Exercise Class Week 45

Problem 1, May 22, 2006 Exam

A factory has n production machines. The probability that a machine works at time t breaks down in the time interval $(t, t+h)$ is given by $\lambda h + o(h)$. The probability that a machine in repair at time t is fixed during the time interval $(t, t+h)$ is $\mu h + o(h)$. The n machines break down and are repaired independetly.

Let $N(t)$, $t \geq 0$, denote the number machines currently being repaired at time t.

a) Explain why $N(t)$ is a birth-death process with state space $\Omega = \{0, 1, \dots, n\}$, where the birth rates λ_i and death rates μ_i given by

$$\lambda_i = (n-i)\lambda \qquad \text{For } i = 0, 1, \dots, n$$
$$\mu_i = i\mu$$

Solution:

- There are n machines so the state space is obviously $\Omega = \{0, 1, \dots, n\}$
- The probabilities specified by the $o(h)$-notation implies that the time until breakdown is exponentially distributed with rate λ for each machine. And the time until repaired is exponentially distributed with rate μ for each broken machine.

Thus if $N(t) = i$, there are $n-i$ working machines and i broken machines. Each machine breaks with the same rate, so the rate for first break among the $n-i$ machines is

$$\lambda_i = (n-i)\lambda$$

Similarily for the broken machine each contribute with rate μ, so the total rate is

$$\mu_i = i\lambda$$

It is a birth-death process with the given rates since each "birth" and "death" is exponentially distributed with the specified rates.

b) Assume $N(0) = 0$. Let S denote the time until the first breakdown. Derive the probability density of S.

Solution:

The time untill the first transition is exponentially distributed. The only possible transition, $0 \to 1$, has rate $\lambda_0 = n\lambda$, which means

$$S \sim \text{Exp}(n\lambda) \implies f(s) = n\lambda e^{-n\lambda s}, \qquad s \geq 0$$

C) Consider one specific machine. Define a Markov chain $X(t)$, $t \geq 0$, by

$$X(t) = \begin{cases} 1, & \text{if machine is under repair at time } t \\ 0, & \text{if machine is working at time } t \end{cases}$$

Derive a set of differential equations that $P_{00}(t)$ and $P_{01}(t)$ satisfy. Solve these and show that

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$
$$P_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

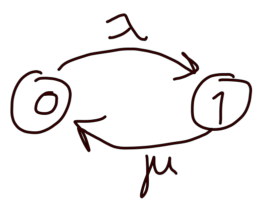
<u>Solution:</u>

We have a birth-death process with states 0 and 1

Birth rates are $\qquad$ Death rates are

$\lambda_0 = \lambda \qquad \mu_0 = 0$

$\lambda_1 = 0 \qquad \mu_1 = \mu$



We set-up the forward kolmogorov equations

[The following was given on the formula sheet of the exam

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$]

For our birth-death process

$$q_{01} = \lambda_0 = \lambda$$
$$q_{10} = \mu_1 = \mu$$

and

$$v_0 = q_{01} = \lambda$$
$$v_1 = q_{10} = \mu$$

We thus find the system of equations:

$$P_{00}'(t) = \mu P_{01}(t) - \lambda P_{00}(t) \quad (\text{I})$$
$$P_{01}'(t) = \lambda P_{00}(t) - \mu P_{01}(t) \quad (\text{II})$$

At time t we must either be in state 0 or 1, so

$$P_{00}(t) + P_{01}(t) = 1$$

Insert this into (I) to find

$$P_{00}'(t) = \mu(1 - P_{00}(t)) - \lambda P_{00}(t)$$

$$\Downarrow$$

$$P_{00}'(t) + (\lambda+\mu)P_{00}(t) = \mu$$

Need to know how to solve this type of equation!

Can use following formula from formula sheet on the exam

$$\left[\begin{array}{l} f'(t) + \alpha f(t) = g(t), \quad t \geq 0 \\ \text{with initial condition } f(0) = a \text{ has solution} \\ f(t) = a e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\,ds \end{array}\right]$$

Obviously $P_{00}(0) = 1$.

Further, $\alpha = \lambda + \mu$

$g(s) \equiv \mu$

We find

$$\begin{aligned} P_{00}(t) &= e^{-(\lambda+\mu)t} + \int_0^t e^{-(\lambda+\mu)(t-s)} \cdot \mu\,ds \\ &= e^{-(\lambda+\mu)t} + \left[\frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)(t-s)}\right]_{s=0}^{s=t} \\ &= e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}\left[1 - e^{-(\lambda+\mu)t}\right] \\ &= \frac{\mu}{\lambda+\mu} + \left[1 - \frac{\mu}{\lambda+\mu}\right] e^{-(\lambda+\mu)t} \\ &= \underline{\underline{\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}} \end{aligned}$$

Thus

$$\begin{aligned} P_{01}(t) &= 1 - P_{00}(t) \\ &= \underline{\underline{\frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}} \end{aligned}$$