P3) for $z \in K_q(\delta)$ where $q, \delta < \infty$. From Appendix B.4 we know that there exist q_d, δ_d such that for $z \in \mathbb{K}_{q_d}(\delta_d)$ $u(t,x)$ and its derivatives takes the following form:

$$
\begin{aligned}
u(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}dy \\
u_{x_i}(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_i}(t-|y-x|,y)}{|y-x|}dy \\
u_t(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(t-|y-x|,y) + \nabla_x f(t-|y-x|,y)\cdot(y-x)}{|y-x|^2}dy \\
u_{x_ix_j}(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_ix_j}(t-|y-x|,y)}{|y-x|}dy \\
u_{tx_i}(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_i}(t-|y-x|,y) + \nabla_x f_{x_i}(t-|y-x|,y)\cdot(y-x)}{|y-x|^2}dy \\
u_{tt}(t,x) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{\nabla^2 f(t-|y-x|,y)}{|y-x|}dy + f(t,x)
\end{aligned}
$$

Since $F(t,x) \in C^3(\mathbb{R}\times\mathbb{R}^3, (\mathcal{S})_{-1})$, $f(t,x)$, $Df(t,x)$, $D^2f(t,x)$, and $D^3f(t,x)$ satisfy (E1) - (E3) for some q_1, δ_1 on $\mathbb{R}\times\mathbb{R}^3$. Note that the highest order derivative appearing in the integrands of $u(t,x)$ and its derivatives up to second order is of order 2. This means that all these integrands are functions $\frac{h(t-|y-x|,y)g(x,y)}{|y-x|}$, $\frac{h(t-|y-x|,y)g(x,y)}{|y-x|^2}$, or a sum of such functions, satisfying the following conditions:

1) $h(t,x)$ satisfies (E1) - (E3) on $\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{K}_{q_1}(\delta_1)$.

2) $Dh(t,x)$ is continuous in $\mathbb{R}^+\times\mathbb{R}^3$ (since Dh satisfies (E2)).

3) $g(x,y)$ is continuous (since either $g(x,y)=1$ or $g(x,y) = y_i - x_i$, $i = 1,2,3$).

Let $i_1(t,x) \equiv \frac{1}{4\pi t^2}\int_{B(x,t)} \frac{h(t-|y-x|,y)g(x,y)}{|y-x|}dy$ and $i_2(t,x) \equiv \frac{1}{4\pi t}\int_{B(x,t)} \frac{h(t-|y-x|,y)g(x,y)}{|y-x|^2}dy$. These functions satisfy (P1) - (P3) on $K_1\times K_2$ for some q_2, δ_2. This follows from Lemma 3.13 since conditions 1) - 3) hold.

Now we integrate the expressions for $u(t,x)$ and its derivatives term wise. All expressions can be written as linear combinations of $t^2 i_1(t,x)$, $t i_2(t,x)$, and $f(t,x)$. Then by Lemmas 3.7 a) and 3.5 a) $u(t,x)$ and its derivatives up to second order satisfy (P1) - (P3) for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_{q_2}(\delta_2)$.

Choose $q = \max(q_d, q_1, q_2)$ and $\delta = \min(\delta_d, \delta_1, \delta_2)$. For this choice of q, δ the conditions in Theorem 2.12 are satisfied for $(t,x,z) \in K_1 \times K_2 \times \mathbb{K}_q(\delta)$. This means that there exists a $(\mathcal{S})_{-1}$-process:

$$\begin{aligned} U(t,x) &= \mathcal{H}^{-1} u(t,x) \\ &= \frac{1}{4\pi} \mathcal{H}^{-1} \int_{B(x,t)} \frac{f(y, t-|y-x|)}{|y-x|} dy \\ &= \frac{1}{4\pi} \int_{B(x,t)} \frac{F(y, t-|y-x|)}{|y-x|} dy, \end{aligned}$$

where the last inequality follows from Lemma 3.12 a).

Smoothness of the solution

We are now going to show that $U(t,x) \in C^2(K_1 \times K_2, (\mathcal{S})_{-1})$. Let us look at $U_{x_i x_j}(t,x)$. If we take the Hermite transform we get:

$$u_{x_i x_j}(t,x) = \tfrac{1}{4\pi} \int_{B(x,t)} \tfrac{f_{x_i x_j}(t-|y-x|,y)}{|y-x|} dy = \tfrac{t^2}{4\pi} \int_{B(0,1)} \tfrac{f_{x_i x_j}(t-t|y|, x+ty)}{|y|} dy$$

Since $D^3 F$ is continuous, $D^3 f$ is continuous in the following norm, $|\cdot|_z \equiv \sup_{z \in \mathbb{K}_q(\delta)} |\cdot|$, by Lemma 3.1. By the same lemma we need to show that $u_{x_i x_j}(t,x)$ is continuous in $|\cdot|_z$ then $U_{x_i x_j}(t,x)$ is continuous. Define $\tilde{f}(t,x,y) \equiv \nabla^2 f(t - t|y|, x+ty)$.

$$D_{t,x} \tilde{f}(t,x,y) = ((\partial_1 \nabla^2 f)(t-t|y|, x+ty)|y| + (D_2 \nabla^2 f)(t-t|y|, x+ty) \cdot y, (D_2 \nabla^2 f)(t-t|y|, x+ty))$$

Since $D^3 f$ is continuous in $|\cdot|_z$, then $D_{t,x}\tilde{f}$ is also continuous in $|\cdot|_z$. Let $B \equiv \{y : y \in B(x,t) \text{ for } (t,x) \in K_1 \times K_2\}$. Since $K_1 \times \overline{B}$ is compact, $M_1 \equiv \sup_{K_1 \times \overline{B}} |D_{t,x} \tilde{f}(t,x)|_z < \infty$. Let $T \equiv \sup_{K_1} |t|$, $T < \infty$ since K_1 is compact. Let $i_1(t,x)$ be defined as before. Then since $i_1(t,x)$ satisfies (E1) $M_2 \equiv \sup_{K_1 \times K_2} |i_1(t,x)|_z < \infty$. Now we use the mean value theorem and the Cauchy-Schwartz inequality:

$$|i_1(t+\Delta t, x+\Delta x) - i_1(t,x)|_z \leq \tfrac{1}{4\pi} \int_{B(0,1)} \tfrac{|\tilde{f}(t+\Delta t, x+\Delta x, y) - \tilde{f}(t,x,y)|_z}{|y|} dy$$

$$\leq \tfrac{1}{4\pi} M_1 |(\Delta t, \Delta x)| \int_{B(0,1)} \tfrac{1}{|y|} dy = \tfrac{1}{4\pi} M_1 \pi |(\Delta t, \Delta x)|$$

$$|u_{x_i x_j}(t+\Delta t, x+\Delta x) - u_{x_i x_j}(t,x)|_z \leq |(t+\Delta t)^2 i_1(t+\Delta t, x+\Delta x) - t^2 i_1(t,x)|_z$$

$$\leq |t^2| |i_1(t+\Delta t, x+\Delta x) - i_1(t,x)|_z + |\Delta t| |\Delta t + 2t| |i_1(t+\Delta t, x+\Delta x)|_z$$

$$\leq \tfrac{1}{4} T^2 M_1 |(\Delta t, \Delta x)| + 3 T M_2 |\Delta t|$$

This means that $u_{x_i x_j}(t,x)$ is Lipschitz continuous in $|\cdot|_z$, which implies continuity in $|\cdot|_z$.

By Lemma 3.1 $U_{x_ix_j}(t,x)$ is continuous. In a similar fashion we could show that U_{tt} and U_{tx_i} are continuous. Thus $D^2U(t,x)$ is continuous. By Lemma 1.1 $U(t,x)$ and $DU(t,x)$ is also continuous, so $U(t,x)$ is C^2. □

Remark: We have assumed one extra degree of differentiability, compared to the classical case. See the comment on this in Chapter 6.

5.2.2 The general Initial Value Problem

Now we turn to the general initial value problem for the inhomogeneous wave equation. In this section both the initial values and the forcing term will be stochastic distributions.

$$\boxed{\begin{array}{rcl} U_{tt}(t,x) - \nabla^2 U(t,x) & = & F(x,t) \\ U(0,x) & = & G(x) \\ U_t(0,x) & = & H(x) \\ \\ F(x,t) & \in & C^3(\mathbb{R}\times\mathbb{R}^3,(\mathcal{S})_{-1}) \\ G(x) & \in & C^4(\mathbb{R}^3,(\mathcal{S})_{-1}) \\ H(x) & \in & C^3(\mathbb{R}^3,(\mathcal{S})_{-1}) \end{array}} \tag{5.7}$$

Theorem 5.3 *The initial value problem (5.7) has a solution in $C^2_{loc}(\mathbb{R}^+\times\mathbb{R}^3,(\mathcal{S})_{-1})$ which takes the form*

$$U(t,x) = \frac{1}{4\pi t}\int_{\partial B(x,t)}[tH(y)+G(y)+\nabla G(y)\cdot(y-x)]dS(y) + \frac{1}{4\pi}\int_{B(x,t)}\frac{F(y,t-|y-x|)}{|y-x|}dy.$$

Proof:

Existence, Formula and Smoothness

Let $U^h(t,x)$ be be a solution of (5.1) and $U^p(t,x)$ be a solution of (5.4). We see that $U(t,x) = U^h(t,x)+U^p(t,x)$, and that this function therefore is C^2. We need to show that this function satisfies the initial value problem (5.7).

$$\begin{aligned} U_{tt}(t,x) - \nabla^2U(t,x) &= [U^h_{tt}(t,x)+U^p_{tt}(t,x)] - [\nabla^2U^h(t,x)+\nabla^2U^p(t,x)] \\ &= [U^h_{tt}(t,x)-\nabla^2U^h(t,x)] + [U^p_{tt}(t,x)-\nabla^2U^p(t,x)] \\ &= 0 + F(x,t) \;=\; F(x,t) \end{aligned}$$

$$\begin{aligned} U(0,x) &= U^h(0,x)+U^p(0,x) \;=\; G(x)+0 \;=\; G(x) \\ U_t(0,x) &= U^h_t(0,x)+U^p_t(0,x) \;=\; H(x)+0 \;=\; H(x) \end{aligned}$$

Uniqueness

Let $U(t,x)$ and $V(t,x)$ be two solutions of (5.7). Then the difference $U(t,x)-V(t,x)$ satisfies the homogeneous initial value problem (5.1) with $G(x)=0$ and $H(x)=0$. By Theorem 5.1 this problem has a unique solution taking form $U(t,x)-V(t,x)=0$. So $U(t,x)=V(t,x)$ and this solution is unique. $\square$

Chapter 6

Discussion

So far we have studied the stochastic wave equation in 1D and 3D. It was shown that properties like existence, uniqueness, formula, and smoothness in the classical case extends to the stochastic case using the setting described in [HØUZ96]. With the exception that we added extra smoothness to the initial values and forcing terms in the stochastic case. This will be discussed in section 6.3. Other than this we have complete analogy between the classical and the stochastic case.

But what about the 2D case? Well, as we will see in section 6.2, we can not use the same approach as in the 1D and 3D cases. Therefore I was only able to get partial results for the 2D problem. By the method of descent, we will get a solution for the homogeneous 2D problem. This is described in section 6.1.

An important thing that should be noted is the following. Even though the solutions that have been described so far are stochastic distributions, they are still strong C^2 solutions in the setting we used.

6.1 Application: 2D Homogeneous Wave Equation

Let us show an application of our 3D results. It is called the method of descent. This method is used to get a solution of the classical 2D homogeneous problem from the solution of the classical 3D homogeneous problem. Note that this method only works for homogeneous problems.

If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then define $\bar{x} = (x_1, x_2)$. Now let us assume that $U(t, x)$ is a solution of the following initial value problem.

$$\begin{aligned} U_{tt}(t,x) - \nabla_x^2 U(t,x) &= 0 \\ U(0,x) &= G(\bar{x}) \\ U_t(0,x) &= H(\bar{x}) \end{aligned}$$

$$\begin{aligned} G(\bar{x}) \in C^4(\mathbb{R}^2, (\mathcal{S})_{-1}) \\ H(\bar{x}) \in C^3(\mathbb{R}^2, (\mathcal{S})_{-1}) \end{aligned}$$

By Theorem 5.1 $U(t,x)$ is unique and has the following form

$$U(t,x) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [tH(\bar{y}) + G(\bar{y}) + \nabla_{\bar{x}} G(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(y).$$

You can see from this formula that $U(t,x)$ does not depend on x_3, which means that $U_{x_3x_3}(t,x) = 0$. So $V(t,\bar{x}) \equiv U(t,x)$ solves:

$$\begin{aligned} V_{tt}(t,\bar{x}) - \nabla_{\bar{x}}^2 V(t,\bar{x}) &= 0 \\ V(0,\bar{x}) &= G(\bar{x}) \\ V_t(0,\bar{x}) &= H(\bar{x}) \\ G(\bar{x}) \in C^4(\mathbb{R}^2, (\mathcal{S})_{-1}) & \\ H(\bar{x}) \in C^3(\mathbb{R}^2, (\mathcal{S})_{-1}) & \end{aligned} \tag{6.1}$$

Now it can be shown that

$$\begin{aligned} V(t,\bar{x}) &= \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [tH(\bar{y}) + G(\bar{y}) + \nabla_{\bar{x}} G(\bar{y}) \cdot (\bar{y} - \bar{x})] dS(y) \\ &= \frac{1}{2} \int_{B(\bar{x},t)} \frac{tG(y) + t^2 H(y) + tDG(y) \cdot (y - \bar{x})}{|y - \bar{x}|} dy \end{aligned} \tag{6.2}$$

This is done in [Eva94] and [Col88]. So now we have existence and uniqueness of, and a formula for the solution of the stochastic 2D homogeneous wave equation. Furthermore this formula is the same as in the classical case.

6.2 Problems with the 2D Wave Equation

I did try to solve the general 2D problem, using the same method as in the 1D and 3D cases. But I ran into a problem. The integrand in (6.2) is not continuous in the entire domain of integration. There is a singularity on the boundary of this domain, and this singularity does not disappear in spherical coordinates. This case is different from the 3D case, where you also have a singularity, but this singularity disappears in spherical coordinates.

So, what is the problem with this singularity? It is integrable, the integral causing trouble is $\int_0^1 \frac{r^2}{(1-r^2)^{\frac{1}{2}}} dr$, but according to *Maple* this integral equals $\frac{\pi}{4}$. Well, let us try to solve this problem the way did the 1D and 3D case. We take the Hermite transform of (6.1), solve and get a formula like (6.2). Now in order to use our existence theorem, Theorem 2.12, we need to show that $v(t,x) = \mathcal{H}V(t,x)$ and its derivatives up to second order satisfies (P1) - (P3) for some q, δ. Let us look at $v(t,x)$. Since the singularity is integrable, (P1) and (P2) follows in a way similar to what we did in 1D and 3D. But in order to show (P3) it is necessary to have an integrand that is continuous in on the closure of the domain of integration. This is a consequence of using Lemma 2.11 (Lemma 2.8.5 in [HØUZ96]), see Lemma 3.8. I did not find an alternative to using Lemma 2.11.

This discussion so far only concerned the homogeneous problem, and this we already know the solution to. But the same kind of singularity appears in the non homogeneous problem, see [Col88] so the problem remains the same.

6.3 The strong Smoothness Assumptions

If you compare the stochastic results with the classical ones, you will find that the initial functions and forcing terms are assumed to be smoother in all cases except from the 1D homogeneous case. To be precise, there is one degree of differentiability more than in the classical case. It is my *belief* that this is not necessary, that it would be enough the assume as much smoothness as in the classical case.

The stronger smoothness assumption was needed to show continuity. We used this assumption to show that the solutions to the Hermite transformed problems were C^2 and to show continuity of $D^2U(t,x)$ uniformly in z. Here $U(t,x)$ is the solution of whatever problem at hand. To get these results we used the mean value theorem and therefore had to assume more smoothness than necessary. Since there is so much similarity between the classical and the stochastic cases, the method used in the classical smoothness argument should also apply to the stochastic case. There might be some more difficulties in the case where we showed that $D^2U(t,x)$ is continuous in (t,x) uniformly in z.

I was not able to find any smoothness proofs for the classical case in the literature. In the literature the authors just "look" at the formula and declare it to be C^2. I have not managed to show that yet.

6.4 Further Work

There are still work to be done concerning the stochastic wave equation. Both with respect to improving and extending this work. Let me mention at least a few subject for further study:

- The smoothness assumptions should be improved if possible.

- The 2D problem is not completely solved. It remains to show that there is a formula for the inhomogeneous problem.
- It is well known that there are existence and uniqueness theorems and formulas for the classical wave equation in more than 3 dimensions. It might be possible to apply similar methods to the ones used in this thesis to the stochastic analogs of such higher dimensional problems. On the other hand, the calculations could get very long and complicated since the higher dimensional formulas get more and more complicated.
- We have only considered strong solutions, so weak and distribution solutions could also be studied.

A final problem would be to look at the interpretation of these solutions and to find applications for them.

Appendix A

Two basic Results

In this appendix we will prove two basic results that we are going to need repeatedly in this text. These results are stated without proofs as Lemma 1.1 and 1.2 in the Introduction. The last result is well known, but I have not been able to find a precise statement and proof of that result.

Lemma A.1 *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a function and $\alpha \in \mathbb{N}^d$ a multi index. Assume $D^\alpha u(x)$ exists and is continuous for all α such that $|\alpha| = k$. Then $D^\beta u(x)$ is continuous for all multi indexes β such that $|\beta| < k$.*

Proof:

Claim: Assume $D^\gamma u(x)$ exists and is continuous for all γ such that $|\gamma| = j$. Then $D^\beta u(x)$ is continuous for all β such that $|\beta| = j - 1$.

Proof:

Fix x_0 and $r > 0$ such that for all $\gamma, |\gamma| = j$ $D^\gamma u(x)$ is continuous for $x \in \overline{B(x_0, r)}$. Since $\overline{B(x_0, r)}$ is compact $\sup_{x \in \overline{B(x_0,r)}} |D^\gamma u(x)| \equiv M_\gamma < \infty$. By the mean value theorem $D^\beta u(x + h) - D^\beta u(x) = \sum_{i=1}^d D^{\beta+e_i} u(\xi) h_i$, where $\xi_i \in (x_i, x_i + h_i)$.

$$|D^\beta u(x + h) - D^\beta u(x)| \le \sum_{i=1}^d |D^{\beta+e_i} u(\xi)||h_i| \le d \max_i \{M_{\beta+e_i}\} \|h\|_1$$

Fix $\epsilon > 0$ and let $\delta = \frac{\epsilon}{d \max_i \{M_{\beta+e_i}\}}$, then $\|h\|_1 < \delta \Rightarrow |D^\beta u(x + h) - D^\beta u(x)| < \epsilon$. Since x_0 was arbitrary, $D^\beta u(x)$ is continuous. *QED*

Since the claim holds when $|\gamma| = k$, the lemma follows by induction. □

Lemma A.2 *Let $f : V \times \mathbb{R} \to \mathbb{R}$, $u : V \to \mathbb{R}$, and $l : V \to \mathbb{R}$ be functions, and let $V \subset \mathbb{R}^d$ be* *an open set. If $u(x)$, $l(x)$, and $f(x,s)$ are C^1, then*

$$\frac{\partial}{\partial x_i}\int_{l(x)}^{u(x)} f(x,s)ds = u_{x_i}(x)f(x,u(x)) - l_{x_i}(x)g(x,l(x)) + \int_{l(x)}^{u(x)} f_{x_i}(x,s)ds.$$

Proof:

Let us show that $\frac{\partial}{\partial x_i}\int_0^{a(x)} f(x,s)ds = a_{x_i}(x)f(x,a(x)) + \int_0^{a(x)} f_{x_i}(x,s)ds$ when $a(x)$ is C^1. Then the lemma holds since $\frac{\partial}{\partial x_i}\int_{l(x)}^{u(x)} f(x,s)ds = \frac{\partial}{\partial x_i}\int_0^{u(x)} f(x,s)ds - \frac{\partial}{\partial x_i}\int_0^{l(x)} f(x,s)ds$.

Let $b(x) = x$, and write $\int_0^{a(x)} f(x,s)ds = \int_0^{a(x)} f(b(x),s)ds$.

$$\frac{\partial}{\partial x_i}\int_0^{a(x)} f(b(x),s)ds = a_{x_i}(x)\partial_a\int_0^{a(x)} f(b(x),s)ds + b_{x_i}(x)\partial_{b_i}\int_0^{a(x)} f(b(x),s)ds.$$

By the fundamental theorem of calculus $\partial_a\int_0^{a(x)} f(b(x),s)ds = f(b(x),a(x)) = f(x,a(x))$.

Since $f_{x_i}(b(x),x)$ is continuous, it is bounded on a compact subset K of V containing an open ball around x. This allows us to use the bounded convergence theorem in K:

$$\begin{aligned}
\partial_{b_i}\int_0^{a(x)} f(b(x),s)ds &= \lim_{\Delta b_i\to 0}\frac{1}{\Delta b_i}\int_0^{a(x)} f(b(x)+\Delta b(x),s) - f(b(x),s)ds \\
&= \lim_{\Delta b_i\to 0}\int_0^{a(x)}\frac{1}{\Delta b_i}\Delta b_i f_{b_i}(b(x),s)\mid_{b_i=\xi} ds \\
&= \int_0^{a(x)}\lim_{\Delta b_i\to 0} f_{b_i}(b(x),s)\mid_{b_i=\xi} ds \\
&= \int_0^{a(x)} f_{b_i}(b(x),s)ds \;=\; \int_0^{a(x)} f_{x_i}(x,s)ds
\end{aligned}$$

□

Appendix B

Calculations of the Derivatives in Chapter 4 and 5

In this appendix we will derive the formulas for the derivatives that we used in the proofs in Chapters 4 and 5.

Note. In this appendix we will never mention the z-variable or the spaces $\mathbb{K}_q(\delta)$. Every time we use Theorem 3.3, we have to consider a specific $\mathbb{K}_{q'}(\delta')$. But if we let $\delta = \min(\delta')$ and $q = \max(q')$, then all results in this appendix holds when $z \in \mathbb{K}_q(\delta)$.

B.1 The Derivatives in Section 4.1

We are now going to find the second derivatives of equation (4.3), that is the following equation:

$$u(t,x) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds \tag{B.1}$$

Let $F(y)$ be a function such that $F'(y) = h(y)$, such a function exists since $h(y)$ is integrable. By the fundamental Theorem of calculus (see [Rud76]) $\int_{x-t}^{x+t} h(s)ds = F(x+t) - F(x-t)$.

$$\begin{aligned}
\frac{\partial}{\partial x}\int_{x-t}^{x+t} h(s)ds &= \frac{\partial}{\partial x}F(x+t) - \frac{\partial}{\partial x}F(x-t) \\
&= h(x+t) - h(x-t) \\
\frac{\partial}{\partial t}\int_{x-t}^{x+t} h(s)ds &= \frac{\partial}{\partial t}F(x+t) - \frac{\partial}{\partial t}F(x-t) \\
&= h(x+t) + h(x-t)
\end{aligned}$$

With this in mind, it is now straight forward to calculate the second derivatives of equation

(B.1).

$$\begin{aligned}u_t(t,x) &= \tfrac{1}{2}(g'(x+t)-g'(x-t))+\tfrac{1}{2}(h(x+t)+h(x-t))\\ u_x(t,x) &= \tfrac{1}{2}(g'(x+t)+g'(x-t))+\tfrac{1}{2}(h(x+t)-h(x-t))\end{aligned}$$

$$\begin{aligned}u_{tt}(t,x) &= u_{xx}(t,x) = \tfrac{1}{2}(g''(x+t)+g''(x-t))+\tfrac{1}{2}(h'(x+t)-h'(x-t))\\ u_{xt}(t,x) &= u_{tx}(t,x) = \tfrac{1}{2}(g''(x+t)-g''(x-t))+\tfrac{1}{2}(h'(x+t)+h'(x-t))\end{aligned}$$

B.2 The Derivatives in Section 4.2

In this section we are going to find the derivatives of equation (4.6), that is the following equation:

$$u(t,x)=\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds \tag{B.2}$$

Let us define $g(t,x,s)\equiv\int_{x-(t-s)}^{x+(t-s)} f(y,s)dy$.

Lemma B.1

1) $g_t(t,x,s)=f(x+(t-s),s)+f(x-(t-s),s)$

2) $g_x(t,x,s)=f(x+(t-s),s)-f(x-(t-s),s)$

Proof:

We introduce the following functions, $l(t,x,s)=x-(t-s)$ and $u(t,x,s)=x+(t-s)$.

$$g(t,x,s)=\int_{x-(t-s)}^{x+(t-s)} f(y,s)dy=\int_{l(t,x,s)}^{u(t,x,s)} f(y,s)dy$$

Note that l and u are C^∞-functions. Since F is a C^2 $(\mathcal{S})_{-1}$ process, by Theorem 3.3 $f=\mathcal{H}F$ is C^2. Now the conditions of Lemma 1.2 are satisfied. By this lemma:

$$\begin{aligned}\partial_t g(t,x,s) &= f(u,s)u_t(t,x,s)-f(l,s)l_t(t,x,s)+\int_{x-(t-s)}^{x+(t-s)}\partial_t f(y,s)dy\\ &= f(x+(t-s),s)+f(x-(t-s),s)\end{aligned}$$

$$\begin{aligned}\partial_x g(t,x,s) &= f(u,s)u_x(t,x,s)-f(l,s)l_x(t,x,s)+\int_{x-(t-s)}^{x+(t-s)}\partial_x f(y,s)dy\\ &= f(x+(t-s),s)-f(x-(t-s),s)\end{aligned}$$

□

Lemma B.2 *$g_t(t,x,s)$ and $g_x(t,x,s)$ are bounded and continuous for $(t,x,r)\in K_1\times K_2\times K_3\subset\mathbb{R}^3$ where K_1, K_2, and K_3 are compact sets.*

Proof:

Since $F(x,t) \in C^1(\mathbb{R}^2, (\mathcal{S})_{-1})$, we have from Theorem 3.3 that $f(\cdot,\cdot)$ is bounded and continuous. Let us show that $f(x+(t-s),s)$ is bounded and continuous.

- Let $h(x,t,s) = x+t-s$ and $f(x+t-s,s) = f(h(x,t,s),s)$. Fix $\epsilon > 0$, then since $f(y_1,y_2)$ is continuous exist a δ_f such that if $\|(\Delta y_1, \Delta y_2\|_1 < \delta_f$ then $|\partial_1 f(y_1+\Delta y_1, y_2+\Delta y_2) - \partial_1 f(y_1,y_2)| < \epsilon$. $h(x,t,s)$ is also continuous, so there exist δ_h such that if $\|(\Delta x, \Delta t, \Delta s)\|_1 < \delta_h$ then $|h(x+\Delta x, t+\Delta t, s+\Delta s) - h(x,t,s)| \equiv |\Delta h(x,t,s)| < \frac{\delta_f}{2}$. Now let $\delta \equiv \min(\frac{\delta_f}{2}, \delta_h)$, and note that $\|(\Delta x, \Delta t, \Delta s)\|_1 < \delta$ implies that $|\Delta s| < \delta$. With $\Delta y_1 = \Delta h(x,t,s)$ and $\Delta y_2 = \Delta s$ we get

$$\begin{aligned}
\|(\Delta x, \Delta t, \Delta s)\|_1 < \delta \quad &\Rightarrow \quad \|(\Delta h(x,t,s), \Delta s)\|_1 < \frac{\delta_f}{2} + \frac{\delta_f}{2} = \delta_f \\
&\Rightarrow \quad |f(h(x,t,s) + \Delta h(x,t,s), s+\Delta s) - f(h(x,t,s),s)| < \epsilon.
\end{aligned}$$

- Let $s \in K_3 \subset \mathbb{R}$, where K_3 is compact. Now $\overline{K_1+K_2} \times K_3$ is compact. From Theorem 3.3 we get that

$$\sup_{z;(t,x,s)\in K_1\times K_2\times K_3} |f(x+t-s,s)| \leq \sup_{z;(y_1,y_2)\in \overline{K_1+K_2}\times K_3} |f(y_1,y_2)| < \infty.$$

The proof in the case of $f(x-(t-s),s)$ is similar. Since both $g_t(t,x,s)$ and $g_x(t,x,s)$ are linear combinations of $f(x+(t-s),s)$ and $f(x-(t-s),s)$ by Lemma B.1, they are bounded and continuous by Lemma 3.5. □

Proposition B.3

1) $u_t(t,x) = \frac{1}{2}\int_0^t [f(x+(t-s),s) + f(x-(t-s),s)]ds$

2) $u_x(t,x) = \frac{1}{2}\int_0^t [f(x+(t-s),s) - f(x-(t-s),s)]ds$

Proof:

Claim 1: $g(t,x,s)$ is continuous.

Proof:

Let $y = (t,x,s)$ and $\Delta y = (\Delta t, \Delta x, \Delta s)$. In the following we will use the mean value theorem.

$$
\begin{aligned}
|g(y+\Delta y)-g(y)| &= \Big|\int_{x+\Delta x+t+\Delta t-s-\Delta s}^{x+\Delta x-t-\Delta t+s+\Delta s} f(y,s+\Delta s)dy - \int_{x+t-s}^{x-t+s} f(y,s)dy\Big| \\
&= \Big|\Big(\int_{x+\Delta x+t+\Delta t-s-\Delta s}^{x+\Delta x-t-\Delta t+s+\Delta s} - \int_{x+t-s}^{x-t+s}\Big) f(y,s+\Delta s)dy \\
&\quad + \int_{x-t+s}^{x+t-s} [f(y,s+\Delta s)-f(y,s)]dy\Big| \\
&\leq \Big|\Big(\int_{x+t-s}^{x+\Delta x-t-\Delta t+s+\Delta s} + \int_{x+\Delta x+t+\Delta t-s-\Delta s}^{x-t+s}\Big) f(y,s+\Delta s)dy\Big| \\
&\quad + \Big|\int_{x-t+s}^{x+t-s} \Delta s f_s(y,\xi)dy\Big|, \text{ where } t<\xi<t+\Delta t \\
&\leq 2\|\Delta y\|_1 \sup_y |f(y,s+\Delta s)| + \|y\|_1 |\Delta s| \sup_y ||f_s(y,s)| \\
&\leq 2\|\Delta y\|_1 \sup_{y,s} |f(y,s)| + \|y\|_1 |\Delta s| \sup_{y,s} ||f_s(y,s)|
\end{aligned}
$$

Since $F(t,x) \in C^1(\mathbb{R}^2, (\mathcal{S})_{-1})$, it follows from Theorem 3.3 that there exists M_1 and M_2 such that $\sup_{y,s}|f(y,s)| \leq M_1$ and $\sup_{y,s}|f_s(y,s)| \leq M_2$.

Note that if $\|\Delta y\|_1 < \delta$ then $|\Delta s| < \delta$. For every $\epsilon > 0$ choose $\delta = \min(\frac{\epsilon}{4M_1}, \frac{\epsilon}{2\|y\|_1 M_2})$. If $\|\Delta y\|_1 < \delta$ then $|g(y+\Delta y)-g(y)| < \epsilon$. In other words $g(t,x,s)$ is continuous. *QED*

Claim 2: $\partial_a \int_0^{b(t)} g(a(t),x,s)ds = \int_0^{b(t)} \partial_a g(a(t),x,s)ds$

Proof:

In the following we will use the mean value theorem. By Lemma B.2 $g_t(t,x,s)$ is bounded and continuous, and this boundedness will allow us to use the bounded convergence theorem.

$$
\begin{aligned}
\partial_t \int_0^u g(t,x,s)ds &= \lim_{\Delta t\to 0} \int_0^u \frac{1}{\Delta t}[g(t+\Delta t,x,s)-g(t,x,s)]ds \\
&= \lim_{\Delta t\to 0} \int_0^u \partial_t g(\xi,x,s)ds, \text{ where } t<\xi<t+\Delta t \\
&= \int_0^u \lim_{\Delta t\to 0} g_t(\xi,x,s)ds \;=\; \int_0^u g_t(t,x,s)ds
\end{aligned}
$$

QED

Let $a(t)=t$ and $b(t)=t$, then $u(t,x) = \frac{1}{2}\int_0^t g(t,x,s)ds = \frac{1}{2}\int_0^{b(t)} g(a(t),x,s)ds$.

Let $K(t,x,s) = \int_0^s g(t,x,r)dr$, then $\partial_s K(t,x,s) = g(t,x,s)$. This integral is well defined since $g(t,x,s)$ is continuous by Claim 1.

$$
\begin{aligned}
\partial_t u(t,x) &= \frac{1}{2}\partial_t K(a(t),x,b(t)) \;=\; \frac{1}{2}[K_a(a(t),x,b(t))a_t(t) + K_b(a(t),x,b(t))b_t(t)] \\
&= \frac{1}{2}[\partial_a \int_0^{b(t)} g(a(t),x,s)ds + g(t,x,t)] \;=\; \frac{1}{2}\int_0^{b(t)} g_a(a(t),x,s)ds \\
&= \frac{1}{2}\int_0^t [f(x+(t-s),s) + f(x-(t-s),s)]ds
\end{aligned}
$$

The last two equalities follows from Claim 2 and Lemma B.1 respectively. $\partial_x u(t,x)$ is calculated in a similar way.

$$
\begin{aligned}
\partial_x u(t,x) &= \frac{1}{2}K_x(t,x,t) \;=\; \frac{1}{2}\partial_x \int_0^t g(t,x,s)ds \;=\; \frac{1}{2}\int_0^t g_x(t,x,s)ds \\
&= \frac{1}{2}\int_0^t [f(x+(t-s),s) - f(x-(t-s),s)]ds
\end{aligned}
$$

□

Proposition B.4

1) $u_{tt}(t,x) = f(t,x) + \frac{1}{2}\int_0^t [f_t(x+(t-s),s) + f_t(x-(t-s),s)]ds$
2) $u_{tx}(t,x) = \frac{1}{2}\int_0^t [f_x(x+(t-s),s) + f_x(x-(t-s),s)]ds = u_{xt}(t,x)$
3) $u_{xx}(t,x) = \frac{1}{2}\int_0^t [f_x(x+(t-s),s) - f_x(x-(t-s),s)]ds$

Proof:

We will start by showing that $\partial_t \int_0^u f(x+(t-s),s)ds = \int_0^u \partial_t f(x+(t-s),s)ds$. By the mean value theorem we get

$$
\left|\frac{1}{\Delta t}[f(x+t+\Delta t-s,s) - f(x+t-s,s)\right| \leq |f_t(x+\xi-s,s)| \leq \sup_{x,t,s}|f_t(x+t-s,s)|.
$$

Here $t < \xi < t+\Delta t$. The domain of (t,x,s) is the compact set $K_1 \times K_2 \times K_3$. By Theorem 3.3 $\partial_1 f(\cdot,\cdot)$ is bounded on compact sets since $F(t,x) \in C^1(\mathbb{R}^2,(\mathcal{S})_{-1})$. The set $K \equiv \overline{\{(x+t-s,s)|(t,x,s) \in K_t \times K_x \times K_s\}}$ is compact, so

$$
\sup_{(t,x,s)\in K_t\times K_x\times K_s} |f_t(x+t-s,s)| \leq \sup_{(y_1,y_2)\in K} |f_t(y_1,y_2)| < \infty.
$$

Note that $f(x+t-s,s)$ is continuous. By the bounded convergence theorem we get

$$
\begin{aligned}
\partial_t \int_0^u f((x-t+s,s)ds &= \lim_{\Delta t\to 0}\int_0^u \frac{1}{\Delta t}[f(x+t+\Delta t-s,s) - f(x+t-s,s)]ds \\
&= \int_0^u \lim_{\Delta t\to 0}\frac{1}{\Delta t}[f(x+t+\Delta t-s,s) - f(x+t-s,s)]ds \\
&= \int_0^u f_t(x+t-s,s)ds
\end{aligned}
$$

A similar proof will show that $\partial_x \int_0^u f(x+(t-s),s)ds = \int_0^u \partial_x f(x+(t-s),s)ds$. Interchange $f(x+(t-s),s)$ with $f(x-(t-s),s)$ and the results still holds. With these results and proposition B.3 in mind let us calculate the second derivatives.

$$\begin{aligned}
u_{xx}(t,x) &= \partial_x u_x(t,x) = \partial_x \frac{1}{2}\int_0^t [f(x+(t-s),s) - f(x-(t-s),s)]ds \\
&= \frac{1}{2}\int_0^t \partial_x [f(x+(t-s),s) - f(x-(t-s),s)]ds \\
&= \frac{1}{2}\int_0^t [f_x(x+(t-s),s) - f_x(x-(t-s),s)]ds
\end{aligned}$$

$$\begin{aligned}
u_{tx}(t,x) &= \partial_x u_t((t,x) = \partial_x \frac{1}{2}\int_0^t [f(x+(t-s),s) + f(x-(t-s),s)]ds \\
&= \frac{1}{2}\int_0^t \partial_x [f(x+(t-s),s) + f(x-(t-s),s)]ds \\
&= \frac{1}{2}\int_0^t [f_x(x+(t-s),s) + f_x(x-(t-s),s)]ds
\end{aligned}$$

Let $u(t) = t$, and let $K^b(t,x,s) = \int_0^s \partial_b g(t,x,s)ds$. Then $\partial_s K^b(t,x,s) = \partial_b g(t,x,s)$ where $b \in \{x,t\}$. These functions are well defined since $g_t(t,x,s)$ and $g_x(t,x,s)$ are continuous by Lemma B.2 and therefore integrable.

$$\begin{aligned}
u_{tt}(t,x) &= \partial_t u_t(t,x) \;=\; \frac{1}{2}\partial_t K^t(t,x,u) \;=\; \frac{1}{2}[K^t_u(t,x,u)u_t + K^t_t(t,x,u)] \\
&= \frac{1}{2}[g_t(t,x,u) + \partial_t \frac{1}{2}\int_0^t [f(x+(t-s),s) + f(x-(t-s),s)]ds \\
&= \frac{1}{2}[f(x+(t-u),u) + f(x-(t-u),u)] \\
&\quad + \frac{1}{2}\int_0^t \partial_t [f(x+(t-s),s) + f(x-(t-s),s)]ds \\
&= f(x,t) + \frac{1}{2}\int_0^t [f_t(x+(t-s),s) - f_t(x-(t-s),s)]ds
\end{aligned}$$

$$\begin{aligned}
u_{xt}(t,x) &= \partial_t u_x(t,x) \;=\; \frac{1}{2}\partial_t K^x(t,x,u) \;=\; \frac{1}{2}[K^x_u(t,x,u)u_t + K^x_t(t,x,u)] \\
&= \frac{1}{2}[g_x(t,x,u) + \partial_t \frac{1}{2}\int_0^t [f(x+(t-s),s) - f(x-(t-s),s)]ds \\
&= \frac{1}{2}[f(x+(t-u),u) - f(x-(t-u),u)] \\
&\quad + \frac{1}{2}\int_0^t \partial_t [f(x+(t-s),s) - f(x-(t-s),s)]ds \\
&= \frac{1}{2}\int_0^t [f_t(x+(t-s),s) + f_t(x-(t-s),s)]ds
\end{aligned}$$

$\square$

B.3 The Derivatives in Section 5.1

In this section we are going to find the derivatives of equation (5.3), that is the following equation:

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} [th(y) + g(y) + \nabla g(y)\cdot(y-x)]dS(y). \tag{B.3}$$

This time we are luckier than in the 1D case. We were able to find part of the solution in the literature. We are now going to use results from the books by Zachmanoglou and Thoe [ZT76] and Colton [Col88].

Define $u^f(t,x) \equiv \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y)dS(y)$. According to [ZT76] and [Col88] this function is a solution of the homogeneous wave equation. Furthermore the solution to the initial value problem (5.2) is given by $u(t,x) = u^h(t,x) + \partial_t u^g(t,x)$. The derivatives of $u^f(t,x)$ are as follows:

$$u^f_{x_i}(t,x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f_{x_i}(y)dS(y) \tag{B.4}$$

$$u^f_t(t,x) = \frac{1}{t}u^f(t,x) + \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla f(y)\cdot(y-x)dS(y) \tag{B.5}$$

$$u^f_{x_i x_j}(t,x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f_{x_i x_j}(y)dS(y) \tag{B.6}$$

$$u^f_{tt}(t,x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} \nabla^2 f(y)dS(y) \tag{B.7}$$

We will also need $u^f_{tx_i}$ $(= u^f_{x_i t})$.

Lemma B.5 *Assume that there exists a C^3 $(\mathcal{S})_{-1}$-process $F(x)$ such that $\mathcal{H}F(x) = f(x)$ for $x \in \mathbb{R}$, then*

$$u^f_{tx_i} = \frac{1}{t}u^f_{x_i}(t,x) + \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla f_{x_i}(y)\cdot(y-x)dS(y), \tag{B.8}$$

for $(t,x) \in K_t \times K$, where K and K_t are compact.

Outline of proof:

$$\partial_{x_i} u^f_{x_i}(t,x) = \tfrac{1}{t}u^f_{x_i}(t,x) + \tfrac{1}{4\pi t^2}\textstyle\int_{\partial B(x,t)} \nabla f_{x_i}(y)\cdot(y-x)dS(y)$$

This follows from (B.5). We need to show that we could interchange integration and differentiation. Let us change to spherical coordinates. Let $h(\phi,\theta) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, then

$$\textstyle\int_{\partial B(x,t)} \nabla f(y)\cdot(y-x)dS(y) = t^2\int_0^{2\pi}\int_0^{\pi} \nabla f(x + th(\phi,\theta))\cdot th(\phi,\theta)\sin\theta d\theta d\phi.$$

By Theorem 3.3 ∇f satisfies (E1) - (E3) for some q, δ on $\mathbb{R}$. Since $g(t,x,\phi,\theta) \equiv x + th(\phi,\theta)$ is continuous and takes bounded sets to bounded sets, $\nabla f(g(t,x,\phi,\theta))$ satisfies (E1) - (E3) on $g^{-1}(\mathbb{R})$ for q, δ by Lemma 3.6 b). By Lemma 3.7:

(A) $\nabla f(x + th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta$ satisfies (E1) - (E3) on $g^{-1}(\mathbb{R})$ for q, δ.

The set $B \equiv K \times K_t \times [0, 2\pi] \times [0, \pi] \subset g^{-1}(\mathbb{R})$, and it is compact. Note that

(B) $D[\nabla f(x + th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta]$ is continuous.

This fact follows from a similar argument as in the proof of Lemma 3.8. By (A) and (B), Lemma 3.9 yields that $\int_0^\pi \nabla f(x+th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta d\theta$ satisfies (P1) - (P3) on B. By (P1) and the bounded convergence theorem and an argument similar to the one in the proof of Lemma 1.2, we get

$$\partial_{x_i} \int_0^{2\pi} \int_0^\pi \nabla f(x+th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta d\theta d\phi = \int_0^{2\pi} \partial_{x_i} \int_0^\pi \nabla f(x+th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta d\theta d\phi.$$

By (A) and the bounded convergence theorem we get

$$\partial_{x_i} \int_0^\pi \nabla f(x + th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta d\theta = \int_0^\pi \partial_{x_i} \nabla f(x + th(\phi,\theta)) \cdot h(\phi,\theta)\sin\theta d\theta. \qquad \square$$

If you set $f(y) = h(y)$, then you get the derivatives of $u^h(t,x)$. Let $f(y) = \frac{1}{t}[g(y) + \nabla g(y) \cdot (y-x)] \equiv \bar{g}(y)$ and note that $u(t,x) = u^h(t,x) + u^{\bar{g}}(t,x)$, that is $u^{\bar{g}}(t,x) = u_t^g(t,x)$. This means that

$$\begin{aligned} u_{x_i}^{\bar{g}}(t,x) &= u_{tx_i}^g(t,x), \text{and} && \text{(B.9)} \\ u_t^{\bar{g}}(t,x) &= u_{tt}^g(t,x). && \text{(B.10)} \end{aligned}$$

$$u_{x_ix_j}^{\bar{g}}(t,x) = u_{tx_ix_j}^g(t,x) = \frac{1}{t}u_{x_ix_j}^g(t,x) + \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla g_{x_ix_j}(y) \cdot (y-x) dS(y) \quad \text{(B.11)}$$

$$u_{tx_i}^{\bar{g}}(t,x) = u_{ttx_i}^g(t,x) = \frac{1}{4\pi t}\int_{\partial B(x,t)} \nabla^2 g_{x_i}(y) dS(y) \quad \text{(B.12)}$$

$$u_{tt}^{\bar{g}}(t,x) = u_{ttt}^g(t,x) = \frac{1}{t}\nabla^2 u^g(t,x) + \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla(\nabla^2 g)(y) \cdot (y-x) dS(y) \quad \text{(B.13)}$$

(B.11) follows from (B.9) after interchanging integration and differentiation. The argument for this is similar to the one in the proof of Lemma B.5. The situation is similar for (B.12) which follows from (B.10). (B.13) follows from (B.11) since $u^{\bar{g}}$ satisfies the wave equation.

All this leads to the following formulas for the derivatives of $u(t,x)$:

$$\begin{aligned}
u_{x_i}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th_{x_i}(y)+g_{x_i}(y)+\nabla g_{x_i}(y)\cdot(y-x)]dS(y)\\
u_t(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[h(y)+\nabla h(y)\cdot(y-x)+t\nabla^2 g(y)]dS(y)\\
u_{x_ix_j}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th_{x_ix_j}(y)+g_{x_ix_j}(y)+\nabla g_{x_ix_j}(y)\cdot(y-x)]dS(y)\\
u_{tx_i}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[h_{x_i}(y)+\nabla h_{x_i}(y)\cdot(y-x)+t\nabla^2 g_{x_i}(y)]dS(y)\\
u_{tt}(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[t\nabla^2 h(y)+\nabla^2 g(y)+\nabla(\nabla^2 g)(y)\cdot(y-x)]dS(y)
\end{aligned}$$

B.4 The Derivatives in Section 5.2

In this section we are going to find the derivatives of equation (5.6), that is the following equation:

$$u(t,x)=\frac{1}{4\pi}\int_{B(x,t)}\frac{f(y,t-|y-x|)}{|y-x|}dy \qquad \text{(B.14)}$$

In this section we will use results stated in [Col88]. Let $v(t,x)$ be the solution of

$$\begin{aligned}
v_{tt}(t,x;s)-\nabla^2 v(t,x;s) &= 0\\
v(s,x;s) &= 0\\
v_t(s,x;s) &= f(s,x)
\end{aligned}$$

Then by [Col88] $u(t,x)=\int_0^t v(t,x;s)ds$ solves the the initial value problem (5.5). Furthermore the derivatives of $u(t,x)$ are:

$$\begin{aligned}
u_{x_i}(t,x) &= \int_0^t v_{x_i}(t,x;s)ds\\
u_t(t,x) &= \int_0^t v_t(t,x;s)ds\\
u_{x_ix_j}(t,x) &= \int_0^t v_{x_ix_j}(t,x;s)ds\\
u_{tx_i}(t,x) &= \int_0^t v_{tx_i}(t,x;s)ds\\
u_{tt}(t,x) &= \int_0^t v_{tt}(t,x;s)ds+f(t,x)
\end{aligned}$$

The function $u_{tx_i}(t,x)$ is not given in [Col88], so we must show that you can interchange differentiation and integration in $\partial_{x_i}u_t(t,x)$. Since $v(t,x;s)$ is C^2, $v_t(t,x;s)$ is C^1 and by

Lemma 1.2 the interchanging can be done. From section B.3 we have expressions for $v(t,x;s)$ and its derivatives. Note the time shift s.

$$v(t,x;s) = \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(s,y)dS(y)$$

$$v_{x_i}(t,x;s) = \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f_{x_i}(s,y)dS(y)$$

$$v_t(t,x;s) = \frac{1}{4\pi(t-s)^2} \int_{\partial B(x,t-s)} [f(s,y) + \nabla_x f(s,y)\cdot(y-x)]dS(y)$$

$$v_{x_ix_j}(t,x;s) = \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f_{x_ix_j}(s,y)dS(y)$$

$$v_{tx_i}(t,x;s) = \frac{1}{4\pi(t-s)^2} \int_{\partial B(x,t-s)} [f_{x_i}(s,y) + \nabla_x f_{x_i}(s,y)\cdot(y-x)]dS(y)$$

$$v_{tt}(t,x;s) = \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} \nabla^2 f(s,y)dS(y)$$

Let us change coordinates and rewrite the expressions for $u(t,x)$. We start by normalizing the domain of integration. Then we introduce a new variable $r = t-s$ and write the expression as a volume integral.

$$\begin{aligned}
u(t,x) &= \int_0^t v(t,x;s)ds = \int_0^t \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(s,y)dS(y)ds \\
&= \int_0^t \frac{t-s}{4\pi} \int_{\partial B(0,1)} f(s,x+(t-s)y)dS(y)ds = \frac{1}{4\pi}\int_0^t r^2 \int_{\partial B(0,1)} \frac{f(t-r,x+ry)}{r}dS(y)dr \\
&= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(t-|y-x|,y)}{|y-x|}dS(y)dr = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(t-|y-x|,y)}{|y-x|}dy
\end{aligned}$$

We rewrite all the derivatives of $u(t,x)$ in this fashion. The result is:

$$u_{x_i}(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_i}(t-|y-x|,y)}{|y-x|}dy$$

$$u_t(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(t-|y-x|,y) + \nabla_x f(t-|y-x|,y)\cdot(y-x)}{|y-x|^2}dy$$

$$u_{x_ix_j}(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_ix_j}(t-|y-x|,y)}{|y-x|}dy$$

$$u_{tx_i}(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f_{x_i}(t-|y-x|,y) + \nabla_x f_{x_i}(t-|y-x|,y)\cdot(y-x)}{|y-x|^2}dy$$

$$u_{tt}(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{\nabla^2 f(t-|y-x|,y)}{|y-x|}dy + f(t,x)$$

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