Optimisation of Production

(Trotman, pp. 72–74)

• $L(t)$: Inventory (= amount in storage)

• $P(t)$: Production per time unit

• $S(t)$: Sales per time unit

Equation for the inventory (system dynamics):

$$\frac{dL(t)}{dt} = P(t) - S(t) - \alpha L(t).$$

Here, $-\alpha L$ is loss (destruction) at the storage per time unit.
• $S(t)$: Sales prognosis (Expected sales)

**Optimal** inventory ($\mathcal{L}$) and production ($\mathcal{P}$) are matching the sales prognosis $S$,

$$\frac{d\mathcal{L}(t)}{dt} = \mathcal{P}(t) - S(t) - \alpha\mathcal{L}(t).$$

E.g., if we want $\mathcal{L}(t)$ to be constant, we need a production $\mathcal{P}(t) = S(t) + \alpha\mathcal{L}(t)$.

At $t = 0$, the inventory is off the ideal inventory, $L(0) = L_0 \neq \mathcal{L}(0)$.

**How do we plan the production $P(t)$ so as to minimize the extra cost of being off the ideal situation?**
The Cost Functional

(what we suffer from not being on the ideal track)

- $L(t) - L(t)$: Deviation from optimal inventory
- $P(t) - P(t)$: Deviation from optimal production

Common formulation:

$$C(P) = \int_0^T \left[ \beta^2 (L(t) - L(t))^2 + (P(t) - P(t))^2 \right] dt$$

This is an *Optimal Control problem*:

$$\min_P C$$

$$L'(t) = P(t) - S(t) - \alpha L(t),$$

$$L(0) = L_0.$$
- The variable $P(t)$ is the control variable.

- We expect the sales prognosis $S(t)$ to be true, and try to adjust the inventory towards the optimal $L(t)$.

- Inventory is forced to follow the system dynamics:

$$\frac{dL(t)}{dt} = P(t) - S(t) - \alpha L(t), \quad L(0) = L_0.$$ 

We are starting off the ideal curve (which we think we know), and want to minimize the cost of getting there.
Observe that $P$ may be expressed as

\[ P = L' + \alpha L + S. \]

Let $y = L(t) - \mathcal{L}(t)$. Then, since

\[ L' = P - S - \alpha L, \]
\[ \mathcal{L}' = \mathcal{P} - S - \alpha \mathcal{L}, \]

we have

\[ L' - \mathcal{L}' = P - \mathcal{P} - \alpha (L - \mathcal{L}), \]

or

\[ P - \mathcal{P} = \frac{dy}{dt} + \alpha y. \]

This is inserted into the cost functional:

\[ C(y) = \int_0^T \left[ \beta^2 y^2 + (y' + \alpha y)^2 \right] dt \]

- $\beta^2 y^2$ is strongly convex and $(z + \alpha y)^2$ is convex

Thus, $C$ is strictly convex.
Solution

The optimal solution is found by solving the Euler equation for $C'$:

$$\frac{d}{dx} f_y' - f_y = \frac{d}{dx} \left[ 2 \left( y' + \alpha y \right) \right] - \left[ 2y\beta^2 + 2 \left( y' + \alpha y \right) \alpha \right]$$

$$= 2 \left( y'' - (\alpha^2 + \beta^2) y \right) = 0.$$  

with a fixed value at $t = 0$,

$$y(0) = y_0 = L_0 - L_0.$$  

and a natural condition at $t = T$ (we have no particular restriction on $y$ at $T$):

$$f_y' (y(T)) = 2 \left( y'(T) + \alpha y(T) \right) = 0.$$  

With $\gamma^2 = (\alpha^2 + \beta^2)$, the problem may now be formulated as

$$y'' - \gamma^2 y = 0,$$

$$y(0) = y_0,$$

$$y'(T) + \alpha y(T) = 0.$$
From the general solution of the Euler equation

\[ y(t) = A \exp(\gamma t) + B \exp(-\gamma t), \]

it is easy to find the unique optimal solution

\[ y^*(t) = y_0 \frac{e^{\gamma t} + \rho e^{-\gamma t}}{1 + \rho}, \quad \rho = \frac{\gamma + \alpha}{\gamma - \alpha} e^{2\gamma T}. \]

The minimal value for the functional may also be computed:

\[ C(y^*) = y_0^2 \left[ \frac{\gamma (\rho - 1)}{\rho + 1} - \alpha \right]. \]
Application: The Ice Cream Factory

The factory is planning for the summer. Today, $t = 0$, is April 19th, and $t = 4$ is August 19th.

- Sales prognosis: $S(t) = 1 + t$

- Optimal inventory: $L(t) = 4$

- The optimal production rate for $L(t) = 4$ is $P(t) = S(t) + \alpha \times 4$.

For a decay rate, $\alpha \approx 0.1$, and $\beta \approx 1.5$ in the cost functional, the best production and inventory for various values of $L_0$ are shown on the plots.
Ideal and best possible production rate

Ideally and best possible inventory

Inventory < 4

Inventory > 4