Exercise 4.1 Calculate the Fourier series expansions of the following functions and verify the symmetric properties of the coefficients:

(a) \( f \) has period 2 and \( f(t) = |t| \) if \( |t| < 1 \).

(b) \( f \) has period \( a \) and \( f(t) = \frac{t}{a} \) if \( 0 \leq t < a \).

(c) \( f(t) = |\sin t| \).

(d) \( f(t) = \sin^3 t \).

Exercise 4.6 Find the Fourier series expansion of the function \( f \) with period \( a = 2 \) defined on \([-1, 1)\) for \( \omega \in \mathbb{C} \setminus \mathbb{Z} \) by

\[
f(t) = e^{i\omega t}.
\]

Deduce the relation

\[
\frac{\pi^2}{\sin^2 \pi x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}
\]

for all \( x \in \mathbb{R} \setminus \mathbb{Z} \) from Parseval's equality.

Exercise 5.10 Let \( f \) be the \( 2\pi \)-periodic function defined on \([-\pi, \pi)\) by

\[
f(x) = \cosh(ax), \quad a > 0.
\]

(a) Show that the Fourier series of \( f \) converges uniformly to \( f \).

(b) Compute the expansion of \( f \) in a series of cosines.

(c) Conclude from this that

\[
\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a} \left[ \coth(\pi a) - \frac{1}{\pi a} \right], \quad a \in \mathbb{R} \setminus \{0\}.
\]

(d) Justify the term-by-term differentiation of the series for \( f \) and show that

\[
\sinh(ax) = \frac{2\sinh(ax)}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + a^2} \sin nx, \quad x \in (-\pi, \pi).
\]

Exercise 5.11

(a) Show that if \( f \in C^2_0[0,a] \), then \( |c_n(f)| \leq \frac{K}{n^2} \).

(b) Show that \( f \in C^0_0[0,a] \) implies \( \lim_{n \to \infty} |n^k c_n(f)| = 0 \) for all \( k \in \mathbb{N} \).

Exercise 5.12 Take \( f \in L^1_0(0,a) \) and let \( f_k \) be a sequence in \( L^1_0(0,a) \) such that

\[
\lim_{k \to \infty} \int_0^a |f(t) - f_k(t)| \, dt = 0.
\]

Show that for fixed \( n \), \( \lim_{k \to \infty} c_n(f_k) = c_n(f) \).