

### III

## A Bayesian framework for integrated reservoir characterization



# A Bayesian Framework for Integrated Reservoir Characterization

Håkon Tjelmeland      Henning Omre

Department of Mathematical Sciences  
Norwegian University of Science and Technology  
Trondheim, Norway

## Abstract

At the stage of reservoir evaluation, general reservoir information and reservoir specific observations are available. The inherent difference between these types of information is emphasized. The integration of the information is done in a Bayesian framework. The requirement that sampling from the posterior must be possible put serious constraints on the model formulation. These constraints are thoroughly discussed and illustrated by two examples. The first describes a model for petrophysical variables within a Gaussian family and substantial analytical treatment is feasible. For the second, in which a fairly complex stochastic model for shale units is a shale matrix is defined, analytical computations are intractable. However, the posterior distribution can be explored by sampling-based techniques.

## 1 Introduction

The primary objective of petroleum reservoir evaluation is to forecast future production of oil and gas in the reservoir under study. The procedure for providing these forecasts consists of two steps. First, establish a representation of relevant reservoir characteristics based on available information. Second, simulate the petroleum production from the reservoir by numerical simulation of fluid flow through this representation of the reservoir characteristics. In order to assess the uncertainty in the production forecasts, a stochastic model for the reservoir characteristics is defined. The predicted uncertainty is then obtained by Monte Carlo sampling from this model and thereafter simulation of fluid flow; see Lia et al (1995). The focus of this article is on stochastic modeling of the reservoir characteristics and the sampling techniques available.

The stochastic model for characteristics of the reservoir under study must be based on two types of information: (i) general reservoir information and (ii) reservoir-specific observations. The former consists of geological and petrophysical process understanding and observations in geological analogues and comparable reservoirs. The latter consists of observations in the reservoir under study as measurements in wells, data from seismic surveys and previous production history. Note that the former is experience-based while the latter are observations in the actual reservoir. The challenge is to combine these two types of information in an optimal manner for production forecasts and from a statistical point of view a Bayesian approach seems reasonable.

Heterogeneity modeling along the lines of Haldorsen and Lake (1984) concerns bias correction of the production forecasts. A large number of publications have been presented on this topic the last decade; see Haldorsen and Damsleth (1990), Omre (1991) and Dubrule (1993) for good overviews. Assessment of uncertainty in production forecasts has only recently been thoroughly discussed in the petroleum literature; see Omre et al (1993a) and

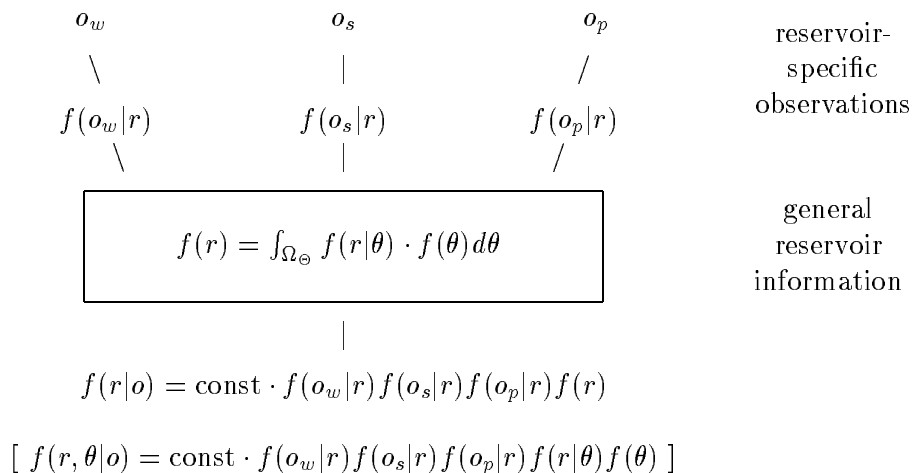


Figure 1: Outline of model used to obtain the posterior distribution

Journal (1994). The objective of this article is to contribute to formalization of uncertainty assessments in production forecasts through formalizing the stochastic modeling of reservoir characterization.

This article contains a discussion and formalization of how the general reservoir information and reservoir-specific observations should be combined. This is done in a Bayesian framework. It is required that sampling can be performed from the resulting stochastic model and this puts constraints on the complexity of the model. These constraints are thoroughly discussed. Two examples are presented, one where substantial analytical treatment is feasible and one where sampling-based methods must be used. Finally, we put forward some recommendations for stochastic modeling of reservoir characterization.

## 2 Model formulation

Consider a particular petroleum reservoir under study. Let  $R$  denote the reservoir characteristics relevant for the evaluation. The random field  $R$  is usually multivariate to represent characteristics like porosity, horizontal and vertical permeability, initial oil and gas saturations, fluid properties etc. Moreover, it must be spatial in order to represent the spatial variability of the characteristics mentioned above.

The reservoir characteristics,  $R$ , is largely unknown at the stage of evaluation. The available information consists of general reservoir information and reservoir-specific observations, listed above as (i) and (ii), respectively. Based on the former, a prior stochastic model for the reservoir characteristics,  $R$ , must be established. The model must be conditioned on the actual reservoir-specific observations made. Hence, the goal of the evaluation is to obtain properties of the posterior stochastic model for the reservoir characteristics conditioned on the available observations, either analytically or at least by sampling. An outline for obtaining this posterior stochastic model is given in Figure 1. The reservoir-specific observations are of three types: observations in wells, denoted by  $O_w$ ; seismic data, denoted by  $O_s$ ; and production tests and history, denoted by  $O_p$ . Denote them by  $O = (O_w, O_s, O_p)$ . For the

reservoir under study, let the observed values be  $o_w$ ,  $o_s$  and  $o_p$ , respectively. In order to link these reservoir-specific observations to the reservoir characteristics,  $R$ , the acquisition procedures must be modeled through likelihood functions. For well observations, the likelihood function  $f(o_w|r)$  specifies the pdf of observing  $o_w$  given that the true reservoir characteristics are  $r$ . The associated conditional stochastic variable can be written as  $O_w = g_w(r) + U_w$ , where  $g_w(r)$  is the expected response in the well data acquisition equipment to the reservoir characteristics  $r$ , and  $U_w$  is some stochastic error term. The function  $g_w(r)$  is often termed the transfer function and it is a subject of intensive research among petrophysicists. Correspondingly,  $f(o_s|r)$  and  $f(o_p|r)$  are likelihood functions for seismic data and production test and history, respectively. The associated  $g_s(r)$  for seismic data can be derived from wave equations along the lines adopted in the geophysics community. The  $g_p(r)$  is defined from Darcy's law and may be represented by a fluid flow simulator. Note that the formalism allows a trade-off between refinement in the mathematical modeling of  $g_w(\cdot)$ ,  $g_s(\cdot)$  and  $g_p(\cdot)$  and the size of the variance in the respective error terms  $U_w$ ,  $U_s$  and  $U_p$ . If rough models for the former is used with associated larger variance in the latter, the reservoir-specific observations,  $o_w$ ,  $o_s$  and  $o_p$ , are less appreciated in the evaluation.

The available reservoir-specific observations are associated with the reservoir characteristics,  $R$ , through the likelihood functions, see Figure 1. Hence, to determine  $R$  appears as an inverse problem. Unfortunately, the reservoir characteristics,  $R$ , are of very high dimension due to its multivariate, spatial nature and the problem appears ill-posed. This entails that  $R$  cannot be uniquely determined by the reservoir-specific observations,  $o_w$ ,  $o_s$  and  $o_p$ . The statistical solution to this involves assigning a prior stochastic model to the reservoir characteristics.

The prior stochastic model for  $R$  is represented by a pdf  $f(r)$ . It must be assessed from general reservoir information consisting of geological process understanding, observations from analogues and experience from comparable reservoirs. Hence, it is based on the prior conceptions of the geoscientists before measurements in the reservoir under study are made. It is crucial for the evaluation that  $f(r)$  gives a realistic description of the prior uncertainty about  $R$ . This is most easily obtained by using a hierarchical model for  $f(r)$ , i.e. to let the distribution for  $R$  depend on a set of model parameters  $\theta$ , which are themselves considered as stochastic with an associated prior pdf,  $f(\theta)$ . In order to make the assessment of  $f(\theta)$  easier, it is important to parameterize the distribution for  $R$  so that  $\theta$  is easily interpretable, for example as expected values for poro-perm characteristics, net-to-gross values and expected sizes of facies objects. The prior distribution for  $\theta$  can then, for example, be assessed by requesting geoscientists to identify analogues and other reservoirs that are comparable to the reservoir of interest and to utilize experience and observations from these to determine  $f(\theta)$ . Based on this, the following relation is defined

$$f(r) = \int_{\Omega_{\Theta}} f(r|\theta) \cdot f(\theta) d\theta, \quad (1)$$

where  $\Omega_{\Theta}$  is the sample space of the model parameters  $\Theta$ .

The conditional pdf  $f(r|\theta)$  represents the uncertainty in the reservoir characteristics for given values of the model parameters. This corresponds to the geometrical rearrangements of characteristics and can be termed heterogeneity uncertainty. The pdf  $f(\theta)$  represents the prior uncertainty in the model parameters and can be termed model parameter uncertainty. Note that the former, heterogeneity uncertainty, corresponds to the heterogeneity evaluation approach as initiated by Haldorsen and Lake (1984). The latter, model parameter uncertainty,

appears to be very important whenever assessment of uncertainty is the objective; see Omre et al (1993a).

With likelihood and prior as defined above, the posterior distribution for the reservoir characteristics is given by

$$f(r|o) = \text{const} \cdot f(o_w|r) \cdot f(o_s|r) \cdot f(o_p|r) \cdot \int_{\Omega_\Theta} f(r|\theta) \cdot f(\theta) d\theta, \quad (2)$$

by assuming  $O_w$ ,  $O_s$  and  $O_p$  to be conditionally independent for given  $R$ . Note that  $O_w$ ,  $O_s$  and  $O_p$  are actually three blocks of observations from the well, seismic and production data acquisition tools, respectively. Each block contains multivariate observations that are highly dependent. The three blocks,  $O_w$ ,  $O_s$  and  $O_p$  are based on different tools and procedures and hence, it is reasonable to consider them to be conditionally independent given the reservoir characteristics.

The posterior pdf of the reservoir characteristics,  $f(r|o)$ , is obtained from the stochastic models previously defined. Analytical treatment of the posterior distribution is feasible in very few cases only. However, sampling is possible from a fairly large class of models. But in the general case even sampling is totally intractable. This is the topic of the following sections.

The framework for stochastic reservoir evaluation defined in this section is along the lines of traditional statistical modeling. Ill-posed inverse problems are frequently observed and introduction of a prior to compensate for this is often done. Bayesian approaches are in frequent use whenever few observations and extensive expert experience are available. The framework is familiar but the problem of reservoir characterization has several special features. First of all, the target variable in reservoir characterization,  $R$ , is of extremely high dimension due to its multivariate and spatial nature. The amount of information in the reservoir observations normally vary considerably, spatially. Close to wells the observations almost uniquely define the reservoir characteristics but further away from wells the observations carry much less information. This makes specification of the prior very important. The likelihood functions, including the transfer functions, normally are very complex and require the solution of large sets of differential equations. Reliable numerical solutions to these require large computer resources. Luckily, considerable geoscientific knowledge is available about reservoir characteristics. Moreover, the reservoir is usually only a small unit in a larger regional geological structure and hence analogues can be identified. Once the posterior stochastic model for the reservoir characteristics is established, there is a need to sample from it in order to make further processing possible, rather than identifying an optimal reservoir characterization based on, for example, the maximum a posteriori criterion. Finally, reservoir evaluation is a dynamic task with frequent updates of the posterior stochastic model as more reservoir-specific observations become available through infill drilling, additional seismic surveys and current production history. The real challenge is to adapt the framework defined above so that all features mentioned are accounted for in a balanced manner.

### 3 Stochastic simulation

The stochastic model of interest is the posterior pdf for the reservoir characteristics given the reservoir-specific observations,  $f(r|o)$ , specified in equation (2). For most models relevant in reservoir characterization, the integral in this expression cannot be determined analytically

and this makes direct sampling from the distribution problematic. But the problem can be avoided by instead consider the joint posterior pdf  $f(r, \theta|o)$ . This distribution is given by

$$f(r, \theta|o) = \text{const} \cdot f(o_w|r) \cdot f(o_s|r) \cdot f(o_p|r) \cdot f(r|\theta) \cdot f(\theta) \quad (3)$$

and contains no problematic integrals. Moreover, if  $(r, \theta)$  is a sample from  $f(r, \theta|o)$ ,  $r$  is obviously a sample from the marginal pdf  $f(r|o)$  of interest.

Sampling from a pdf is most efficiently done whenever the pdf can be factorized into lower dimensional pdf's, preferable one-dimensional ones. If  $T = (R, \Theta) = (T^1, \dots, T^n)$  is a decomposition of the reservoir characteristics and associated model parameters into univariate random variables, the joint posterior distribution above can be expressed as

$$f(r, \theta|o) = f(t|o) = f(t^1|o) \cdot f(t^2|t^1, o) \cdot \dots \cdot f(t^n|t^{n-1}, \dots, t^1, o). \quad (4)$$

The sequential simulation algorithm of Gómez-Hernández and Journel (1993) or Omre et al (1993b), can then be used. But this requires that the following integrals can be determined analytically for  $i = 2, \dots, n$

$$\int_{\Omega_{T^i}} \dots \int_{\Omega_{T^n}} f(t|o) dt^{i+1} \cdot \dots \cdot dt^n, \quad (5)$$

where  $\Omega_{T^i}$  is the sample space of the random variable  $T^i$ . This is a serious constraint and can be dealt with only for very particular models, most commonly specific models from the Gaussian and Poisson families. The next section provide an example in the Gaussian case.

Sampling from complex stochastic models has in recent years been a subject of intensive research in the statistical community. For reviews of the theory and techniques, see Besag and Green (1993) and Neal (1993). Markov chain Monte Carlo techniques have been the central topic and the Metropolis-Hastings algorithm is the most prominent class of algorithms. All these algorithms have in common that they are iterative and that convergence, i.e. sampling from the correct stochastic model, is ensured in the limit only. The advantage is that sampling from relatively complex stochastic models can be performed. Note that the simulated annealing algorithm, in the geostatistical community frequently used for simulation (Deutsch, 1993; Hegstad et al, 1993), belongs to the class of Markov chain Monte Carlo techniques.

The Metropolis-Hastings algorithm is iterative and each iteration consists of two steps. Let  $t = (r, \theta)$  denote the current state in an algorithm for sampling from  $f(t|o)$ . Each iteration then consists of first (i) drawing a potential new state  $\tilde{t}$  according to a transition matrix  $Q(t \rightarrow \tilde{t})$  and (ii) accepting  $\tilde{t}$  as the new current state with probability

$$A(t \rightarrow \tilde{t}) = \min \left\{ 1, \frac{f(\tilde{t}|o) Q(\tilde{t} \rightarrow t)}{f(t|o) Q(t \rightarrow \tilde{t})} \right\}, \quad (6)$$

otherwise  $t$  is kept as the current state. Only very weak restrictions exist on the choice of  $Q(t \rightarrow \tilde{t})$  and this makes it a very flexible algorithm. An important property of the procedure is also that the pdf  $f(t|o)$  need to be known up to a normalizing constant only because the constant cancels in the expression for the acceptance probability,  $A(t \rightarrow \tilde{t})$ . Hence, the posterior pdf must be of the form

$$f(r, \theta|o) = \text{const} \cdot h(r, \theta), \quad (7)$$

where  $h(r, \theta)$  is a known function. This requires the normalizing constants in the pdf's  $f(r|\theta)$ ,  $f(o_w|r)$ ,  $f(o_s|r)$  and  $f(o_p|r)$  to be known because they are functions of the conditioning variables,  $\theta$  and  $r$ . However, the fact that no similar restriction is necessary for the normalizing constant in the pdf for the model parameters,  $f(\theta)$ , still gives a lot of flexibility in the choice of prior stochastic model.

To determine the normalizing constant in  $f(r|\theta)$  analytically is difficult in the general case, since integrating over the multivariate, spatial random variable  $r$  is usually non-trivial. Hence, this constitutes a real constraint. In practice, one has to assume that  $f(r|\theta)$  belongs to the Gaussian or Poisson families of random fields with parameters  $\theta$ . Note, however, that the pdf of the stochastic model parameters  $\Theta$  can be chosen arbitrarily without constraints. This entails that the prior pdf for the reservoir characteristics,  $f(r) = \int f(r|\theta) \cdot f(\theta) d\theta$ , still belongs to a fairly large class of pdf's.

One could envisage to determine the normalizing constant in  $f(r|\theta)$  by sampling-based techniques (Heikkinen and Högmänder, 1994; Higdon et al, 1995) but this is computationally prohibitive unless  $\theta$  is very low-dimensional.

To determine the normalizing constants in  $f(o_w|r)$ ,  $f(o_s|r)$  and  $f(o_p|r)$  analytically is of course also impossible in the general case but with reasonable assumptions this can normally be done. If the observation error follows familiar parametric classes of pdf's, for example Gaussian, with parameters dependent on  $r$ , the normalizing constant is usually analytically available. But note that the assumptions, in the case of additive noise, do not involve the transfer functions,  $g_w(r)$ ,  $g_s(r)$  and  $g_p(r)$ . An example where the Metropolis-Hastings algorithm is used for sampling is provided in the next section.

As previously mentioned, Markov chain Monte Carlo algorithms only give samples from the specified distribution in the limit. It therefore becomes essential to decide when a sufficiently good approximation is reached and much research has concentrated on this topic. One approach used is to look for theoretical bounds on the number of iterations necessary to reach within a given distance from the specified distribution (Meyn and Tweedie, 1994; Rosenthal, 1994) but so far no good bounds seem to exist for models of the complexity necessary in reservoir characterization. A technique frequently used also for complex models is output analysis; see Ripley (1981) and references therein. One approach is to plot important univariate characteristics of the realizations against number of iterations and to look for when they seem to have stabilized statistically. Alternatively, if some property of the pdf is analytically known, this can be utilized by comparing the theoretically known properties with corresponding estimated values. The latter approach is used in the last example in the next section.

## 4 Examples

In this section we present two examples of Bayesian models for reservoir characteristics. The first example uses a Gaussian random field to model the spatial distribution of petrophysical variables and this makes analytical decomposition of the posterior distribution possible. In the second example, a Poisson marked point field is used to model small shale units and here we must resort to iterative algorithms for sampling. It should be noted that the primary intention of including the examples is to illustrate the discussion of the previous sections, not to introduce completely realistic models for reservoir evaluation. Still, however, the models contain important aspects for reservoir characterization.

## 4.1 Petrophysical variables

Let the reservoir characteristics of interest,  $R = (R(x_1), \dots, R(x_n))^T$ , be the value of a petrophysical variable, porosity say, within a facies, where  $R(x_i)$  denotes the porosity value at a location  $x_i$ , contained in a domain  $\mathcal{V}$ . The locations  $x_1, \dots, x_n$  are typically nodes in a 2D-, or 3D-lattice but this is not crucial in what follows. Modeling of the spatial distribution of petrophysical variables are frequently addressed in the geostatistical literature; see, for example, Haldorsen and Damsleth (1990).

As frequently done in reservoir characterization, we assume the petrophysical variables to be Gaussian distributed, possibly after a suitable transformation. Thus, the porosity values  $R(x_1), \dots, R(x_n)$  are a subset of a Gaussian random field  $\{R(x); x \in \mathcal{V}\}$  given by

$$R(x) = \sum_{i=1}^k \mu_i \cdot f_i(x) + \varepsilon(x) \quad (8)$$

$$\text{Cov}\{\varepsilon(x), \varepsilon(x') | \theta\} = \sigma^2 \cdot \rho(x, x'), \quad (9)$$

where  $\theta = (\mu, \sigma)$ ,  $\mu = (\mu_1, \dots, \mu_k)^T$  are model parameters,  $f_1(\cdot), \dots, f_k(\cdot)$  are known trend functions and  $\varepsilon(\cdot)$  is a Gaussian random field with zero expectation and covariance function given by equation (9).  $\rho(\cdot, \cdot)$  is a correlation function. Thus, the vector  $R$  has a multi-Gaussian distribution with expectation vector and covariance matrix given by

$$\text{E}\{R | \theta\} = F\mu \quad (10)$$

$$\text{Cov}\{R | \theta\} = \sigma^2 \Sigma_\varepsilon, \quad (11)$$

where  $F$  is an  $n \times k$ -matrix with element  $(i, j)$  equal to  $f_j(x_i)$  and  $\Sigma_\varepsilon$  is an  $n \times n$ -correlation matrix with element  $(i, j)$  equal to  $\rho(x_i, x_j)$ .

The mean porosity level and the variability of the porosity values vary substantially between different petroleum reservoirs. Hence, the model parameters,  $\theta = (\mu, \sigma)$ , are considered stochastic and assigned a prior pdf,  $f(\theta)$ , which we assume to be of the form

$$f(\theta) = f(\sigma) \cdot f(\mu | \sigma), \quad (12)$$

where  $f(\sigma)$  is a univariate prior distribution for  $\sigma$  and  $f(\mu | \sigma)$  is a multi-Gaussian distribution with an expectation vector  $\mu_0$  and covariance matrix  $\sigma^2 \Sigma_\mu$ , where  $\Sigma_\mu$  is a given  $k \times k$ -matrix.

Observations of the three types,  $O_w, O_s$  and  $O_p$ , are available and we assume the associated transfer functions to be linear and the noise to be additive and multi-Gaussian, i.e.

$$O_w = A_w r + \eta_w, \quad (13)$$

$$O_s = A_s r + \eta_s \quad (14)$$

and

$$O_p = A_p r + \eta_p, \quad (15)$$

where  $A_w, A_s$  and  $A_p$  are given matrices of dimensions  $m_w \times n, m_s \times n$  and  $m_p \times n$ , and  $\eta_w, \eta_s$  and  $\eta_p$  are  $m_w$ -,  $m_s$ - and  $m_p$ -dimensional multi-Gaussian random vectors with zero expectation and covariance-matrices  $\Sigma_\eta^w, \Sigma_\eta^s$  and  $\Sigma_\eta^p$ , respectively. Of course, the matrices  $A_w$  and  $A_s$  are sparse with a lot of zero elements. For row number  $i$  of  $A_w$ , for example, the non-zero elements  $(A_w)_{ij}$  correspond to locations  $x_j$  in the vicinity of the position for well observation number  $i$ ,  $(O_w)_i$ .

The assumptions in equation (13) and (14) for linear transfer functions for well observations and the seismic data are quite frequently made and it is often a good approximation to reality. See, for example, Haas and Dubrule (1994) where the seismic data in each vertical trace are assumed to be given from petrophysical variables by convolution with a wavelet. For the production data,  $o_p$ , the linear assumption is more questionable but is made here to enable analytical treatment of the posterior distribution.

To simplify notation, write

$$O = Ar + \eta, \quad (16)$$

where

$$A = \begin{bmatrix} A_w \\ A_s \\ A_p \end{bmatrix} \quad (17)$$

is a  $m \times n$  matrix, for  $m = m_w + m_s + m_p$ , and  $\eta$  is multi-Gaussian with zero expectation and covariance matrix

$$\Sigma_\eta = \begin{bmatrix} \Sigma_\eta^w & 0 & 0 \\ 0 & \Sigma_\eta^s & 0 \\ 0 & 0 & \Sigma_\eta^p \end{bmatrix}. \quad (18)$$

The model considered here is a minor generalization of a model discussed in Hjort and Omre (1994), where the observations are assumed exact, i.e.  $\Sigma_\eta = 0$ ; see also Omre and Halvorsen (1989) and Omre et al (1989).

With the likelihood functions and the prior pdf specified above, the posterior pdf becomes

$$\begin{aligned} f(r, \theta | o) &\propto f(\theta) \cdot f(r | \theta) \cdot f(o | r) \\ &\propto f(\sigma) \cdot \frac{1}{\sigma^k} \exp \left\{ -\frac{1}{2\sigma^2} (\mu - \mu_0)^T \Sigma_\mu^{-1} (\mu - \mu_0) \right\} \\ &\cdot \frac{1}{\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} (r - F\mu)^T \Sigma_\varepsilon^{-1} (r - F\mu) \right\} \cdot \exp \left\{ -\frac{1}{2} (o - Ar)^T \Sigma_\eta^{-1} (o - Ar) \right\}. \end{aligned} \quad (19)$$

Using that any linear combination of multi-Gaussian distributed variables are Gaussian, it is straightforward to factorize this posterior distribution into

$$f(r, \theta | o) = f(\sigma | o) \cdot f(\mu | \sigma, o) \cdot f(r | \theta, o), \quad (20)$$

where  $f(\sigma | o)$  is given by

$$\begin{aligned} f(\sigma | o) &\propto \frac{f(\sigma)}{\sqrt{|\sigma^2 A (F \Sigma_\mu F^T + \Sigma_\varepsilon) A^T + \Sigma_\eta|}} \\ &\cdot \exp \left\{ -\frac{1}{2} (o - AF\mu_0)^T \left( \sigma^2 A (F \Sigma_\mu F^T + \Sigma_\varepsilon) A^T + \Sigma_\eta \right)^{-1} (o - AF\mu_0) \right\} \end{aligned} \quad (21)$$

and  $f(\mu | \sigma, o)$  and  $f(r | \theta, o)$  are both multi-Gaussian distributions with expectation vectors and covariance matrices given by

$$\begin{aligned} E\{\mu | \sigma, o\} &= \left( \frac{1}{\sigma^2} \Sigma_\mu^{-1} + F^T A^T \left( \sigma^2 A \Sigma_\varepsilon A^T + \Sigma_\eta \right)^{-1} A F \right)^{-1} \\ &\cdot \left( \frac{1}{\sigma^2} \Sigma_\mu^{-1} \mu_0 + F^T A^T \left( \sigma^2 A \Sigma_\varepsilon A^T + \Sigma_\eta \right)^{-1} o \right) \end{aligned} \quad (22)$$

$$\text{Cov}\{\mu|\sigma, o\} = \left( \frac{1}{\sigma^2} \Sigma_\mu^{-1} + F^T A^T (\sigma^2 A \Sigma_\varepsilon A^T + \Sigma_\eta)^{-1} A F \right)^{-1} \quad (23)$$

$$\text{E}\{R|\theta, o\} = \left( \frac{1}{\sigma^2} \Sigma_\varepsilon^{-1} + A^T \Sigma_\eta^{-1} A \right)^{-1} \cdot \left( \frac{1}{\sigma^2} \Sigma_\varepsilon^{-1} F \mu + A^T \Sigma_\eta^{-1} o \right) \quad (24)$$

$$\text{Cov}\{R|\theta, o\} = \left( \frac{1}{\sigma^2} \Sigma_\varepsilon^{-1} + A^T \Sigma_\eta^{-1} A \right)^{-1}. \quad (25)$$

Thus, simulation from the posterior distribution can be done by first sampling a univariate  $\sigma$  from  $f(\sigma|o)$ , thereafter sampling a  $k$ -variate  $\mu$  from  $f(\mu|\sigma, o)$  and finally an  $r$  from  $f(r|\theta, o)$ . To sample from  $f(\sigma|o)$ , a Markov chain Monte Carlo algorithm can be used. But note that each iteration involve matrix operations on matrices of size  $m \times m$  and is time consuming if the number of observations,  $m$ , is large. In the case of exact observations the expression for  $f(\sigma|o)$  simplifies and matrix operations in each iteration become unnecessary; see Hjort and Omre (1994). The parameter vector  $\mu$  can easily be simulated from the  $k$ -variate Gaussian distribution  $f(\mu|\sigma, o)$  and  $R$  can be simulated from the multi-Gaussian distribution  $f(r|\theta, o)$  by use of geostatistical simulation techniques; see Journel and Huijbregts (1978) and Gómez-Hernández and Journel (1993).

## 4.2 Shale units

A stochastic model for shale units in a sand reservoir was first presented in the trend-setting paper of Haldorsen and Lake (1984). They used a simple marked Poisson point field conditioned on observations in wells; see also Lien et al (1992) where a similar model is used. Here, we discuss a model inspired by the model of Haldorsen and Lake but allow for clustering of shale units and conditioning on seismic- and production-data. We consider the 2D-case but a similar 3D-model is also possible.

The prior model for the shale units is defined by a marked point field; see Stoyan and Stoyan (1994). The set of shales in a domain  $\mathcal{D}$  is defined by  $R = \{N; R_1, \dots, R_N\}$  where  $N$  is the number of shale units and  $R_i = (X_i, A_i, L_i)$  specifies shale unit number  $i$ . Each shale unit is in the model represented as a line segment centered at a position  $X_i \in \mathcal{D}$ , it has a dip angle  $A_i$  relative to the horizontal plane and length  $L_i$ .

The shale units are assumed to appear in clusters which are repulsive to each other. This is included in the prior model,  $f(r)$ , by assuming the locations in  $R$  to be within the class of doubly stochastic Poisson point fields or Cox processes; see Cox (1955) and Grandell (1976). More precisely, let the number of shale units,  $N$ , and their locations,  $X_1, \dots, X_N$ , conditioned on a parameter vector  $\theta$ , be a non-stationary Poisson point field with intensity function given by

$$\lambda(\theta, x) = \lambda_0 + \lambda_1 \sum_{i=1}^q I \left[ \left( \frac{u_i - x^1}{a_i} \right)^2 + \left( \frac{v_i - x^2}{b_i} \right)^2 \leq 1 \right] \quad (26)$$

for  $x = (x^1, x^2) \in \mathcal{D}$ , where  $\theta = (\lambda_0, \lambda_1, \psi)$ ,  $\psi = \{q; (u_i, v_i, a_i, b_i), i = 1, \dots, q\}$  and  $I[\cdot]$  is the indicator function. Thus,  $\psi$  represents a set of ellipses in  $\mathcal{D}$ ,  $q$  is the number of ellipses and  $(u_i, v_i)$  and  $(a_i, b_i)$  are the location and half axis, respectively, of ellipse number  $i$ . The intensity at location  $x \in \mathcal{D}$  is thus given as a background intensity  $\lambda_0$  plus an intensity  $\lambda_1$  for each ellipse covering  $x$ ; see Figure 2. The marks  $(A_i, L_i)$  are assumed to be independent of location and to have pdf  $f(a_i, l_i)$ . This defines the conditional pdf for the

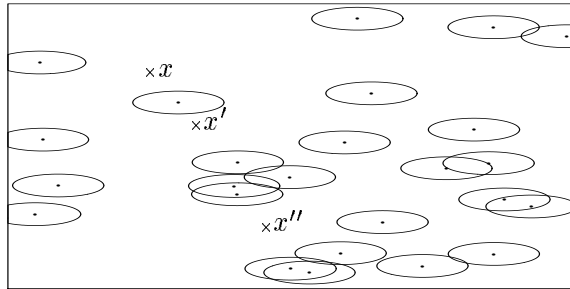


Figure 2: A realization of the marked point field  $\psi$ . The intensity function at locations  $x$ ,  $x'$  and  $x''$  are  $\lambda_0$ ,  $\lambda_0 + \lambda_1$  and  $\lambda_0 + 2\lambda_1$ , respectively.

reservoir characteristics,  $R$ ,

$$f(r|\theta) = \exp\{-\Lambda(\theta)\} \prod_{i=1}^n [\lambda(\theta, x_i) f(a_i, l_i)], \quad (27)$$

where  $\Lambda(\theta) = \int_{\mathcal{D}} \lambda(\theta, x) dx$  is the integrated intensity.

The stochastic model parameters  $\Theta = (\lambda_0, \lambda_1, \psi)$  are assigned a prior pdf according to

$$f(\theta) = f(\lambda_0) \cdot f(\lambda_1) \cdot f(\psi), \quad (28)$$

where  $f(\lambda_0)$  and  $f(\lambda_1)$  are univariate pdf's for the intensities of the location model and  $f(\psi)$  is the pdf of a marked point field defining the configuration of the ellipses in the intensity model; again see Stoyan and Stoyan (1994). This defines the prior model for the reservoir characteristics,  $f(r)$ . Note that the conditional pdf  $f(r|\theta)$  must allow analytical computation of the normalizing constant, whereas the pdf of the model parameters,  $f(\theta)$ , can be arbitrarily chosen.

The three types of observations,  $O = (O_w, O_s, O_p)$ , are available and the required associated likelihood functions are specified. The former,  $O_w$ , represents observations in wells according to

$$O_w = g_w(r) + U_w, \quad (29)$$

and consists of two parts, corresponding to observations of shale units crossing the wells and that no non-observed shales cross the well traces. For the former, the  $g_w(r)$  are the intersections between the well traces and the shale units defined by  $r$  and  $U_w$  have a multi-Gaussian pdf with zero expectation and covariance matrix  $\Sigma_w$  representing the uncertainty in the location of the intersection along the well trace. The latter type of observations in  $O_w$  are assumed to be exact, so that  $U_w$  corresponding to this part is zero.

The seismic observations,  $O_s$ , are located in a lattice over the reservoir according to

$$O_s = g_s(r) + U_s, \quad (30)$$

where  $g_s(r)$ , for each node in the lattice, is a weighted sum over the lengths of the shale units located within a circle of radius  $v$  around the node. The weights vary linearly from one, for

shales located at the center of the circle, to zero at the boundary. The  $U_s$  is assumed to have a multi-Gaussian distribution with zero expectation and covariance matrix  $\Sigma_s$ .

The latter,  $O_p$ , represents observations of production history according to

$$O_p = g_p(r) \cdot U_p, \quad (31)$$

where  $g_p(r)$  is the accumulated production after time  $t_0$ , represented by  $k \cdot t_0 / (1 + l(r))$ , in which  $l(r)$  is the tortuosity around the shale units in the reservoir  $r$  and  $U_p$  is Gaussian with unit expectation and variance  $\tau_p^2$ .

The likelihood functions are defined so that the associated normalizing constants can be computed. Otherwise, the transfer functions can be arbitrarily chosen, linear or non-linear. It is important, however, that they can be computed quickly.

The posterior pdf to be sampled from is then given by

$$\begin{aligned} f(r, \theta | o) &= \text{const} \cdot f(\theta) \cdot f(r | \theta) \cdot f(o | r) \\ &= \text{const} \cdot f(\lambda_0) \cdot f(\lambda_1) f(\psi) \cdot \exp\{-\Lambda(\theta)\} \prod_{i=1}^n [\lambda(\theta, x_i) f(a_i, l_i)] \\ &\cdot \exp\left\{-\frac{1}{2}[o_w - g_w(r)]^T \Sigma_w^{-1} [o_w - g_w(r)]\right\} \cdot I[\text{no unobserved shales crossing wells}] \\ &\cdot \exp\left\{-\frac{1}{2}[o_s - g_s(r)]^T \Sigma_s^{-1} [o_s - g_s(r)]\right\} \cdot \exp\left\{-\frac{1}{2\tau_p^2}[o_p/g_p(r) - 1]^2\right\}. \end{aligned} \quad (32)$$

This expression is too complex for most analytical computations and sequential simulation is also infeasible. However, sampling by Metropolis-Hastings algorithm is possible.

Figure 3 contains displays from an example based on the model presented above. The domain  $\mathcal{D}$  is taken to be the unit square. The dip angle and shale length are assumed independent with pdf's that are Gaussian with zero expectation and variance  $(\frac{\pi}{50})^2$  and gamma distributed with expectation 0.05 and variance  $0.03^2$ , respectively. The prior intensity model is defined by the two intensity parameters  $\lambda_0$  and  $\lambda_1$  and a repulsive marked point model  $f(\psi)$ . The prior shale intensities,  $\lambda_0$  and  $\lambda_1$ , are assigned gamma distributions with expectations 50 and 2500 and variances  $20^2$  and  $500^2$ , respectively; see Figure 3(a) and (b). The marked point model is defined by

$$\begin{aligned} f(\psi) &= \text{const} \cdot \exp\left\{5 \cdot q - 0.1 \sum_{i=1}^q \sum_{j=i+1}^q \frac{1}{|(u_i, v_i), (u_j, v_j)|}\right\} \\ &\cdot I[a_1 = \dots = a_q = 0.08 \cap b_1 = \dots = b_q = 0.04], \end{aligned} \quad (33)$$

where  $|\cdot, \cdot|$  denotes the distance between two locations. Note that the prior pdf for the number of shale units,  $f(n)$ , is implicitly defined and can be estimated by sampling from the prior distribution; see Figure 3(c).

This defines the prior model for the reservoir characteristics,  $f(r)$ . A 'true' reservoir is generated from this prior model with a Metropolis-Hastings algorithm; see Figure 3(d). The clustering, the varying dip angle and the length dispersion of the shale units are easily observed. The corresponding 'true'  $\lambda_0$  and  $\lambda_1$  are 41.5 and 2041, respectively, and the number of shale units,  $n$ , is 199.

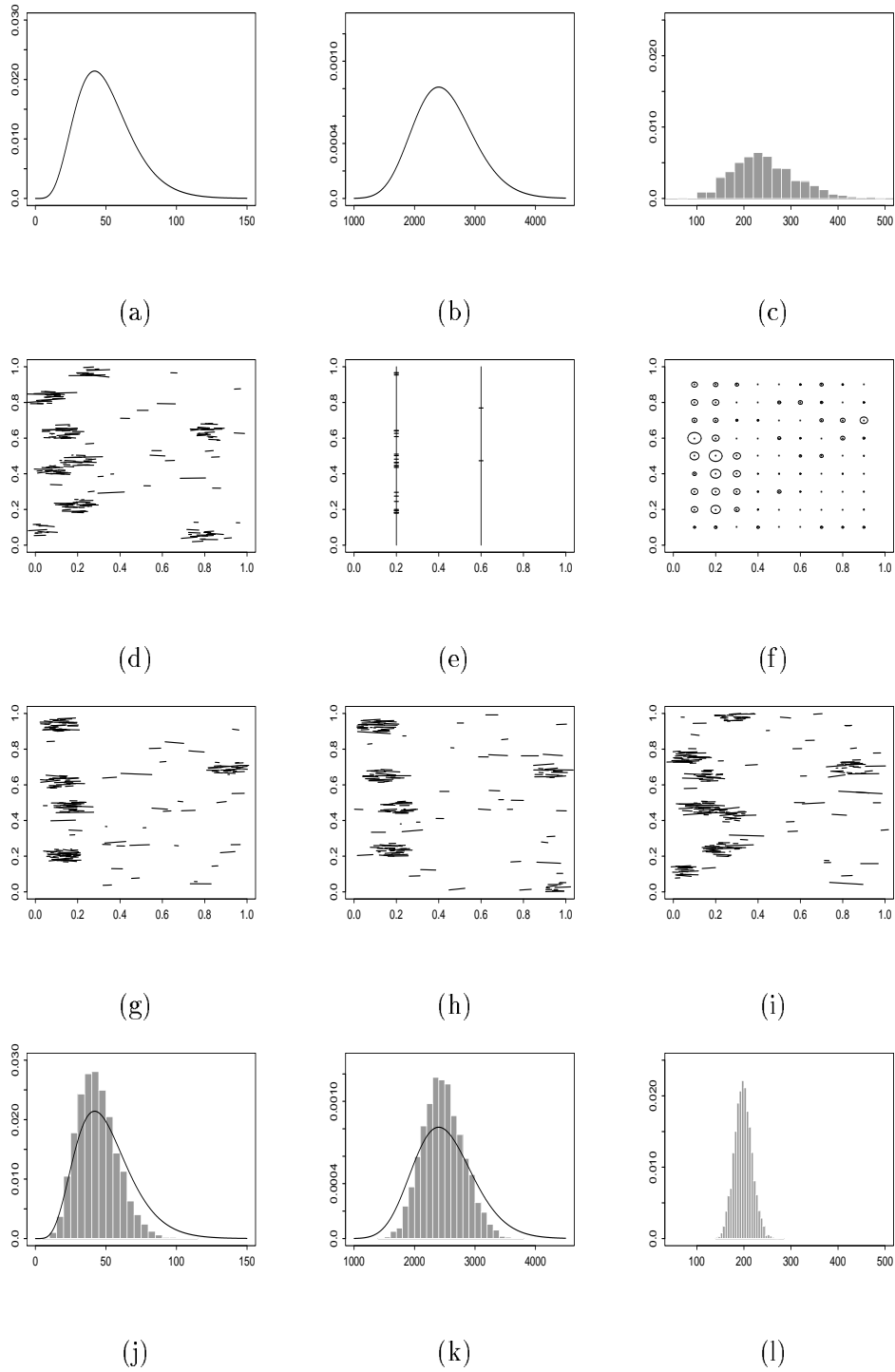


Figure 3: Simulation example for shale units. (a), (b) and (c) Prior distributions for  $\lambda_0$ ,  $\lambda_1$  and  $n$ , respectively. (d) ‘true’ reservoir sampled from  $f(r)$ . (e) and (f) observations  $o_w$  and  $o_s$  sampled from the likelihood functions. (g), (h) and (i) realizations from the posterior distribution  $f(r|o)$ . (j), (k) and (l) posterior distributions for  $\lambda_0$ ,  $\lambda_1$  and  $n$ , respectively.

The reservoir-specific observations are obtained from the ‘true’ reservoir characteristics in Figure 3(d). The well observations are in two wells located at  $x^1 = 0.2$  and  $0.6$ , respectively, see Figure 3(e), and the Gaussian vertical observation errors are independent with variances  $\tau_w^2 = 0.03^2$ . The actual well observations,  $o_w$ , are displayed in Figure 3(e). The seismic observations are on a  $9 \times 9$  lattice over the reservoir; see Figure 3(f). The radius of influence to each lattice node,  $v$ , is  $0.2$  and the Gaussian observation errors are independent with variances  $\tau_s^2 = 0.35^2$ . The actual seismic observations,  $o_s$ , are displayed as circles with radii proportional to observed values in Figure 3(f). The production observation is the accumulated production with  $k \cdot t_0$  set to  $1000.0$  and turtousity is computed as the total length of all shale units. The multiplicative Gaussian observation error is assigned variance  $\tau_p^2 = 0.2^2$ . The actual production observation,  $o_p$ , is  $82.8$ .

The specification above defines the posterior pdf,  $f(r, \theta | o)$ , up to a constant. Hence, the Metropolis-Hastings algorithm can be used to generate samples from it. The actual implementation of the algorithm is crucial to obtain an acceptable convergence rate. The technicalities of the implementation are presented below.

In Figures 3(g), (h) and (i), three realizations from the posterior pdf  $f(r|o)$  are presented. They reflect the uncertainty in the prior model constrained by the observations actually made. One can recognize the clustering and other characteristics of the prior model. Moreover, the reservoir-specific observations,  $o = (o_w, o_s, o_p)$ , carry information and the clusters are located in reasonable accordance with the ‘true’ reservoir in Figure 3(d). The posterior pdf for the parameters,  $f(\lambda_0|o)$  and  $f(\lambda_1|o)$ , and for the number of shale units,  $f(n|o)$ , can be estimated through sampling and the corresponding histograms are presented in Figures 3(j), (k) and (l), respectively. In Figures 3(j) and (k), also the corresponding prior pdf’s are indicated. One can observe that the posterior pdf’s are shifted towards the ‘true’ values,  $41.5$ ,  $2041$  and  $199$ . Moreover, the associated variances are smaller than in the corresponding prior pdf’s since the information in the reservoir-specific observations is included.

The realizations from the prior and the posterior pdf’s presented above are generated by a Metropolis-Hastings algorithm. The choice of transition matrix  $Q(t \rightarrow \tilde{t})$  in this type of algorithms is crucial for the rate of convergence. It is common to use transition matrices, which propose to change only one component of  $t$ . Such a choice would, however, give extremely slow convergence for the pdf’s considered here because of the hierarchical structure of the prior. However, the same hierarchical structure can be utilized to propose changes on two different scales; changes on single components and changes on clusters. The  $Q(t \rightarrow \tilde{t})$  used can be expressed as

$$Q(t \rightarrow \tilde{t}) = 0.1 \cdot Q_1(t \rightarrow \tilde{t}) + 0.1 \cdot Q_2(t \rightarrow \tilde{t}) + 0.5 \cdot Q_3(t \rightarrow \tilde{t}) + 0.3 \cdot Q_4(t \rightarrow \tilde{t}), \quad (34)$$

where  $Q_1(t \rightarrow \tilde{t})$ ,  $Q_2(t \rightarrow \tilde{t})$ ,  $Q_3(t \rightarrow \tilde{t})$  and  $Q_4(t \rightarrow \tilde{t})$  correspond to four different classes of changes, the first three for single components and the last for a cluster of shales. The transition  $Q_1(t \rightarrow \tilde{t})$  changes  $\lambda_0$  only and the potential new value,  $\tilde{\lambda}_0$ , is sampled from the uniform distribution over  $[\lambda_0 - 10, \lambda_0 + 10]$ .  $Q_2(t \rightarrow \tilde{t})$  correspondingly changes  $\lambda_1$  and  $\tilde{\lambda}_1$  is sampled from the uniform distribution over  $[\lambda_1 - 250, \lambda_1 + 250]$ .  $Q_3(t \rightarrow \tilde{t})$  adds, removes or changes one shale unit, with probabilities one third for each. If a shale unit is to be added, the new location is sampled from the distribution  $\lambda(\theta, \cdot) / \Lambda(\theta)$  and the dip angle and length from  $f(a_i, l_i)$ . If a shale unit is to be removed, it is drawn uniformly from the  $n$  currently present. If a shale unit is to be changed, the unit to be changed is first sampled uniformly from the  $n$  present. If this shale unit corresponds to a well observation, the new dip angle

and length are sampled from  $f(a_i, l_i)$  and the intersection with the well is sampled from a Gaussian distribution centered at the observed intersection and with variance equal to  $\tau_w^2$ . This defines a line segment of length equal to the length of the shale, on which the center of the shale must be located. The location of the shale is then sampled uniformly over this line segment. If the shale unit to be changed does not correspond to a well observation, the new location, dip angle and length are sampled as in the case of adding a new shale unit.  $Q_4(t \rightarrow \tilde{t})$  adds, moves or removes a cluster of shales. This is done by adding, moving or removing one ellipse in  $\psi$  and changing the shales in  $R$  accordingly as specified by the prior distribution. The precise algorithm used to generate  $\tilde{t} = (\tilde{r}, \tilde{\theta})$  from  $t = (r, \theta)$  according to  $Q_4(t \rightarrow \tilde{t})$  is as follows.

1. Set  $\tilde{\lambda}_0 := \lambda_0$  and  $\tilde{\lambda}_1 := \lambda_1$ .
2. With probability 1/6 try to add a cluster, i.e.
  - (a) Draw a position  $(u, v)$  from a uniform distribution over  $\mathcal{D}$  and set  $\tilde{\psi} := \{q + 1; (u_1, v_1, a_1, b_1), \dots, (u_q, v_q, a_q, b_q), (u, v, 0.08, 0.04)\}$ .
  - (b) Draw the number of new shales,  $m$ , from a Poisson distribution with parameter  $\lambda_1 \cdot \pi \cdot 0.04 \cdot 0.08$  and set  $\tilde{r} := \{n + m; r_1, \dots, r_n, r_{n+1}, \dots, r_{n+m}\}$ , where the new shales  $r_{n+1}, \dots, r_{n+m}$  are sampled independently with locations from a uniform distribution over the ellipse with center at  $(u, v)$  and with dip angles and lengths from  $f(a_i, l_i)$ .

or, with probability 1/6, try to remove a cluster, i.e.

- (a) Draw an integer  $i$  from a uniform distribution from 1 to  $q$  and set  $\tilde{\psi} := \{q - 1; (u_1, v_1, a_1, b_1), \dots, (u_{i-1}, v_{i-1}, a_{i-1}, b_{i-1}), (u_{i+1}, v_{i+1}, a_{i+1}, b_{i+1}), \dots, (u_q, v_q, a_q, b_q)\}$ .
- (b) For  $j = 1, \dots, n$ , if the location of the shale unit  $r_j = (x_j, a_j, l_j)$  is inside the ellipse with center at  $(u_i, v_i)$ , remove the point with probability  $\lambda(\tilde{\theta}, x_j) / \lambda(\theta, x_j)$ . Let  $\tilde{R}$  be the set of the remaining shales.

or, with probability 2/3, try to move a cluster, i.e.

- (a) Draw an integer  $i$  from a uniform distribution from 1 to  $q$  and draw a new position  $(\tilde{u}_i, \tilde{v}_i)$  where  $\tilde{u}_i$  and  $\tilde{v}_i$  are sampled from uniform distributions over  $[\max(0, u_i - a_i/4), \min(1, u_i + a_i/4)]$  and  $[\max(0, v_i - b_i/4), \min(1, v_i + b_i/4)]$ , respectively. Let the set of potential new ellipses be  $\tilde{\psi} := \{q; (u_1, v_1, a_1, b_1), \dots, (u_{i-1}, v_{i-1}, a_{i-1}, b_{i-1}), (\tilde{u}_i, \tilde{v}_i, a_i, b_i), (u_{i+1}, v_{i+1}, a_{i+1}, b_{i+1}), \dots, (u_q, v_q, a_q, b_q)\}$ .
- (b) For  $j = 1, \dots, n$ , if the location of the shale unit  $r_j = (x_j, a_j, l_j)$  is inside the old ellipse with center at  $(u_i, v_i)$  but outside the new ellipse with center at  $(\tilde{u}_i, \tilde{v}_i)$ , remove the shale unit with probability  $\lambda(\tilde{\theta}, x_j) / \lambda(\theta, x_j)$ .
- (c) For the area in  $\mathcal{D}$  which is inside the new ellipse with center at  $(\tilde{u}_i, \tilde{v}_i)$  but outside the old ellipse with center at  $(u_i, v_i)$ , draw new shale units according to a Poisson point model with intensity  $\tilde{\lambda}_1$  and add them to  $R$ . The dip angles and lengths for the new shale units are sampled from  $f(a_i, l_i)$ . Let  $\tilde{R}$  be the resulting set of shale units.

One should note that with the  $Q(t \rightarrow \tilde{t})$  used here, the proposed new state vector,  $\tilde{t}$ , may be in conflict with the well observations,  $o_w$ , because  $\tilde{t}$  may include non-observed shale units that

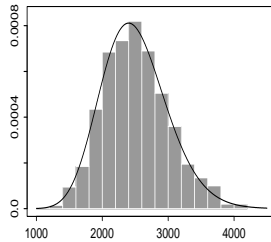


Figure 4: Histogram of simulated values for  $\lambda_1$  in conditional realizations together with the corresponding prior distribution  $f(\lambda_1)$ .

cross the well traces. But this cause no problems in the algorithm because no such states will be accepted. If  $\tilde{t}$  and  $o_w$  are inconsistent,  $f(\tilde{t}|o)$ , and thereby also the acceptance probability  $A(t \rightarrow \tilde{t})$ , is zero.

To check the convergence, we plotted important characteristics as the function of iteration number as discussed in section 3. The number of iterations necessary to reach convergence varies considerably dependent on the observations. Especially the well observations,  $o_w$ , are of importance. In most cases less than 500,000 is sufficient but when the number of observed shale units is large, as much as 5,000,000 iterations are sometimes necessary.

The simulation algorithm described above is fairly complex and it becomes important to be able to check the implementation of the algorithm. To do this we performed the following one thousand times.

1. Draw an  $r^*$  and a  $\theta^*$  from the prior distribution  $f(r^*, \theta^*)$  by the Metropolis-Hastings algorithm.
2. Draw observations,  $o_w$ ,  $o_s$  and  $o_p$ , according to the likelihood functions  $f(o_w|r^*)$ ,  $f(o_s|r^*)$  and  $f(o_p|r^*)$ , respectively.
3. Draw  $r$  and  $\theta$  from the corresponding posterior distribution  $f(r, \theta|o)$  by the Metropolis-Hastings algorithm.

The  $(r, \theta)$ -samples should then, unconditionally, be distributed according to the prior distribution  $f(r, \theta)$  and this can be used to check both the implementation of the algorithm and that the number of iterations used is sufficient. Figure 4 contains a histogram of the one thousand values simulated for  $\lambda_1$ , together with the corresponding prior distribution. One can observe that the histogram gives a good approximation to the theoretical distribution.

## 5 Closing Remarks

Stochastic modeling in reservoir characterization is characterized as being an extremely high-dimensional problem, having many types of observations linked to the characteristics through complex transfer functions with considerable expert knowledge about the processes generating

the reservoir available and the need for dynamic updating due to incoming observations. A solution in a statistical spirit can be found in a Bayesian framework. Hence, a formalism for integration of all types of reservoir information has been defined. Sampling from the stochastic model may be very computer intensive but to have an established formalism makes it easier to utilize the computer efficiency when it becomes available.

The stochastic model for the reservoir characteristics has to be a simplified representation of the truth. Firstly, different variables and layers which can be considered to be independent should be modeled separately. Secondly, only the characteristics of the reservoir that are crucial for fluid flow should be represented in the model. Thirdly, the transfer functions for the observations should be defined so that they are relatively quickly computed and the error term be defined accordingly. It is important that the approximations made are balanced with respect to the final objective of the study. Guidelines for this need to be developed. Note that linearization is not required at any stage, only the capacity for fast computation.

Integration of reservoir information will remain a challenge but applying the formalism defined above will at least make it a structured task.

## References

- Besag, J. and Green, P.J. (1993). "Spatial statistics and Bayesian computation", *J. Royal Statist. Soc. B*, **55**, 3-23.
- Cox, D.R. (1955). "Some statistical models connected with series of events", *J. Royal Stat. Soc. B*, **17**, 129-164.
- Cressie, N. (1993). *Statistics for spatial data*, Wiley, New York.
- Deutsch, C.V. (1993). "Conditioning reservoir models to well test information", in Soares, A. (ed.) *Geostatistics Tróia '92*, Kluwer Academic Publishers, 505-518.
- Dubrule, O. (1993). "Introducing more geology in stochastic reservoir modelling", in Soares, A. (ed.) *Geostatistics Tróia '92*, Kluwer Academic Publishers, 351-369.
- Gómez-Hernández, J. and Journel, A. (1993). "Joint sequential simulation of multiGaussian fields", in Soares, A. (ed.) *Geostatistics Tróia '92*, Kluwer Academic Publishers, 85-94.
- Grandell, J. (1976). *Doubly Stochastic Poisson processes*, Lecture Notes in Mathematics, Springer Verlag, Berlin.
- Haas, A. and Dubrule, O. (1994). "Geostatistical inversion — a sequential method of stochastic reservoir modelling constrained by seismic data", *First Break*, **12**, 561-569.
- Haldorsen, H.H. and Damsleth, E. (1990). "Stochastic modeling". *J. of Petroleum Technology*, 404-412.
- Haldorsen, H.H. and Lake, L.W. (1984). "A new approach to shale management in field-scale models", SPEJ, August 1984, 447-457.
- Hegstad, B.K., Omre, H., Tjelmeland, H. and Tyler, K. (1993). "Stochastic simulation and conditioning by annealing in reservoir description", in Armstrong, M. and Dowd, P.A. (eds.) *Geostatistical Simulations, Proceedings of the Geostatistical Simulation Workshop, Fontainebleau, France, 27-28 May 1993*, Kluwer Academic Publishers.
- Heikkinen, J. and Högmänder, H. (1994). "Fully Bayesian approach to image restoration with an application in biogeography", *Appl. Statist.*, **43**, 569-582.

- Higdon, D.M., Johnson, V.E., Turkington, T.G., Bowsher, J.E., Gilland, D.R. and Jaszczak, R.J. (1995). "Fully Bayesian estimation of Gibbs hyperparameters for emission computed tomography data", Discussion paper, ISDS, Duke University.
- Hjort, N.L. and Omre, H. (1994). "Topics in spatial statistics (with discussion)", *Scand. J. of Stat.*, **21**, 289-357.
- Journal, A.G. (1994). "Modeling uncertainty: some conceptual thoughts", in Dimitrakopoulos, R. (ed.) *Geostatistics for the next century*, Kluwer Acad. Press, 30-43.
- Journal, A.G. and Huijbregts, C.J. (1978). *Mining geostatistics*, Academic Press, New York.
- Lia, O., Omre, H., Tjelmeland, H., Holden, L., Egeland, T., Andersen, T., MacDonald, A., Hustad, O.S. and Qi, Y. (1995). "The great reservoir uncertainty study – GRUS", in Olsen, J., Olaussen, S., Jensen, T.B., Landa, G.H. and Hinderaker, L. (eds.) *PROFIT (Program for research on field oriented improved recovery technology) — Project summary reports: Reservoir characterization, Near well flow*. Norwegian Petroleum Directorate, Stavanger, Norway.
- Lien, S.C., Haldorsen, H.H. and Manner, M. (1992). "Horizontal wells: Still appealing in formations with discontinuous vertical permeability barriers", *J. of Petroleum Technology*, 1364-1370.
- Meyn, S.P. and Tweedie, R.L. (1994). "Computable bounds for convergence rates of Markov chains", *Ann. Appl. Probab.* **4**, 981-1011.
- Neal, R.M. (1993). "Probabilistic inference using Markov chain Monte Carlo methods", Technical Report CRG-TR-93-1, Department of Computer Science, University of Toronto.
- Omre, H. (1991). "Stochastic models for reservoir characterization", in Kleppe, J. and Skjæveland, S.M. (eds.). *Recent advances in improved oil recovery methods for North Sea sandstone reservoirs*, Norwegian Petroleum Directorate, Stavanger, Norway.
- Omre, H. and Halvorsen, K.B. (1989). "The Bayesian bridge between simple and universal Kriging", *Math. Geol.*, **21**, 767-786.
- Omre, H., Halvorsen, K.B. and Berteig, V. (1989). "A Bayesian approach to Kriging", in Armstrong, M. (ed.) *Geostatistics I*, Kluwer Academic Publishers, 109-126.
- Omre, H., Tjelmeland, H., Qi, Y. and Hinderaker, L. (1993a). "Assessment of uncertainty in the production characteristics of a sand stone reservoir", in Linville, B., Burchfield, T.E. and Wesson, T.C. (eds.) *Reservoir characterization III*, PennWell Publishing Company, Tulsa, Oklahoma.
- Omre, H., Sølna, K. and Tjelmeland, H. (1993b). "Simulation of random functions on large lattices", in Soares, A. (ed.) *Geostatistics Tróia '92*, Kluwer Academic Publishers, 179-199.
- Ripley, B.D. (1981). *Spatial statistics*. Wiley, New York.
- Rosenthal, J.S. (1994). "Minorization conditions and convergence rates for Markov chain Monte Carlo", Technical Report 9321, Department of Statistics, University of Toronto.
- Stoyan, D. and Stoyan, H. (1994). *Fractals, random shapes and point fields: methods of geometrical statistics*, Wiley, Chichester.