Exercise 4.1 Calculate the Fourier series expansions of the following functions and verify the symmetric properties of the coefficients:

(a) f has period 2 and $f(t) = |t|$ if $|t| < 1$.

(b) f has period a and $f(t) = \frac{t}{a}$ if $0 \leq t < a$.

(c) $f(t) = |\sin t|$.

(d) $f(t) = \sin^3 t$.

Exercise 4.6 Find the Fourier series expansion of the function f with period $a = 2$ defined on $[-1, +1)$ for $z \in \mathbb{C}\backslash\mathbb{Z}$ by

$$f(t) = e^{i\pi z t}.$$

Deduce the relation

$$\frac{\pi^2}{\sin^2 \pi x} = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}$$

for all $x \in \mathbb{R}\backslash\mathbb{Z}$ from Parseval's equality.

Exercise 5.10 Let f be the 2π-periodic function defined on $[-\pi, \pi)$ by

$$f(x) = \cosh(ax), \quad a > 0.$$

(a) Show that the Fourier series of f converges uniformly to f.

(b) Compute the expansion of f in a series of cosines.

(c) Conclude from this that

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a}\left[\coth(\pi a) - \frac{1}{\pi a}\right], \quad a \in \mathbb{R}\backslash\{0\}.$$

(d) Justify the term-by-term differentiation of the series for f and show that

$$\sinh(ax) = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + a^2} \sin nx, \quad x \in (-\pi, \pi).$$

Exercise 5.11

(a) Show that if $f \in C_p^2[0, a]$, then $|c_n(f)| \leq \frac{K}{n^2}$.

(b) Show that $f \in C_p^\infty[0, a]$ implies $\lim_{|n| \to \infty} |n^k c_n(f)| = 0$ for all $k \in \mathbb{N}$.

Exercise 5.12 Take $f \in L_p^1(0, a)$ and let f_k be a sequence in $L_p^1(0, a)$ such that

$$\lim_{k \to \infty} \int_0^a |f(t) - f_k(t)|\, dt = 0.$$

Show that for fixed n, $\lim_{k \to \infty} c_n(f_k) = c_n(f)$.