Exercise Class Week 43

Extension of Problem 4.66

(d) We have a branching process with

$P_0 = \frac{1}{5}$ and $P_1 = \frac{4}{5}$

Calculate π_0.

Solution:

Mean number of offspring of a single individual

$$\mu = 0 \cdot P_0 + 1 \cdot P_1$$

$$\mu = 4/5$$

$\mu < 1 \Rightarrow$ Probability 1 that population dies out

$$\pi_0 = 1$$

(e) We have a branching process with

$P_0 = 1/5$ and $P_2 = 4/5$

Calculate π_0.

Solution:

Mean number of offspring of a single individual

$$\mu = 0 \cdot P_0 + 2 \cdot P_2$$

$$= 8/5$$

$\mu > 1 \Rightarrow$ Probability less than 1 that the population dies out.

In this case π_0 is the smallest positive number that solves

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^{\,j} P_j$$

$$\pi_0 = \pi_0^{\,0} \cdot \frac{1}{5} + \pi_0^{\,2} \cdot \frac{4}{5}$$

$$\frac{4}{5}\pi_0^2 - \pi_0 + \frac{1}{5} = 0$$

$\Downarrow$ Quadratic Formula

$$\pi_0 = \frac{5}{8}\left(1 \pm 3/5\right)$$

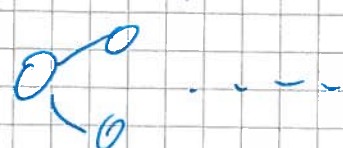
Smallest solution ~~is~~ found with negative sign

$$\pi_0 = \frac{5}{8}\left(1 - 3/5\right) = \underline{\underline{1/4}}$$

(F) What is the probability that the population dies out if it starts with 2 individuals?

Solution:

Consider the chain started by each individual individually.

Individual 1  Probability π_0 to die out

Answer is probability that each branch dies out

$$\pi_0^2 = \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{16}$$

Problem 4.70

A total of m white and m black balls are distributed among two urns, with each urn containing m balls. At each stage, one ball is randomly selected from each urn and the two selected balls are interchanged. Let X_n denote the number of black balls in urn 1 after n interchanges.

(a) Give the transition probabilities of the Markov chain. X_n, $n \geq 0$.

Solution:

Observations to make:

- There are m+1 states: 0, 1, 2, ..., m
- Each transition consist of
  - (1) Black ball urn 1 → urn 2 + Black ball urn 2 → urn 1

    Net change = 0
  - (2) White ball urn 1 → urn 2 + White ball urn 2 → urn 1

    Net change = 0

(3) Black ball urn 1 → urn 2

White ball urn 2 → urn 1

Net change = −1

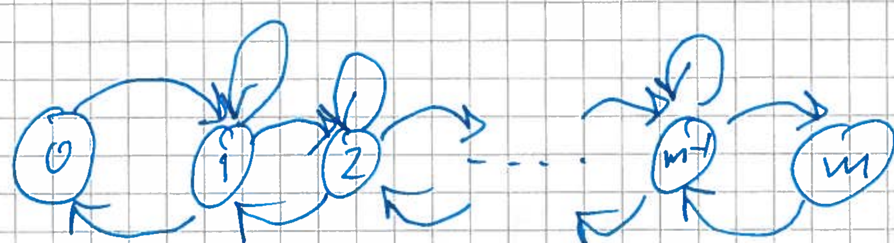
(4) White ball urn 1 → urn 2

Black ball urn 2 → urn 1

Net change = 1

This means

- If current state is 0 or m we must move to 1 and $m-1$, respectively
- If current state is $i \neq 0, m$, we can move to $i-1$, i and $i+1$



Transitions from $i=0$

$P_{01} = 1$ and $P_{0j} = 0$ for $j \neq 1$

Transitions from $i=m$

$P_{m,m-1} = 1$ and $P_{m,j} = 0$ for $j \neq m-1$

For $0 < i < m$

$i \to i-1$:

Need to select one of the i black balls in urn 1 and one of the i white balls in urn 2

$i \to i$

Select one of i black balls in urn 1 and one of the $m-i$ black balls in urn 2

or

Select one of the $m-i$ white balls in urn 1 and one of the i ~~bla~~ white balls in urn 2

$$P_{i,i} = \frac{i(m-i)}{m^2} + \frac{(m-i)i}{m^2} = 2 \cdot \frac{(m-i) \cdot i}{m^2}$$

$i \to i+1$:

select one of $m-i$ white balls in urn 1 and one of the $m-i$ black balls in urn 2.

$$P_{i,i+1} = \frac{(m-i)^2}{m^2}$$

~~(b) Withou~~

(b) Without any computations what do you think are the limiting probabilities of this chain?

Solution:

Forget everything you did in (a)!!

Ask instead:

What do I expect to happen after randomly switching 2 ~~and 2 b.~~ balls for a long time?

Answer:

I expect that any (ordered) selection ~~one~~ of m balls from the $2m$ balls is equally likely.

More specifically:

There are m black balls and m white balls what is the probability of selecting i black balls and (m-i) white balls?

Hyper geometric probability!

$$\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

(c) Find the limiting probabilities and show that the stationary chain is time reversible.

Solution:

IF the equation

$$\pi_i P_{ij} = \pi_j P_{ji} \quad *$$

holds for all i and j, the chain is time reversible and the numbers $\pi_0, \pi_1, \ldots, \pi_m$ are the limiting probabilities.

Just need to verify that our guess is correct!

Step 1: $i=j$

Trivial $\pi_i P_{ii} = \pi_i P_{ii}$

Step 2: $j = i+1$

Need to show

$$\pi_i \, P_{i,i+1} = \pi_{i+1} \, P_{i+1,i} \qquad (1)$$

~~Show First that~~ Note First that

$$(*) \quad \binom{m}{i+1} = \frac{m(m-1)\cdots(m-i)}{(i+1)\cdot i \cdots 2\cdot 1} = \frac{(m-i)}{i+1} \, \frac{m(m-1)\cdot\cdots\cdot(m-i+1)}{i\cdot(i-1)\cdots 2\cdot 1}$$

$$= \frac{(m-i)}{i+1} \binom{m}{i}$$

we need to show that

$$\frac{\binom{m}{i}^2}{\cancel{\binom{2m}{m}}} \cdot \left(\frac{m-i}{\cancel{m}}\right)^2 = \frac{\binom{m}{i+1}^2}{\cancel{\binom{2m}{m}}} \cdot \left(\frac{i+1}{\cancel{m}}\right)^2$$

$$\Updownarrow$$

$$\binom{m}{i}^2 (m-i)^2 = \binom{m}{i+1}^2 (i+1)^2$$

$$\Updownarrow \quad \text{square root}$$

$$\binom{m}{i} (m-i) = \binom{m}{i+1} (i+1)$$

$$\Updownarrow$$

$$\binom{m}{i+1} = \frac{m-i}{i+1} \binom{m}{i}$$

This is $(*)$ and we are finished

Step 3: $j = i-1$

We just showed this. Direction doesn't matter.

The chain is time reversible and the limiting probabilities are as guessed in (b).

Q.E.D.