Exercise Class — Week 39

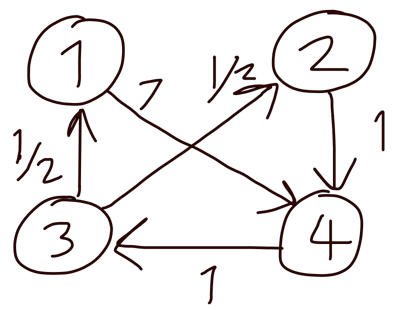
Problem 4.14

b)
$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Specify the classes of the Markov chain and determine wether they are transient or recurrent.

Solution:

Draw the Markov chain



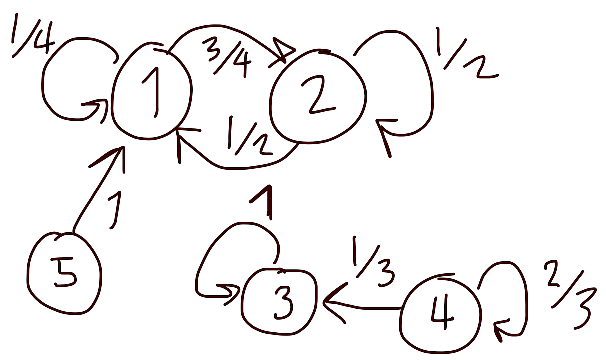
- All states communicate, i.e. it is possible to travel from every node to every node. Therefore, they all belong to the same class $\{1,2,3,4\}$.
- There must always be a recurrent class in a finite dimensional Markov chain. Therefore, $\{1,2,3,4\}$ is recurrent.

d)
$$P = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Specify the classes of the Markov chain and determine whether they are transient or recurrent.

Solution:

Draw the Markov chain



- From 1 you can only get to 2, and from 2 you can ot to 1. Therefore $\{1,2\}$ is one class in the chain. It is impossible to leave this class, so it is recurrent.

a recurrent class.

- From 4 you can only reach 3, but you can not come back. Therefore, $\{4\}$ is a transient class.
- From 5 you can only go to 1, but you cannot come back, so $\{5\}$ is transient

Problem 4.23

Each year has either good weather or bad weather. If the weather is good, the number of storms in is Poisson distributed with mean 1. If the weather is bad, the number of storms is Poisson distributed with mean 3. Any year's weather is only dependent of the previous year's weather. A good year is equally likely to be followed by a good year and a bad year. A bad year is twice as likely to be followed by bad year as by a good year.
Year 0 was good.

a) Find the expected number of storms during years 1 and 2.

Solution:

Let $X_i = \begin{cases} 0, & \text{if year } i \text{ is good} \\ 1, & \text{if year } i \text{ is bad} \end{cases}$

and let S_i = "Number of storms in year i".

We have $S_i \mid X_i = 0 \sim Poiss(1)$ and $S_i \mid X_i = 1 \sim Poiss(3)$.

The transition probability matrix is

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

and the two-step probability matrix is

$$P^{(2)} = \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix}$$

We calculate

1) $E[S_1] = E[S_1 \mid X_1 = 1]P(X_1 = 1 \mid X_0 = 0) + E[S_1 \mid X_1 = 0]P(X_1 = 0 \mid X_0 = 1)$
$= 3 \cdot 1/2 + 1 \cdot 1/2$
$= 2$

2) $E[S_2] = E[S_2 \mid X_2 = 1]P(X_2 = 1 \mid X_0 = 1) + E[S_2 \mid X_2 = 0]P(X_2 = 0 \mid X_0 = 1)$
$= 3 \cdot 7/12 + 1 \cdot 5/12$
$= 26/12$

Thus the expected total number of storms is

$$E[S_1 + S_2] = E[S_1] + E[S_2] = \underline{\underline{25/6.}}$$

b) Strange, skip this.

c) Find the long-run average number of storms per year.

Solution:

We find first the long run proportion of good years, π_1, and of bad, π_0.

The stationary probabilities must satisfy

$$(\mathrm{I})\quad \pi_0 = \pi_0 P(X_1=0|X_0=0) + \pi_1 P(X_1=0|X_0=1)$$
$$= \pi_0/2 + \frac{\pi_1}{3}$$

Additionally, they must sum to 1,

$$(\mathrm{II})\quad \pi_0 + \pi_1 = 1$$

Insert (II) into (I) to find

$$\pi_0 = \pi_0/2 + (1-\pi_0)/3$$
$$\left(1 - \frac{1}{2} + \frac{1}{3}\right)\pi_0 = 1/3$$
$$\underline{\underline{\pi_0 = 2/5}} \Rightarrow \underline{\underline{\pi_1 = 3/5}}$$

This means that in the stationary case

$$E[S] = E[S|X=0]P(X=0) + E[S|X=1]P(X=1)$$
$$= 1 \cdot 2/5 + 3 \cdot 3/5 = \underline{\underline{11/5}}$$

Problem on the web

On any given day Gary is either cheerful (0), so-so (1) or glum (2).
The transition probability matrix is

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

If Gary is currently in a cheerful mood, what is the expected number of days until the next time he is in a glum mood?

Solution:

Let X_i = "Number of days until first glum day starting in state i".
We have $X_2 = 0$.
Then by first step analysis

$$(\mathrm{I})\quad E[X_0] = E[1+X_0]P(0\to0) + E[1+X_1]P(0\to1) + E[1+X_2]P(0\to2)$$
$$= 1 + 0.5E[X_0] + 0.4E[X_1]$$

and

$$(\mathrm{II})\quad E[X_1] = E[1+X_0]P(1\to0) + E[1+X_2]P(1\to1) + E[1+X_2]P(1\to2)$$
$$= 1 + 0.3E[X_0] + 0.4E[X_1]$$

$$0.6\,E[X_1] = 1 + 0.3\,E[X_0]$$

$$E[X_1] = \frac{10}{6} + \frac{3}{6}\,E[X_0]$$

Insert into (I) to find

$$E[X_0] = 1 + 0.5\,E[X_0] + \frac{4}{10}\left(\frac{10}{6} + \frac{3}{6}E[X_0]\right)$$

$$\left(1 - 0.5 - \frac{4}{10}\cdot\frac{3}{6}\right)E[X_0] = 1 + \frac{4}{6}$$

$$\underline{\underline{E[X_0] = 50/9}}$$