CONTENTS

• What is an inverse problem
• Some famous problems
• Examples
Jeopardy - The everyday Inverse problem:

“It was in 1905”

“When did Einstein publish his Theory of Relativity?”

“When did Robert Koch get the Nobel Price in Medicine?”

“When was my grandmother born?”

“When did Norway and Sweden split up from the union?”
• The information is incomplete and non-conclusive

• Different input leads to the same result

• The ”most probable” input depends on circumstances

*direct problem*: given the question, find the answer

*inverse problem*: given the answer, find the question
• To solve an inverse problem is to determine a cause from its effect

• Applications in seismology, geosciences, and many other areas of sciences and engineering

• Optimization provides methods for inverse problems

• The majority of real problems are inverse problems
Well-posed problem:
• a solution always exists
• there is only one solution
• a small change in the problem leads to a small change in the solution

Ill-posed problem:
• a solution may not exist
• there may be more than one solution
• a small change in the problem leads to a big change in the solution

INVERSE PROBLEMS ARE (ALMOST) ALWAYS ILL-POSED!
SOME FAMOUS INVERSE PROBLEMS

• Marc Kac (1966): "Can you hear the shape of a drum?"

• Computer Tomography

• Seismic Inversion

• Image Restoration
NO, you can’t hear the shape of a drum, but you can hear

- the area,
- the length of the boundary,
- and the number of holes!

COMPUTER TOMOGRAPHY

http://www.iwr.uni-heidelberg.de/groups/ngg/Tutorial/TutCT_121203_Lauritsch.pdf

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CT-numbers of tissue in Hounsfield units (HU)

$$CT - number = \frac{\mu - \mu_{\text{water}}}{\mu_{\text{water}}} \cdot 1000$$

$$I_j = I_0 \exp \left( -\int_{\text{line}_j} \mu(x) \, dl \right)$$
Radon transform: Based on line integrals

\[ \mathcal{R}\mu(\rho, \theta) = \hat{\mu}(\rho, \theta) = \int_{-\infty}^{\infty} \mu(\rho \cos \theta - t \sin \theta, \rho \sin \theta + t \cos \theta) dt \]

Inverse problem:

\[ \mu(x) = \mathcal{R}^{-1}(\hat{\mu}(\rho, \theta)) \]
Reconstructive Methods

Data acquisition system

Set of digital projection images

reconstruction algorithm

Tomographic slice images

3-D image

Measured sinogram
INVERSE SCATTERING
Seismic inversion

Salt Intrusion Model, 200 by 300 grid

Salt Intrusion Exploding Reflector Data, dt=0.002

Salt Intrusion Reverse Time Migration, Grid Spacing = 40 Meters

http://www.mgnet.org/~douglas/Classes/cs521-s00/asdf/asdf.ppt
Given the seismic recordings – compute the sea floor motion

Slip

Uplift

(Bjørn Gjevik, UiO. Published by Caltech)
IMAGE RESTORATION

Blur

Max. Entropy restoration

Noise suppression

http://mip.ups-tlse.fr/~noll/noll_tutorial.html
A: Original HST photo.  
B: Enlarged section.  
C: Ground telescope image.  
D: Digitally enhanced image.
Example: The Hilbert Matrix System
The Hilbert matrix system:

\[ Hx = b \]

\[ H_n = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1}
\end{bmatrix} \]

\[ h_{ij} = \int_0^1 x^{i-1} x^{j-1} \, dx \]
Singular values

Exact and direct solutions

Picard-plot

$u_i^* b/\sigma_i$

(cond. Number = $1.5\times10^{18}$)

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Hilbert problem cont. ...

The Picard plot shows $|\alpha_j|$: $x^* = \sum_{j=1}^{n} \alpha_j v_j$, $\alpha_j = \frac{u_j^T b}{\sigma_j}$
Hilbert problem cont. ...

**WARNING:**
A small residual does not mean a good solution for ill-conditioned problems:

![Graph of residual and norm of solution versus iteration number](image)
The cosine taper

The graph shows the cosine taper function, which is a smooth transition from 1 to 0 as the variable $k/k_0$ increases from 0 to 1.5. The taper starts at 1 when $k/k_0 = 0$, remains constant until $k/k_0 = 0.8$, and then gradually decreases to 0 as $k/k_0$ approaches 1.5.
Example: The Hodrick-Prescott Filter
Hodrick-Prescott filter

Yearly mean temperatures and trend curve, Blindern, Oslo

Yearly mean temperatures and trend curve, Blindern, Oslo

Yearly mean temperatures and trend curve, Blindern, Oslo

Yearly mean temperatures and trend curve, Blindern, Oslo

Trend
Residuals
\[ \min_t \left( \sum_{i=1}^{N} (x_i - t_i)^2 + \lambda \sum_{i=2}^{N-1} (t_{i+1} - 2t_i + t_{i-1})^2 \right) \]

\( N \) unknowns!
Blindern, Oslo, $\lambda=100$

$\lambda=1500$

$\lambda=100000$
FREDHOLM INTEGRAL EQUATIONS

\[ y(t) = \int_0^1 K(t, \tau) x(\tau) d\tau \]

\( K(t, \tau) \) is called the kernel

Discretized Fredholm Integral Equations are generally ill-conditioned
A very common situation in image processing:

\[ BI(x) = f_{PS} * I(x) = \int_{\mathbb{R}^2} f_{PS}(x-y)I(y)dy \]

- \( I \) : Image
- \( BI \) : Blurred image
- \( f_{PS} \) : Point Spread Function

\[ BI \rightarrow I : "Deconvolution" \]

The simplest deconvolution is vanCittert/Landweber iteration:

\[ I_{k+1} = I_k + \omega(BI - f_{PS} * I_k) \quad k = 1, 2, \ldots, k_{stop} \]

(NB: Should have some idea about \( f_{PS} \)!)
DE-CONVOLUTION
Measurement

De-convoluted measurement

"Ideal" measurement

Frequency

Spectrum
vanCittert/Landweber iteration is simple and fast!