1 Exercises from the book

2 Exercise 1

Given a homogeneous Markov chain, where the transition matrix **P** depends on a parameter p given by

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & p & 0 & 0.8-p \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.8 \\ 0.1 & p & 0.1 & 0.8-p \end{pmatrix} \end{array}$$

For which value of p is the Markov chain not irreducible?

Solution

A Markov chain is called irreducible if all states are mutually accessible (communicate), i.e. the probability to get from state i to state j at some point is positive for all i, j. In the example given here this property is not fulfilled for $p = 0.8$, as state 3 and 4 cannot be reached from states 1 and 2.

3 Exercise 2

Consider a Markov chain with state space $\Omega = 1, 2, 3, 4, 5, 6$ and transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0.25 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix} \end{array}$$

Determine:

a) The period of each state

b) Which states are transient

c) Which states are ergodic

d) The equivalence classes.

Solution

a) All state have period 1 $\Rightarrow$ the Markov chain is aperiodic.

b) States 3 and 4 are transient, all other states are recurrent.

c) States 1, 2, 5 and 6 are ergodic.

d) There are three equivalence classes are: $\{1,2\},\{3,4\},\{5,6\}$.

4 Exercise 3

Consider a Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

b) Determine the mean time of absorption given the process starts in state 1.

Solution

a) Let $\{X_n, n \geq 0\}$ be a Markov chain with the given transition probability matrix, and let

P_i = "Probability that a chain that starts in state i is absorbed in state 0".

Note immediately that $P_0 = 1$ and $P_3 = 0$.
First step analysis for state 1 gives

$$\begin{aligned} P_1 &= P_0 P(X_1 = 0|X_0 = 1) + P_1 P(X_1 = 1|X_0 = 1) + P_2 P(X_1 = 2|X_0 = 1) + P_3 P(X_1 = 3 X_0 = 1) \\ &= 1 \cdot 0.1 + 0.4 P_1 + 0.1 P_2 + 0. \end{aligned}$$

Similarly for state 2,

$$\begin{aligned} P_2 &= P_0 P(X_1 = 0|X_0 = 2) + P_1 P(X_1 = 1|X_0 = 2) + P_2 P(X_1 = 2|X_0 = 2) + P_3 P(X_1 = 3|X_0 = 2) \\ &= 1 \cdot 0.2 + 0.1 P_1 + 0.6 P_2 + 0 \end{aligned}$$

Solving this linear system of two equations in two variables gives

$$P_1 = 0.2609.$$

b) Define

$$Y_i = \text{"Steps until absorption in state 0 or state 2"}.$$

Then proceed as in the previous problem conditioning on the first step. For state 1 we find

$$\begin{aligned}
\mathrm{E}[Y_1] &= 1 + \mathrm{E}[Y_1]P(X_1 = 1|X_0 = 1) + \mathrm{E}[Y_2]P(X_1 = 2|X_0 = 1) \\
&= 1 + 0.4\mathrm{E}[Y_1] + 0.1\mathrm{E}[Y_2],
\end{aligned}$$

and for state 2 we find

$$\begin{aligned}
\mathrm{E}[Y_2] &= 1 + \mathrm{E}[Y_1]P(X_1 = 1|X_0 = 2) + \mathrm{E}[Y_2]P(X_1 = 2|X_0 = 2) \\
&= 1 + 0.1\mathrm{E}[Y_1] + 0.6\mathrm{E}[Y_2].
\end{aligned}$$

Solving this system gives the mean time of absorption starting in state 1,

$$\mathrm{E}[Y_1] = 2.17.$$