Exercise 6.2 Let f be defined on $[0,1]$ by $f(x) = x(1-x)$.

(a) We wish to consider the expansion of f in a series of sines. Sketch the graph of the periodic (period 2) extension g of f. Is the sine series expansion of g uniformly convergent? Can it be differentiated term by term?

(b) Compute the expansion of g.

(c) Deduce from (b) that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

(d) Compute $\int_0^1 f(x)\,dx$ and deduce that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

(e) Compute the expansion in cosines of

$$f'(x) = 1 - 2x, \quad x \in [0,1],$$

and the expansion of

$$f''(x) = -2, \quad x \in (0,1).$$

(f) Deduce from (e) that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

(g) Expand f in a series of cosines and address the same questions as above.

Exercise 8.1 Consider two consecutive discrete Fourier transforms:

$$(y_k) \overset{\mathcal{F}_N}{\longmapsto} (Y_n) \quad \text{and} \quad (Y_n) \overset{\mathcal{F}_N}{\longmapsto} (z_q).$$

Compute z_q as a function of y_k.

Exercise 8.2 Let (x_k) and (y_k) be two complex periodic sequences (with period N) such that

$$x_{N-k} = \bar{x}_k \quad \text{and} \quad y_{N-k} = \bar{y}_k$$

for all $k \in \mathbb{Z}$. Show that the discrete Fourier transforms (X_n) and (Y_n) are real and that they can be computed with a single transform of order N.

Exercise 8.3 Compute the successive powers of the matrix Ω_N.

Exercise 8.4 Prove Proposition 8.2.4 by computing $(\overline{\Omega}_N \overline{Y})^t\,(\Omega_N Y)$.

Exercise 8.5 Calculate the discrete Fourier transform of the vector $x_k = k$, $k = 0, 1, \ldots, N-1$.