1 Exercises from the book

- Chapter 3: 37, 41, 49, 54
- Chapter 4: 1, 2, 3, 10

2 Problem 4.10

This problem can be solved by making "glum mood" an absorbing state and then calculating the three-step transition probabilities. The modified transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

and the three-step transition probability matrix is

$$\mathbf{P}^3 = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.220 & 0.561 \\ 0 & 0 & 1.000 \end{bmatrix}.$$

The desired probability is then the probability of making a three-step transition from cheerful mood to either cheerful or so-so mood without being absorbed in the glum state, $0.293 + 0.292 = 0.585$.

3 Exercise 1

Simulate (using `R` or `Matlab`) a Markov chain with state space $\Omega = \{1, 2, \ldots, n\}$ with given transition matrix P. Use as initial distribution a uniform distribution on Ω.

Solution

R-code:

```
sim_mc <- function(P, len){

  # get the number of states
  n <- dim(P)[1]
  # define the initial probability vector
  mu <- rep(1/n, n)

  x <- c()
  # initialize the first state
  x[1] <- sample(1:n, 1, prob=mu)
```

```
  # use the Markov property to continue
  for(i in 2:len)
    x[i] <- sample(1:n, 1, prob=P[x[i-1],])

  return(x)
}
```

4 Exercise 2

Consider Example 4.4 on page 193 of the book. Given it **did not rain** on Monday and Tuesday what is the probability that it rains on Thursday?

Solution

The two-step transition matrix is given by:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{pmatrix} \end{array}$$

Since rain on Thursday is equivalent to the process being in either state 0 or 1 on Thursday, the desired probability is given by $P^2_{30} + P^2_{31} = 0.10 + 0.16 = 0.26$.

5 Exercise 3

Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\Omega = \{A, B\}$ and stationary transition matrix

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} \end{array}$$

The initial distribution is given by $P(X_0 = A) = 0.3$ and $P(X_0 = B) = 0.7$. Compute

a) $P(X_3 = A)$

b) $P(X_3 = A|X_0 = A)$

c) $P(X_3 = A|X_1 = B, X_0 = A)P(X_3 = A|X_1 = B, X_0 = A)$

d) $P(X_3 = A|X_2 = B, X_1 = B, X_0 = A)$

e) $P(X_6 = A|X_3 = A)$

f) $P(X_3 = A|X_6 = A)$

Solution

homogeneous $\rightarrow$ stationary, i.e. independent of t.

Hence, we can derive the following transition structures:

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} \end{array} \qquad \mathbf{P}^2 = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.52 & 0.48 \\ 0.36 & 0.64 \end{pmatrix} \end{array} \qquad \mathbf{P}^3 = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.392 & 0.608 \\ 0.456 & 0.544 \end{pmatrix} \end{array}$$

a) $P(X_3 = A) = P(X_0 = A) \cdot P(X_3 = A|X_0 = A) + P(X_0 = B) \cdot P(X_3 = A|X_0 = B) = 0.3 \cdot 0.392 + 0.7 \cdot 0.456 = 0.4368$

b) $P(X_3 = A|X_0 = A) = 0.392$.

c) $P(X_3 = A|X_1 = B, X_0 = A) \overset{\text{Markov}}{=} P(X_3 = A|X_1 = B) \overset{\text{homogeneous}}{=} P(X_2 = A|X_0 = B) = 0.36$

d) $P(X_3 = A|X_2 = B, X_1 = B, X_0 = A) = P(X_3 = A|X_2 = B) = P(X_1 = A|X_0 = B) = 0.6$

e) $P(X_6 = A|X_3 = A) = P(X_3 = A|X_0 = A) = 0.392$.

f)

$$\begin{aligned} P(X_3 = A|X_6 = A) &= \frac{P(X_3 = A, X_6 = A)}{P(X_6 = A)} \\ &= \frac{P(X_6 = A|X_3 = A)P(X_3 = A)}{P(X_6 = A|X_3 = A)P(X_3 = A) + P(X_6 = A|X_3 = B)P(X_3 = B)} \\ &= \frac{0.392 \cdot 0.4368}{0.392 \cdot 0.4368 + 0.456 \cdot 0.5632} = 0.4 \end{aligned}$$