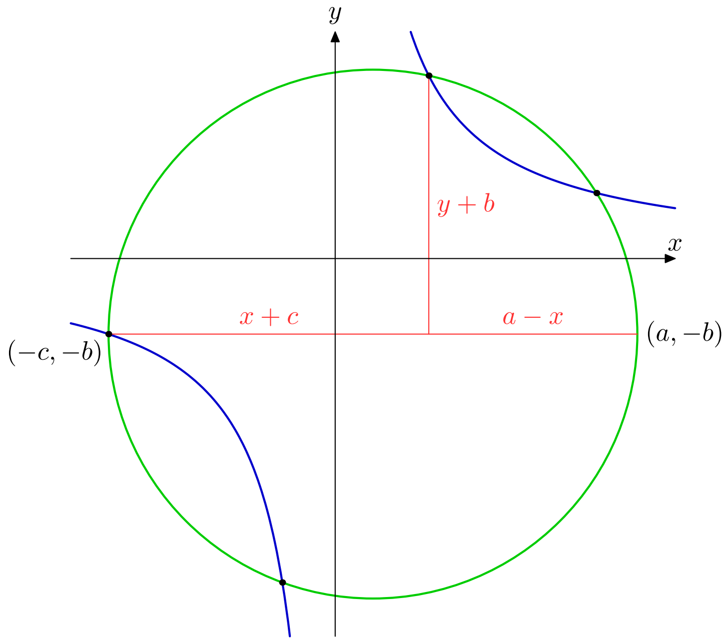
Omar Khayyam and the cubic equation



Omar Khayyam was perhaps not a great poet, but he was a great mathematician! The figure shows a geometric way to find the roots of a certain cubic equation.

In modern language: a, b and c are given, and the circle is constructed so that a diameter has its ends at $(-c,-b)$ and $(a,-b)$. The equation of the circle is then

$$(x+c)(a-x)=(y+b)^2.$$

The hyperbola shown has equation

$$xy=bc,$$

so that the point $(-c,-b)$ is indeed a common point of the two curves.

Substituting $y=bc/x$ from the second equation into the first equation and then multiplying by x^2, we get

$$x^2(x+c)(a-x)=b^2(c+x)^2.$$

The common factor $x+c$ should not surprice us, since $x=-c$ is a known solution to the equation, corresponding to the point $(-c,-b)$. Canceling the common factor and tidying up a little, we end up with the equation

$$x^3+b^2x+b^2c=ax^2$$

for the remaining three roots. This may seem an odd way to write it, but Omar Khayyam did not have access to negative numbers. All quantities here are to be considered positive lengths.

If course, when seen with our modern eyes, we can clearly see that the method works for any real a, b, c so long as b, c and $a+c$ are non-zero. If $a+c=0$, the equation simplifies to $(x+c)(x^2+b^2)=0$, and similar reductions apply in the other cases, so this is not a great loss of generality. However, the method only works when the cofficients of x^3 and x have the same sign. Even that is no great loss of generality though, since replacing x by $x+\xi$ for a sufficiently large ξ will transform any cubic equation to one that can be treated by Khayyam's construction.

We should also appreciate the fact that Omar Khayyam did not have named variables or algebraic notation at his disposal. So the statement of the problem, as well as the solution, would be stated in prose. And *that* is a remarkable accomplishment.