Problem 1

Consider the unconstrained minimization problem

\[ \min_{x=(\xi, \eta) \in \mathbb{R}^2} f(\xi, \eta) = \min_{x=(\xi, \eta) \in \mathbb{R}^2} \{5 - 2\eta - 4\xi + 2\xi\eta + \xi^2 + 2\eta^2\} \].

(a) Compute the gradient and the Hessian of \( f \) in an arbitrary point, and show that \( x^* = (3, -1)' \) is the unique global minimum.

(b) Start at \( x_0 = (0, 0)' \) and verify that one iteration with the Steepest Descent (SD) method brings you to \( x_1 = (1, 1/2)' \).

(c) Explain the Conjugate Gradient (CG) method applied to the general quadratic model problem

\[ Q(x) = \frac{1}{2} x' Ax - b' x, \]

and show that if we start the CG method in \( x_0 = 0 \) with \( d_0 = -\nabla Q(0)' = b \) as the first basis vector, then \( x_1 \) is identical to the first iteration of the SD method applied to the same problem and starting from 0.

(d) Starting from \( x_1 \) in (b), state (without any computations) the result of the next iteration with the CG method applied to the problem in (1), and verify that the corresponding search directions for the two CG iterations are (conjugate) orthogonal with respect to the Hessian of \( f \).
Problem 2

(a) State the Karush-Kuhn-Tucker Theorem for a local minimum $x^*$ of a function $f(x)$ subject to sets of equality, $\{c_i(x) = 0, i \in \mathcal{E}\}$, and inequality, $\{c_i(x) \geq 0, i \in \mathcal{I}\}$, constraints.

In the rest of this problem we consider an inequality constrained optimization problem

\begin{align}
\min_{x \in \Omega} f(x), \\
\Omega &= \{ x : c_i(x) \geq 0, i \in \mathcal{I} \},
\end{align}

where $f(x)$ and $-c_i(x)$ are convex functions for all $i \in \mathcal{I}$.

(b) Prove that $\Omega$ is a convex set.

(c) Assume that $x^*$ is a KKT-point, that is,

\begin{align}
\nabla \mathcal{L}(x^*, \lambda^*) &= 0, \\
\lambda^*_i \cdot c_i(x^*) &= 0, i \in \mathcal{I} \\
x^* &\in \Omega, \lambda^* \geq 0,
\end{align}

where $\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{I}} \lambda_i c_i(x)$.

Prove that $x^*$ is a global minimum for the problem defined in Eqns. (3) and (4).

(d) Consider

\begin{align}
f(x, y, z) &= x + 2y, \\
y &\geq 0, \\
4 - (x - 2)^2 - y^2 &\geq 0, \\
1 - x^2 - y^2 &\geq 0.
\end{align}

and explain why this is a problem of the form above. Make a sketch and guess a solution. Then show that (c) is fulfilled for the point you have found.

Problem 3

(a) Explain what is meant by the standard form of a Linear Programming (LP) problem. Transform the following problem to the standard form:

\begin{align}
\max_{x_1, x_2} \{2x_2 + x_1\}, \\
x_1 &\leq 4 + x_2, \\
x_2 &\geq 1 - 4x_1, \\
x_2 &\geq 0.
\end{align}
(b) Find the solutions to the following problem:

\[
\begin{align*}
\min \{ 7x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 \},
\end{align*}
\]

\[
\begin{align*}
5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 &= 1, \\
x_i &\geq 0.
\end{align*}
\]

Hints: The dual problem of

\[
\begin{align*}
\min c'x, \\
Ax &= b, \\
x &\geq 0,
\end{align*}
\]

is

\[
\begin{align*}
\min_{\lambda} (-b)'\lambda, \\
A'\lambda &\leq c.
\end{align*}
\]

Apply the KKT-equations for the dual problem.

**Problem 4**

(a) Define what is meant by a convex and a strictly convex functional \( J (y) \) defined for all \( y \)-s in a convex domain \( D \) of functions, and show that all functions \( y_0 \in D \) such that \( \delta J (y_0; v) = 0 \) in these cases are global minima.

Many control problems lead to the minimization of a functional of the form

\[
J (y) = \int_0^T \left[ y (t)^2 + \dot{y} (t)^2 \right] dt, \quad \left( \dot{y} = \frac{dy}{dt} \right),
\]

over some subset of \( C^1 [0, T] \).

(b) Show that \( J \) is strictly convex.

(c) Let

\[
D = \left\{ y \in C^1 [0, T] \mid y (0) = 1, \ y (T) \text{ is free} \right\}.
\]
Solve the optimization problem

\[(12) \quad \min_{y \in D} J(y),\]

when

\[(13) \quad G(y) = \int_0^T y(t) \, dt = 0.\]

\(\text{(d) Consider the functional}\)

\[(14) \quad H(y) = \left(\int_0^T y(t) \, dt\right)^2.\]

Show that the functional is convex, but not strictly convex.

\(\text{(e) Solve the following problem when } \mu \text{ is a fixed, positive constant:}\)

\[(15) \quad \min_{y \in D} \{ J(y) + \mu H(y) \}, \quad D = \{ y \in C^1[0,T] ; y(0) = 1, y(T) \text{ is free} \}.\]

What happen to the solution when \(\mu \to \infty?\)

**Hint:** Use partial integration to get rid of \(v'\) in the expression for \(\delta J(y;v)\), and recall the equation in \((c)\).