Exercise Class — Week 38

Problem 3.38

Let $U \sim \mathcal{U}[0,1]$. Perform n trials that conditional on $U=u$ are independent with probability u for success.

1. Compute the mean of the number of successes.
2. Compute the variance of the number of successes.

Solution of 1:

Let X="Number of successes in the n trials". Conditional on $U=u$ there are n independent trials with common probability $p=u$ of success. This is the definition of the binomial distribution, so

$$X \mid U=u \sim \text{bin}(n,u).$$

With this set-up we can apply the double expectation formula together with

$$E[U] = \int_0^1 u \cdot 1\, du = \left[\tfrac{u^2}{2}\right]_0^1 = \tfrac{1}{2} \quad \text{and} \quad E[X \mid U] = n \cdot U.$$

We find

$$E[X] = E[E[X \mid U]] = E[n \cdot U] = n\,E[U] = \underline{\underline{n/2}}$$

Solution of 2:

In addition to properties in 1. we need $Var[X \mid U] = nU(1-U)$

We use the law of total variance

$$\begin{aligned}
Var[X] &= E[Var[X \mid U]] + Var[E[X \mid U]] \\
&= E[nU(1-U)] + Var[n \cdot U] \\
&= n[E[U] - E[U^2]] + n^2[E[U^2] - E[U]^2] \\
&\qquad \Big| \; E[U]=1/2 \text{ and } E[U^2] = \int_0^1 u^2 \cdot 1\, du = \left[\tfrac{u^3}{3}\right]_0^1 = 1/3 \\
&= n[1/2 - 1/3] + n^2[1/3 - 1/4] \\
&= \underline{\underline{n/6 + n^2/12}}
\end{aligned}$$

Problem 3.40

Prisoner trapped in a cell containing three doors.

Door 1: Leads back after 2 days
Door 2: Leads back after 3 days
Door 3: Leads immediately to freedom

a) Let D_i = "Prisoner selects door i". Assume $P(D_1) = 0.5$, $P(D_2) = 0.3$ and $P(D_3) = 0.2$. What is the expected number of days until he reaches freedom.

Solution:

Let X = "Days until the prisoner reaches freedom". Each time there are 3 choices and we may calculate the expected value by conditioning on each of them,

$$E[X] = E[X|D_1]P(D_1) + E[X|D_2]P(D_2) + E[X|D_3]P(D_3)$$
$$= (2+E[X])\cdot 0.5 + (3+E[X])\cdot 0.3 + 0\cdot 0.2$$
$$= 1.9 + 0.8E[X]$$
$$0.2E[X] = 1.9$$
$$\underline{\underline{E[X] = 9.5}}$$

b) Prisoner equally likely to choose each of the doors he hasn't already tried. What is the expected number of days until freedom?

Solution:

Let $X_{1,2,3}$ denote the number of days to freedom given doors 1,2,3 are unopened, $X_{1,3}$ the number of days to freedom given doors 1,3 are unopened and so on.

We then find

$$E[X_{1,2,3}] = \frac{1}{3}(2+E[X_{2,3}]) + \frac{1}{3}(3+E[X_{1,3}]) + \frac{1}{3}\cdot 0$$
$$= \frac{1}{3}\left(2+\frac{1}{2}(3+E[X_3]) + \frac{1}{2}\cdot 0\right) + \frac{1}{3}\left(3+\frac{1}{2}(2+E[X_3]) + \frac{1}{2}\cdot 0\right)$$
$$= \frac{1}{3}(2+3/2) + \frac{1}{3}(3+1)$$
$$= 2/3 + 1/2 + 4/3$$
$$\underline{\underline{= 2.5}}$$

Problem 4.5

Markov chain $\{X_n, n \geq 0\}$ with states 0,1,2 has transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

if $P(X_0 = 0) = P(X_0 = 1) = 1/4$, Find $E[X_3]$

Solution:

There are only 3 states so $P(X_0 = 2) = 1/2$.

To calculate $E[X_3]$ we need $P(X_3 = 0), P(X_3 = 1)$ and $P(X_3 = 2)$.

Each of these can be represented by

$$P(X_3 = i) = P(X_3 = i \mid X_0 = 0) P(X_0 = 0) + P(X_3 = i \mid X_0 = 1) P(X_0 = 1) + P(X_3 = i \mid X_0 = 2) P(X_0 = 2)$$

This expression requires the three-step transition probabilities. Found as the elements of P^3,

$$(P^3)_{kl} = P(X_3 = l \mid X_0 = k).$$

Calculations give

$$P^3 = \begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

Thus $P(X_3 = 0) = \frac{13}{36} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} + \frac{5}{12} \cdot \frac{1}{2} = \frac{59}{144}$

Similarily

$$P(X_3 = 1) = \frac{43}{216}$$

$$P(X_3 = 2) = 164/432$$

These probabilities give all the information needed,

$$E[X_3] = 0 \cdot P(X_3 = 0) + 1 \cdot P(X_3 = 1) + 2 \cdot P(X_3 = 2)$$

$$= 1 \cdot \frac{43}{216} + 2 \cdot \frac{164}{432} = \underline{\underline{53/54}}$$