Exercise Class Week 44

Setting

We have a Poisson process $\{N(t), t \geq 0\}$, where ~~but~~ each event ~~can~~ ~~can~~ ~~takes~~ is one of k types: $1, 2, \ldots, k$.

Ex. [child births]

~~Birth~~

Assume that the birth of children ~~in a family~~ follows a Poisson process $\{N(t), t \geq 0\}$ with rate λ. Each child will be ~~a girl~~ a boy or a girl with probabilities

$$p_G = P(\text{child is a girl}) = 0.488$$

$$p_B = P(\text{child is a boy}) = 0.512$$

What can we say about

$$\{N_b(t), t \geq 0\}$$

the number of boys born by time t and

$$\{N_g(t), t \geq 0\}$$

the number of girls born by time t?

- Are they Poisson processes?
- What are their rates?
- Are they independent?

We will see the answer to this questions soon!

~~ate~~

Proposition 5.3 $\{N(t), t\geq 0\}$ is a Poisson process with rate λ.

If $N_i(t)$, $i=1,\dots,k$, represents the number of type i events occuring by time t, then $N_i(t)$, $i=1,\dots,k$, are independent Poisson random variables having means

$$E[N_i(t)] = \lambda\int_0^t P_i(s)\,ds,$$

~~wherer~~

where $P_i(s)$ is the probability of an event occuring at time s being classified as type i.

Note:

- For those interested this means that each Poisson process $\{N_i(t), t\geq 0\}$ is inhomogeneous with intensity function $\lambda_i(t) = \lambda P_i(t)$.
- ~~Loosely~~
- The functions $P_1, P_2, \dots, P_K$ describe how the full rate is divided into the different types of events at each time.

Ex. [child births] continued

We have constant probabilities for each type of event

$$\text{boy} \quad P_b(t) \equiv 0.512$$
$$\text{girl} \quad P_g(t) \equiv 0.488$$

Thus boys arrive as a Poisson process $\{N_b(t), t \geq 0\}$ with rate $\lambda_B = 0.512\lambda$ which is independent of the arrivals of girls which is a Poisson process $\{N_g(t), t \geq 0\}$ with rate $\lambda_G = 0.488\lambda$.

$\square$

Longer example: Example 5.20 [HIV infections]

HIV has a long incubation time, i.e. it takes a long time until symptoms appear. This makes it hard to determine the amount of people in the population that are infected.

Part 1: set up model

We will use the following model:

1) The number of people with HIV follows a Poisson process with ~~can~~ an unknown rate λ.
2) Each person infected ~~after~~ has a random time until symptoms appear. This time is independant for each person and follows an exponential distribution with mean $\mu = 10$.

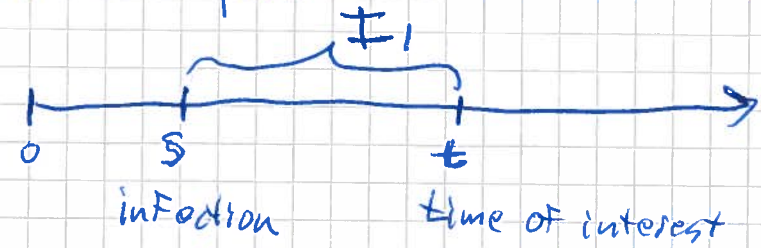
$\Rightarrow$ distribution function

$$G(t) = \int_0^t \frac{1}{10} e^{-y/10}\, dy$$

$$= 1 - e^{-t/10}.$$

Let $N_1(t)$ denote the number of individuals who have shown symptoms by time t. And let $N_2(t)$ denote the number of infected that has not shown symptoms by time t.

Assume person is infected at time $s<t$



Person is in $N_1(t)$ if symptoms appear in I_1, but in $N_2(t)$ if it takes more than $t-s$ time for symptoms to appear. Thus probability of being in 1 is

$$P_1(s) = P(\text{symptoms appear within } t-s)$$
$$= G(t-s)$$

This leads to

$$P_2(s) = 1 - P_1(s) = 1 - G(t-s)$$

By Proposition 5.3

$N_1(t)$ and $N_2(t)$ are independent Poisson variables with means

$$E[N_1(t)] = \lambda \int_0^t P_1(s)\,ds$$
$$= \lambda \int_0^t G(t-s)\,ds = \lambda \int_0^t G(y)\,dy$$
$$= \lambda \int_0^t (1 - e^{-y/10})\,dy$$
$$= \lambda \left[y + 10e^{-y/10} \right]_0^t$$
$$= \lambda (t - 10(1 - e^{-t/10}))$$

and

$$E[N_2(t)] = \lambda \int_0^t (1 - P_1(s))\, ds$$
$$= \lambda \cdot t - E[N_1(t)]$$
$$= \lambda \cdot 10 \left(1 - e^{-t/10}\right)$$

Part 2: Estimate $N_2(16)$

Assume we observe $N_1(16) = 220\,000$. Estimate the number of HIV infected without symptoms at time 16, $N_2(16)$.

2.1) ~~We need to~~ Estimate λ:

We know $N_1(16)$ and make the ~~simple~~ approximation

$$N_1(16) \approx E[N_1(16)]$$
$$= \lambda \left(16 - 10\left(1 - e^{-16/10}\right)\right)$$
$$= 8.02\, \lambda$$

and estimate λ by

$$\hat{\lambda} = \frac{N_1(16)}{8.02} = \frac{220000}{8.02}$$
$$\approx 27431$$

2) Estimate $N_2(16)$:

Make same approximation as in 1)

$$N_2(16) \approx E[N_2(16)]$$
$$= \lambda \cdot 10(1-e^{-16/10})$$
$$\approx 7.98\,\lambda$$

Insert estimate of λ

$$\widehat{N_2(16)} = 7.98\,\hat{\lambda}$$
$$= 7.98 \cdot 27431$$
$$\approx 219\,000$$

~~Kate~~

Part 3: Interpretation.

If 220 thousand people are infected and show symptoms by year 16, then we expect that approximately 219 thousand are infected, but have not yet developed symptoms.