Exercise 6.2 Let $f$ be defined on $[0, 1]$ by $f(x) = x(1 - x)$. 

(a) We wish to consider the expansion of $f$ in a series of sines. Sketch the graph of the periodic (period 2) extension $g$ of $f$. Is the sine series expansion of $g$ uniformly convergent? Can it be differentiated term by term?

(b) Compute the expansion of $g$.

(c) Deduce from (b) that

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3} = \frac{\pi^3}{32}. \]

(d) Compute $\int_0^1 f(x) \, dx$ and deduce that

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^4}{96}. \]

(e) Compute the expansion in cosines of

\[ f'(x) = 1 - 2x, \quad x \in [0, 1], \]

and the expansion of

\[ f''(x) = -2, \quad x \in (0, 1). \]

(f) Deduce from (e) that

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = \frac{\pi}{4}. \]

(g) Expand $f$ in a series of cosines and address the same questions as above.

Exercise 8.1 Consider two consecutive discrete Fourier transforms:

\[ (y_k) \xymatrix{ \xrightarrow{\mathcal{F}} \quad (Y_n) \quad \xymatrix{ \xrightarrow{\mathcal{F}} \quad (z_q).} \]

Compute $z_q$ as a function of $y_k$.

Exercise 8.2 Let $(x_k)$ and $(y_k)$ be two complex periodic sequences (with period $N$) such that

\[ x_{N-k} = \bar{x}_k \quad \text{and} \quad y_{N-k} = \bar{y}_k \]

for all $k \in \mathbb{Z}$. Show that the discrete Fourier transforms $(X_n)$ and $(Y_n)$ are real and that they can be computed with a single transform of order $N$.

Exercise 8.3 Compute the successive powers of the matrix $\Omega_N$.

Exercise 8.4 Prove Proposition 8.2.4 by computing $(\Omega_N \bar{Y})^t (\Omega_N Y)$.

Exercise 8.5 Calculate the discrete Fourier transform of the vector $x_k = k$, $k = 0, 1, \ldots, N - 1$. 