1 Optimalization of production

- $L(t)$: Inventory (= amount in storage)

- $P(t)$: Production per time unit.

- $S(t)$: Sales per time unit.

System model:

$$\frac{dL(t)}{dt} = P(t) - S(t) - \alpha L(t)$$

($-\alpha L$: loss at the storage)

- $S(t)$: Sales prognosis
The optimal inventory ($\mathcal{L}$) and production ($\mathcal{S}$) are the ones matching the sales prognosis:

$$\frac{d\mathcal{L}(t)}{dt} = \mathcal{P}(t) - \mathcal{S}(t) - \alpha \mathcal{L}(t)$$

At $t = 0$, the inventory is off the ideal, $L(0) = L_0 \neq \mathcal{L}(0)$.

How do we control the $P(t)$ so as to minimize the extra cost of being off the ideal?

### 1.1 Cost functional

- $L(t) - \mathcal{L}(t)$: Deviation from optimal inventory
- $P(t) - \mathcal{P}(t)$: Deviation from opt. production rate
\[ C (P) = \int_0^T \left[ \beta^2 (L (t) - \mathcal{L} (t))^2 + (P (t) - \mathcal{P} (t))^2 \right] dt \]

**Optimal control problem:**

\[
\begin{align*}
\min_P C \\
L' (t) &= P (t) - S (t) - \alpha L (t), \\
L (0) &= L_0.
\end{align*}
\]

- The variable \( P (t) \) is our *control variable*

- We expect the sales prognosis \( S (t) \), and need to adjust the inventory towards the optimal \( \mathcal{L} (t) \)

- We are forced to follow the system dynamics

\[
\frac{dL (t)}{dt} = P (t) - S (t) - \alpha L (t)
\]
We are starting off the ideal curve, and want to minimize the cost of getting there.

Observe that $L$ is dependent on $P$, and if we determine $L$, $P$ may be found as

$$P = L' + \alpha L + S$$

Let $y = L(t) - \mathcal{L}(t)$. Then, since

$$L' = P - S - \alpha L,$$
$$\mathcal{L}' = \mathcal{P} - S - \alpha \mathcal{L},$$

we have

$$L' - \mathcal{L}' = P - \mathcal{P} - \alpha (L' - \mathcal{L}') ,$$
or
\[
\frac{dy}{dt} = (P - \mathcal{P}) - \alpha y.
\]
This is inserted in the cost functional:
\[
C (y) = \int_0^T \left[ \beta^2 y^2 + \left( y' + \alpha y \right)^2 \right] dt
\]

Note that
\[
C (y) = \int_0^T \left[ \beta^2 y^2 + \left( y' + \alpha y \right)^2 \right] dt
= \int_0^T \left[ (\beta^2 + \alpha^2) y^2 + \left( y' \right)^2 \right] dt + \int_0^T 2\alpha yy' dy
= \int_0^T \left[ (\beta^2 + \alpha^2) y^2 + \left( y' \right)^2 \right] dt + \alpha y^2 (T) - \alpha y^2 (0)
\]
The first term is strictly convex, the second is convex, and the last is constant ( \( = L_0^2 \)).
1.2 Solution

The solution is found by solving the Euler equation

\[
\frac{d}{dx} f' - f = \frac{d}{dx} \left[ 2 (y' + \alpha y) \right] - \left[ 2 \beta^2 + 2 (y' + \alpha y) \alpha \right] \\
= y'' - (\alpha^2 + \beta^2) y = 0.
\]

with a fixed boundary at \( t = 0 \),

\[ y(0) = y_0 = L_0 - L_0. \]

and a natural condition at \( t = T \):

\[ f_y' (y(T)) = 2 (y'(T) + \alpha y(T)) = 0. \]

With \( \gamma^2 = (\alpha^2 + \beta^2) \), the problem may be formulated as

\[ y'' - \gamma^2 y = 0, \]

\[ y(0) = y_0, \]

\[ y'(T) + \alpha y(T) = 0. \]
From the general solution of the Euler eqn.,

\[ y(t) = A \exp(\gamma t) + B \exp(-\gamma t), \]

it is easy to find the unique optimal solution

\[ y^*(t)) = y_0 \frac{e^{\gamma t} + \rho e^{-\gamma t}}{1 + \rho}, \]

\[ \rho = \frac{\gamma + \alpha}{\gamma - \alpha} e^{2\gamma T}. \]

The minimal value for the functional is

\[ C(y^*) = y_0^2 \left[ \frac{\gamma (\rho - 1)}{\rho + 1} - \alpha \right]. \]

### 1.3 Example: The Icecream Factory

The factory is planning for the summer. \( t = 0 \) is April 1st, \( t = 4 \) is July 31st.

- Sales prediction: \( \mathcal{S}(t) = 1 + t. \)
• Optimal inventory: $L(t) = 4$

• The optimal production rate is $P(t) = S(t) + \alpha \times 4$.

Let $\alpha = 0.1$ and $\beta = 1.5$. 