THE KEY PROBLEMS

• How to select the best/most probable/optimal solution

• How to dampen "ill-conditionness"
  (regularize - but not exaggerate!)

Example: Tikhonov regularization:

\[ X^* = \arg \min_{X} \left\{ \| T(X) - Y \|^2 + \mu \| X - X_0 \|^2 \right\} \]

\( \mu \) small: solution unstable (ill-conditioned problem)

\( \mu \) large: \( X^* \approx X_0 \) (even if our "knowledge" \( X_0 \) is rubbish!)
The effect of Tikhonov Regularization for the "test case":

\[ f_\mu (x) = \| Ax - b \|^2 + \mu \| x \|^2 = x' (A' A + \mu I) x - 2 (A' b)' x + b' b \]

**Solution:**

\[ x^* = \sum_{j=1}^{n} \frac{\sigma_j (b' u_j)}{\sigma_j^2 + \mu} v_j = \sum_{j=1}^{n} \frac{\sigma_j^2 (b' u_j)}{\sigma_j^2 + \mu \sigma_j} v_j = \sum_{j=1}^{n} f_\mu (\sigma_j) \frac{(b' u_j)}{\sigma_j} v_j \]
FREDHOLM INTEGRAL EQUATIONS

\[ y(t) = \int_0^1 K(t, \tau) x(\tau) d\tau \]

\( K(t, \tau) \) is called the \textit{kernel}

Discretized Fredholm Integral Equations are generally ill-conditioned
L-CURVE ANALYSIS
(Per Chr. Hansen, DTU)

Plot of the two terms in the Tikhonov objective function:

\[ \log\|L(X_\mu)\| \]

Penalty

\[ \log\|T(X_\mu) - Y\| \]

Error

Increasing \( \mu \) decreasing
\[ \int_{-\pi/2}^{\pi/2} K(t,s)x(s) \, ds = b(t) \]

\[ K(t,s) = (\cos s + \cos t) \frac{\sin^2(\pi(\sin s + \sin t))}{\pi^2(\sin s + \sin t)^2} \]

\[ x_{sol}(t) = a_1 \exp\left(-\frac{(t-t_1)^2}{c_1^2}\right) + a_2 \exp\left(-\frac{(t-t_2)^2}{c_2^2}\right) \]
ITERATIVE METHODS

The basic iterative method for a linear equation

\[ Ax = b, \ A > 0, \]

is

\[ x_{k+1} = x_k + \omega(b - Ax_k), \ k = 0, 1, \ldots \]

(“fix-point iteration”)

\( \omega \) is called a relaxation parameter.

Convergence if \( \omega \) is chosen so that

\[ \|I - \omega A\| < 1 \]
Fix point iteration is used for inverse problems as a \textit{general} technique where the \textit{number of iterations is the regularization}!

Nonlinear filter

\begin{align*}
  x_k + 1 &= x_k + \omega \left( b - M(x_k) \right), & k = 0, 1, \ldots, k_{opt}
\end{align*}

Common name: \textit{van Cittert iteration}
DE-CONVOLUTION OF SPECTRAL DATA

- Astronomy
- Mass spectrography
- Optics
- Nuclear Magnetic Resonance

Measurement should consist of narrow “spectral lines”, but the instrument “blurs” the lines:

“De-blurring” is needed!
\[ x_{k+1} = \max \left[ 0, x_k + \omega \left( b - g_{\text{blur}} \ast x_k \right) \right], \quad k = 0, 1, \ldots, k_{\text{opt}} \]

2000 iterations!
A very common situation in image processing:

\[ BI(x) = f_{PS} * I(x) = \int_{\mathbb{R}^2} f_{PS}(x - y)I(y)dy \]

- \( I \): Image
- \( BI \): Blurred image
- \( f_{PS} \): Point Spread Function

**BL \rightarrow I : "Deconvolution"**

\[ I_{k+1} = \max\left[ 0, I_k + \omega( BI - f_{PS} * I_k ) \right], \quad k = 1, 2, \ldots, k_{\text{stop}} \]

(NB: Should have some idea about \( f_{PS} \)!)
vanCittert/Landweber iteration is simple and fast!