OPERATOR SPLITTING FOR THE KDV EQUATION

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ABSTRACT. We provide a new analytical approach to operator splitting for equations of the type $u_t = Au + B(u)$ where A is a linear operator and B is quadratic. A particular example is the Korteweg–de Vries (KdV) equation $u_t - uu_x + u_{xxx} = 0$. We show that the Godunov and Strang splitting methods converge with the expected rates if the initial data are sufficiently regular.

1. INTRODUCTION

The ubiquitous Korteweg–de Vries (KdV) equation

$$u_t - uu_x + u_{xxx} = 0$$

offers the perfect blend of the simplest nonlinear convective term uu_x and the simplest dispersive term u_{xxx}: The well-known smooth soliton solutions of the KdV equation interact in an almost linear fashion apart from a phase shift. Furthermore, the KdV equation is completely integrable with an infinite family of conserved quantities. This is a result of a subtle interaction between the Burgers term uu_x and the Airy term u_{xxx}, as it is a well-known fact that the nonlinear Burgers equation $u_t - uu_x = 0$ generically develops shocks in finite time while the linear Airy equation $u_t + u_{xxx} = 0$ preserves all Sobolev norms.

The initial value problem for the KdV equation with $u|_{t=0} = u_0 \in H^s$, either on the whole real line or in the periodic case, has been extensively studied. An incomplete list of references is [2, 5]. For further discussions we refer to [8, 6].

The method of operator splitting, also called the fractional steps method, remains a very popular method both for analysis and numerical computations of partial differential equations. Instead of including a long list of references to relevant works, we entrust the reader instead with [4] and the overview of the field given therein. However, we refer to [7] for rigorous analysis of the splitting method applied to the Schrödinger–Poisson and the cubic nonlinear Schrödinger equations.

Formally operator splitting can be explained as follows. Let $u(t) = \Phi_C(t)u_0 = \Phi_C(t; u_0) \in X$, where X is some normed space, denote the solution of a given differential equation

$$u_t = C(u), \quad t \in [0, T], \qquad u|_{t=0} = u_0.$$

Here C will typically be a differential operator in the spatial variable. Assuming that we can write $C = A + B$ in a natural way, the idea of operator splitting is that

$$(1) \qquad u(t_n) \approx \big(\Phi_B(\Delta t)\Phi_A(\Delta t)\big)^n u_0, \quad t_n = n\Delta t,$$

where $\Delta t \ll 1$. In the case of linear ordinary differential equations this goes back to Sophus Lie. One of the reasons for its popularity is that operator splitting allows for a separate treatment of the equations $u_t = A(u)$ and $u_t = B(u)$; in particular this applies to the use of dedicated special numerical techniques for each of the equations, again we refer to [4] for a long list of examples and relevant references.

In the context of the KdV equation, the first use of operator splitting was reported in the brief paper by Tappert [9] where it was applied as a numerical method. Apparently the first rigorous results appeared in [3] where a Lax–Wendroff result was proved: If operator splitting converges to some limit function, then the limit function is a weak solution of the KdV equation. In addition a systematic study of operator splitting as a numerical method was undertaken for the KdV equation. More extensive rigorous results, specifically the convergence of the splitting approximations in a suitable functional space, were hampered by the apparent incompatibility between the Burgers equation and the Airy equation.

In the present paper we offer a new analytical approach to operator splitting for the KdV equation that will lead to rigorous convergence results (error estimates). Compared to earlier attempts, two new ingredients enter the present approach. First of all we actively use that the solution of the KdV equation remains bounded in a Sobolev space, that is, if $u_0 \in H^s(\mathbb{R})$, then $\|u(t)\|_{H^s(\mathbb{R})}$ remains bounded for $t \in [0, T]$. This together with a bootstrap argument is used to secure the existence of a uniform choice of time step Δt that prevents the solution from any "Burgers" step from blowing up. Indeed the main problem in this approach is that the Airy equation produces small oscillatory waves that, when used as initial data for the Burgers equation, produce shocks. Secondly, since the splitting approximations are merely defined at the discrete times $t = t_n$, to facilitate the convergence analysis we introduce an extension which is defined for all $t \in [0, T]$. Concretely, we introduce an extension v which depends on an additional time variable $\tau \in [0, T]$, i.e., $v = v(t, \tau)$, and let the evolution corresponding to each time variable be governed by one of the split operators in such a way that at each time level $t = t_n$ the extension $v(t_n, t_n)$ coincides with the regular splitting approximation. This extension approach is different from the conventional one, where one lets "time run twice as fast" in each of the subintervals $[t_n, t_n + \Delta t/2]$ and $[t_n + \Delta t/2, t_{n+1}]$ (cf. the discussion and references in [4]).

Formally the operator splitting (1), called sequential or Godunov splitting, yields a first order approximation in Δt, that is,

$$\left\| u(t_n) - \big(\Phi_A(\Delta t) \Phi_B(\Delta t) \big)^n u_0 \right\| = \mathcal{O}(\Delta t), \quad t_n \to t, \ \Delta t \to 0,$$

in an appropriate norm. We show that this holds rigorously for the KdV equation. More precisely, we prove in Theorem 2.4 that for $u_0 \in H^s(\mathbb{R})$ with $s \geq 5$ we have that for Δt sufficiently small

$$\|v(t, t) - u(t)\|_{H^{s-3}(\mathbb{R})} \leq K \Delta t, \quad t \in [0, T],$$

where K depends on s, T and u_0 only. Here $v(t, t)$ denotes the splitting approximation in our approach evaluated on the "diagonal" $(t, \tau) = (t, t)$.

To obtain second order convergence it is common to apply the Strang splitting formula, thus formally

$$\left\| u(t_n) - \Big(\big(\Phi_B(\Delta t/2) \Phi_A(\Delta t/2) \big) \big(\Phi_A(\Delta t/2) \Phi_B(\Delta t/2) \big) \Big)^n u_0 \right\| = \mathcal{O}(\Delta t^2),$$

where $t_n \to t$ as $\Delta t \to 0$. Here we show rigorously this result for the KdV equation. Indeed, in Theorem 3.5 we prove that if $u_0 \in H^s$ for some $s \geq 17$, then for Δt sufficiently small

$$\|v(t, t) - u(t)\|_{H^{s-9}} \leq K \Delta t^2, \quad t \in [0, T],$$

where the constant K depends on u_0, s and T only. Again $v(t, t)$ denotes the operator splitting approximation in our approach. Observe that we have to increase the regularity of the initial data, and hence of the solution, in order to get increased accuracy of operator splitting. Note also that with this type of operator splitting our error estimates are in a much weaker norm than the assumptions.

It is clear that the present approach applies to several other equations, and this is currently being investigated. Furthermore, for applications to numerical analysis, one would need to replace the exact solution operators Φ_A and Φ_B by numerical approximations, say Φ_A^δ and Φ_B^δ, and study their behavior in the limit as $\delta \to 0$, and also to replace the time derivatives by discrete differences. Again this will be studied separately.

The paper is organized as follows. We start by presenting the Godunov operator splitting method for abstract operators. Next we apply this approach to nonlinear ordinary differential equations where the procedure is fairly transparent, before we discuss Godunov splitting for the KdV equation. Subsequently we present the Strang splitting for abstract operators. Finally we apply this procedure to the KdV equation.

2. Operator splitting

We first present a formal calculation motivating the rigorous analysis that will follow. Consider an abstract differential equation

$$(2) \qquad u_t = C(u), \quad t \in [0, T], \qquad u|_{t=0} = u_0$$

for some fixed positive time T, where C typically will be a differential operator in the spatial variable. We assume that u_0 and u are in some Hilbert space X, and write the solution as

$$u(t) = \Phi_C(t; u_0).$$

Formally, expanding the solution in a Taylor series we find

$$u(t) = u_0 + t u_t(0) + \mathcal{O}(t^2) = u_0 + t C(u_0) + \mathcal{O}(t^2),$$

where we have used the equation (2). Assume that one can write

$$C = A + B$$

in some natural way. Operator splitting (of the Godunov type) then works as follows: Instead of solving the problem with C directly, one alternately solves for small time steps the equations $u_t = A(u)$ and $u_t = B(u)$. Making the time steps finer and finer, the approximation will presumably converge to the solution of the original equation (2).

More precisely, fix a positive time step Δt, let $t_n = n\Delta t$, $n \in \mathbb{N}_0$, and define a family $\{u_{\Delta t}(t_n)\}_n$ of functions

$$u_{\Delta t}(t_{n+1}) = \Phi_A(\Delta t; \Phi_B(\Delta t; u_{\Delta t}(t_n))) = \Phi_A(\Delta t) \circ \Phi_B(\Delta t) u_{\Delta t}(t_n), \quad n \in \mathbb{N}_0,$$

$$u_{\Delta t}(0) = u_0.$$

The traditional method of extending the solution to any $t \in [0, T]$ has been to let "time run twice as fast" in each of the subintervals $[t_n, t_{n+1/2}]$ and $[t_{n+1/2}, t_{n+1}]$, where $t_{n+1/2} = t_n + \Delta t/2$. Thus obtaining

$$(3) \qquad u_{\Delta t}(t) = \begin{cases} \Phi_B\left(2(t - t_n); u_{\Delta t}(t_n)\right) & \text{for } t \in \left[t_n, t_{n+1/2}\right], \\ \Phi_A\left(2(t - t_{n+1/2}); u_{\Delta t}(t_{n+1/2})\right) & \text{for } t \in \left[t_{n+1/2}, t_{n+1}\right]. \end{cases}$$

(In the present approach we will use a different extension to all times t.)

Formally one can show

$$\|u_{\Delta t}(t_n) - u(t_n)\| \leq \mathcal{O}(\Delta t) \text{ as } \Delta t \to 0 \text{ and } t_n \to t$$

in some norm.

The convergence can be improved to second order by using the Strang splitting formula. To this end we let the approximation (this time denoted $v_{\Delta t}$ to distinguish from the previous approximation) read

$$v_{\Delta t}(t_{n+1}) = \Phi_B(\Delta t/2; \Phi_A(\Delta t; \Phi_B(\Delta t/2; v_{\Delta t}(t_n))))$$
$$= \Phi_B(\Delta t/2) \circ \Phi_A(\Delta t) \circ \Phi_B(\Delta t/2) v_{\Delta t}(t_n)$$
$$= \Big(\Phi_B(\Delta t/2) \circ \Phi_A(\Delta t/2) \Big)$$
$$\circ \Big(\Phi_A(\Delta t/2) \circ \Phi_B(\Delta t/2) \Big) v_{\Delta t}(t_n), \quad n \in \mathbb{N}_0,$$
$$v_{\Delta t}(0) = u_0.$$

Formally one now has

$$\|v_{\Delta t}(t_n) - u(t_n)\| \le \mathcal{O}(\Delta t^2) \text{ as } \Delta t \to 0 \text{ and } t_n \to t.$$

To show that the operator splitting solutions are well defined, we shall later make use of the following bootstrap lemma, taken from [8, Prop. 1.21].

Lemma 2.1. *Let $t \in [0, T]$. Consider a continuous function $\phi \colon [0, T] \to [0, \infty)$. If there exists a positive constant α such that:*

    (**a**) *$\phi(0) \le \alpha$;*
    (**b**) *for any t such that $\phi(t) \le \alpha$, we can show that $\phi(t) \le \alpha/2$;*

then $\phi(t) \le \alpha/2$ for all $t \in [0, T]$.

2.1. **Doubling the time variable.** We shall formulate the operator splitting solution by introducing two time variables, and define a function $v = v(t, \tau)$ for (t, τ) in the set

$$\Omega_{\Delta t} = \bigcup_{n=0}^{\lfloor T/\Delta t \rfloor} [t_n, t_{n+1}] \times [t_n, t_{n+1}]$$

by requiring that

$$v(0, 0) = u_0,$$
(4)
$$v_t(t, t_n) = B(v(t, t_n)), \quad t \in (t_n, t_{n+1}],$$
$$v_\tau(t, \tau) = A(v(t, \tau)), \quad (t, \tau) \in [t_n, t_{n+1}] \times (t_n, t_{n+1}],$$

where $n = 0, \ldots, \lfloor T/\Delta t \rfloor$. Observe that

$$u_{\Delta t}(t_n) = v(t_n, t_n), \quad n = 0, \ldots, \lfloor T/\Delta t \rfloor.$$

where $u_{\Delta t}$ is given by (3). The specific extension v of $\{u_{\Delta t}(t_n)\}_n$ to $[0, T]$ will serve as an important technical tool to be utilized in the analysis.

The exact solution of (2) is still denoted by u. Introduce the error function

$$w(t) = v(t, t) - u(t).$$

The aim is to show that

$$\|w(t)\| \le \mathcal{O}(\Delta t), \quad \Delta t \to 0, \quad t \in [0, T]$$

in an appropriate norm.

We introduce second order Taylor expansions of the operators A and B (see [1, p. 29]),

$$A(f + g) = A(f) + dA(f)[g] + \int_0^1 (1 - \alpha) d^{(2)} A(f + \alpha g)[g]^2 d\alpha,$$

$$B(f + g) = B(f) + dB(f)[g] + \int_0^1 (1 - \alpha) d^{(2)} B(f + \alpha g)[g]^2 d\alpha.$$
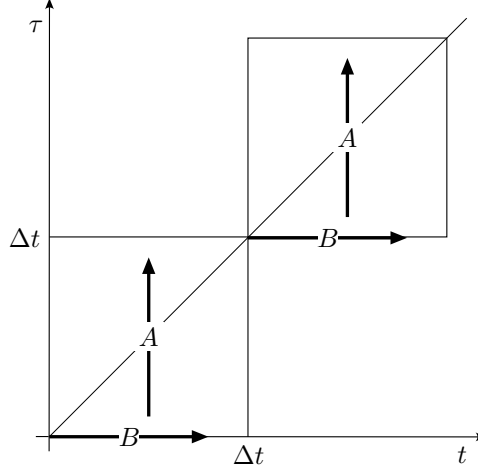
FIGURE 1. Schematic view of the Godunov splitting and the definition of $v(t, \tau)$, cf. equation (4).

Thus we find

$$
\begin{aligned}
w_t - dA(u)[w] - dB(u)[w] &= v_t + v_\tau - u_t - dA(u)[w] - dB(u)[w] \\
&= v_t + A(v) - (A + B)(u) - dA(u)[w] - dB(u)[w] \\
&= v_t - B(v) + \big(A(v) - A(u) - dA(u)[w]\big) \\
&\quad + \big(B(v) - B(u) - dB(u)[w]\big) \\
&= F(t) + \int_0^1 (1 - \alpha)d^{(2)}A(u + \alpha w)[w]^2 d\alpha \\
&\quad + \int_0^1 (1 - \alpha)d^{(2)}B(u + \alpha w)[w]^2 d\alpha
\end{aligned}
$$

(5)

where we have introduced a forcing term $F(t) = F(t, t)$ defined as

(6) $$F(t, \tau) = v_t(t, \tau) - B(v(t, \tau)).$$

We can rewrite (5) as follows

(7) $$w_t - dC(u)[w] = F(t) + \int_0^1 (1 - \alpha)d^{(2)}C(u + \alpha w)[w]^2 d\alpha.$$

The forcing term satisfies the following time development

$$
\begin{aligned}
F_\tau - dA(v)[F] &= v_{t\tau} - B(v)_\tau - dA(v)[v_t - B(v)] \\
&= A(v)_t - dB(v)[v_\tau] - dA(v)[v_t] + dA(v)[B(v)] \\
&= dA(v)[v_t] - dB(v)[A(v)] - dA(v)[v_t] + dA(v)[B(v)] \\
&= [A, B](v, v),
\end{aligned}
$$

(8)

where we have defined the commutator

$$[A, B](f, g) = dA(f)[B(g)] - dB(f)[A(g)].$$

For simplicity we will subsequently be writing $[A, B](v)$ rather than $[A, B](v, v)$.

2.2. **Ordinary differential equations.** As a warm-up we consider the ordinary differential equation

(9) $$u_t = C(u), \quad t > 0, \quad u(0) = u_0 \in \mathbb{R}^n.$$

To simplify the presentation we assume that C is a quadratic function, i.e., that $C''' = d^3 C = 0$ or $d^2 C[f,g]$ is constant. This means that the integral in (7) reduces to a constant. Furthermore, we assume that (9) is such that there is a unique solution $u(t)$ such that $|u(t)| \leq K_{u_0,T}$ for $t \in [0,T]$. Furthermore, we will assume that the operators A and B are two times continuously differentiable and

$$(10) \qquad A', A'' \in L^\infty, \quad A(0) = 0, \qquad |B(u)| \leq K |u|^2, \quad |B'(u)| \leq K |u|.$$

Throughout this paper we use the convention that for a quantity α, K_α denotes a constant depending on α (and perhaps other things). We use this notation to highlight the dependence on α. The actual value of K_α may be different at each occurrence.

Let now α be a positive constant (its precise value will be fixed later). To start the bootstrap argument we assume

$$(11) \qquad\qquad |v(t,\tau)| \leq \alpha, \quad (t,\tau) \in \Omega_{\Delta t}.$$

In this example, since C is quadratic, $d^2 C = \kappa$ for some constant symmetric matrix κ, and w satisfies

$$(12) \qquad\qquad w_t + C'(u)w = F + w^T \frac{\kappa}{2} w, \quad t > 0, \quad w(0) = 0.$$

Furthermore, F satisfies

$$F_\tau - A'(v)F = [A,B](v), \quad (t,\tau) \in [t_n, t_{n+1}] \times (t_n, t_{n+1}],$$

and $F(t,t_n) = 0$, for each n. This means that

$$\frac{\partial}{\partial \tau} |F| \leq K |F| + K\alpha^2.$$

Hence, Gronwall's inequality implies that

$$|F|(t,\tau) \leq K_\alpha \Delta t,$$

where K_α is a constant depending on the assumed bound on v in (11). In view of this bound and (12),

$$\frac{d}{dt} |w| \leq K_\alpha |w| + K_\alpha \Delta t, \quad w(0) = 0.$$

Gronwall's inequality gives that

$$|w(t)| \leq e^{K_\alpha t} t K_\alpha \Delta t \leq K_\alpha \Delta t.$$

Trivially we have

$$|v(t,t) - v(t,\tau)| \leq \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} |A(v(t,s))| \, ds \leq K_\alpha \Delta t,$$

for $(t,\tau) \in [t_n, t_{n+1}] \times (t_n, t_{n+1}]$ for any n. Then we can conclude that

$$|v(t,\tau)| \leq |u(t)| + |w(t)| + |v(t,t) - v(t,\tau)| \leq K + K_\alpha \Delta t.$$

Now we are in a position to choose α so that $K \leq \alpha/4$, this determines K_α. Next choose Δt so small that $K_\alpha \Delta t \leq \alpha/4$, then

$$|v(t,\tau)| \leq \alpha/2 \quad (t,\tau) \in \Omega_{\Delta t}.$$

Hence, by the bootstrap lemma, $|v(t,t)| \leq \alpha/2$ for all t. Consequently,

$$(13) \qquad\qquad |v(t,t) - u(t)| \leq K_{\alpha/2} \Delta t,$$

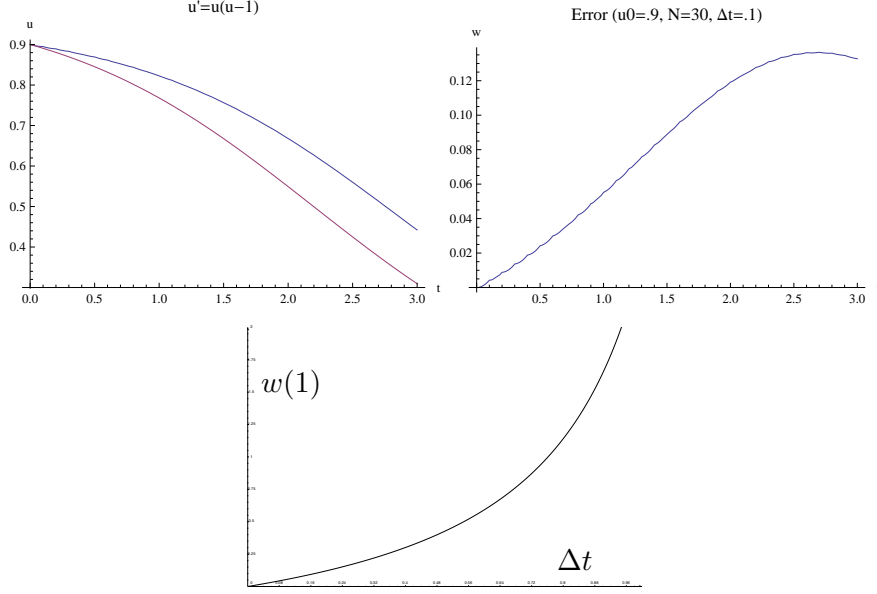i.e., the operator splitting is as expected first order accurate.

FIGURE 2. (Left) The exact (blue) and the approximate (red) solution. (Right) The error. (Below) The error $w(1)$ as a function of Δt.

Remark 2.2. *An interesting example is the logistic equation $u' = u(u-1)$ where we can write $A(u) = -u$ and $B(u) = u^2$. Exact solutions are available for all operators involved, specifically*

$$\Phi_C(t)u_0 = \frac{u_0}{u_0 + e^t(1-u_0)}, \quad \Phi_A(t)u_0 = u_0 e^{-t}, \quad \Phi_B(t)u_0 = \frac{u_0}{1-u_0 t}.$$

Let $u_0 \in (0,1)$. Then there is no blow-up in the full equation, but blow-up for the equation $u_t = b(u)$ at $t^* = 1/u_0$. The function v reads in our case

$$v(t,\tau) = \frac{v(t_n, t_n)e^{-(\tau-t_n)}}{1 - v(t_n, t_n)(t - t_n)}, \quad t, \tau \in [t_n, t_{n+1}],$$

where

$$v(t_n, t_n) = \frac{u_0(1 - e^{-\Delta t})}{(1 - e^{-\Delta t})e^{t_n} + u_0 \Delta t(1 - e^{t_n})}.$$

The quantity $v(t_n, t_n)$ is well-defined on $[0,T]$ if one chooses Δt such that

$$\Delta t < \frac{1 - e^{-\Delta t}}{u_0(1 - e^{-t_n})}.$$

Since $t_n \leq T$, and $e^T < u_0/(u_0 - 1)$, we find that $u_0(1 - e^{-t_n}) \leq u_0(1 - e^{-T}) < 1$. Thus we have to choose Δt such that

$$u_0\big(1 - e^{-T}\big) < \frac{1}{\Delta t}\big(1 - e^{-\Delta t}\big) = 1 - \frac{\Delta t}{2} + \mathcal{O}\big(\Delta t^2\big),$$

or

$$\Delta t < 2\big(1 - u_0(1 - e^{-T})\big).$$

In this case one can verify (13) *directly, namely*

$$|v(t_n, t_n) - u(t_n)|$$

$$= |u_0|^2 \left(e^{t_n} - 1\right) \frac{\left|(\Delta t)^{-1}(1 - e^{-\Delta t}) - 1\right|}{\left|(u_0 + e^{t_n}((\Delta t)^{-1}(1 - e^{-\Delta t}) - u_0))(u_0 + e^{t_n}(1 - u_0))\right|}$$

$$= \mathcal{O}(\Delta t).$$

The example is illustrated in Figure 2.

2.3. **The KdV equation.** Let us now apply this general framework to the KdV equation, that is,

$$u_t = u u_x - u_{xxx}, \quad u|_{t=0} = u_0 \in H^s(\mathbb{R}).$$

In this case $C(u) = u u_x - u_{xxx}$, and the evolution operator $\Phi_C(t; \cdot) \colon H^s \to H^s$ is bounded. Therefore, the Hilbert space $\mathbb{R}^n$ of the previous example is replaced by the Sobolev space $H^s(\mathbb{R})$ with the inner product and norm

$$(f, g)_{H^s(\mathbb{R})} = \sum_{j=0}^{s} \int_{\mathbb{R}} \partial_x^j f(x)\, \partial_x^j g(x)\, dx, \quad \|f\|_{H^s(\mathbb{R})}^2 = (f, f)_{H^s(\mathbb{R})}.$$

We choose A to equal (minus) the Airy operator:

$$A(f) = -f_{xxx},$$
$$dA(f)[g] = -g_{xxx},$$
$$d^{(2)}A(f)[g, h] = 0,$$

and B to equal the Burgers operator:

$$B(f) = f f_x,$$
$$dB(f)[g] = f g_x + f_x g,$$
$$d^{(2)}B(f)[g, h] = h g_x + h_x g,$$
$$d^{(3)}B(f)[g, h, k] = 0.$$

In this case the commutator reads

$$[A, B](f, f) = -\frac{3}{2} \partial_x^2 (f_x)^2.$$

Thus the equations (7), (6), and (8) are

(14)
$$w_t - (uw)_x + w_{xxx} = F + w w_x,$$
$$F = v_t - v v_x,$$

(15)
$$F_\tau + F_{xxx} = -\frac{3}{2} \partial_x^2 (v_x)^2,$$

respectively.

From [2] we recall the classical result: For $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$ there exists a unique solution $u \in C([0, T], H^s(\mathbb{R}))$ of the KdV equation

$$u_t = u u_x - u_{xxx}, \quad u|_{t=0} = u_0.$$

In particular, we can assume that there exists a constant K (depending on T, u_0, and s) such that

$$\|u(t)\|_{H^s(\mathbb{R})} \leq K, \quad t \in [0, T].$$

To save space and typing efforts, we shall write H^k for $H^k(\mathbb{R})$, and ∂ for ∂_x.

Next, let s be an odd integer larger or equal to 5. Assume that there exists a constant α such that

$$\|v(t, \tau)\|_{H^{\hat{k}}} \leq \alpha, \quad (t, \tau) \in \Omega_{\Delta t},$$

where $\hat{k}(s) = (s - 1)/2$.

Let us estimate the behavior of the Airy and Burgers operators. The Airy equation leaves all Sobolev norms invariant, viz.

$$\|v(t,\tau)\|_{H^k} = \|v(t,t_n)\|_{H^k}.$$

By definition we find for the Burgers operator

$$\frac{1}{2}\frac{d}{dt}\|v(t,t_n)\|_{H^s}^2 = (v,v_t)_{H^s} = \sum_{j=0}^{s}\int \partial^j v \partial^j (vv_x)\,dx$$

$$= \sum_{j=0}^{s}\sum_{k=0}^{j}\binom{j}{k}\int \partial^j v\,\partial^{k+1}v\,\partial^{j-k}v\,dx.$$

For $j < s$, any of the above terms can be estimated by

$$\left|\int \partial^j v\,\partial^{k+1}v\,\partial^{j-k}v\,dx\right| \le \left\|\partial^j v\right\|_{L^\infty}\left\|\partial^{\max\{k+1,j-k\}}v\right\|_{L^2}\left\|\partial^{\min\{k+1,j-k\}}v\right\|_{L^2}$$

$$\le K\|v\|_{H^s}^2\|v\|_{H^{\hat k}}$$

$$\le K_\alpha\|v\|_{H^s}^2.$$

For $j = s$ all terms with $k < s$ can be estimated similarly,

$$\left|\int \partial^s v\,\partial^{k+1}v\,\partial^{s-k}v\,dx\right| \le \left\|\partial^s v\right\|_{L^2}\left\|\partial^{\min\{k+1,s-k\}}v\right\|_{L^\infty}\left\|\partial^{\max\{k+1,s-k\}}v\right\|_{L^2}$$

$$\le K\|v\|_{H^s}^2\|v\|_{H^{\hat k}}$$

$$\le K_\alpha\|v\|_{H^s}^2.$$

We are left with the term where $k = s = j$, viz.

$$\left|\int \partial^s v\,\partial^{s+1}v\,v\,dx\right| = \frac{1}{2}\left|\int (\partial^s v)^2\,\partial v\,dx\right|$$

$$\le \|\partial v\|_{L^\infty}\|\partial^s v\|_{L^2}^2$$

$$\le K\|v\|_{H^{\hat k}}\|v\|_{H^s}^2$$

if $\hat k \ge 2$, i.e., if $s \ge 5$.

Thus

(16) $$\frac{d}{dt}\|v(t,t_n)\|_{H^s} \le K_\alpha\|v(t,t_n)\|_{H^s},$$

which implies that

$$\|v(t,t_n)\|_{H^s} \le e^{K_\alpha(t-t_n)}\|v(t_n,t_n)\|_{H^s}.$$

In particular, for any n,

$$\|v(t_n,t_n)\|_{H^s} \le e^{K_\alpha\Delta t}\|v(t_{n-1},t_{n-1})\|_{H^s} \le e^{K_\alpha t_n}\|u_0\|_{H^s}.$$

Thus we have shown the following result.

Lemma 2.3. *We have*

$$\|v(t,\tau)\|_{H^s} \le K_\alpha, \quad (t,\tau) \in \Omega_{\Delta t}.$$

Observe the general result, obtained by integration by parts,

$$(f,f_{xxx})_{H^k} = 0, \quad f \in H^{k+3}.$$

Next we analyze the forcing term that satisfying (15). By taking the H^{s-3} inner product with F in (15) we get

$$\frac{1}{2}\partial_\tau\|F(t,\tau)\|_{H^{s-3}}^2 = -(F,F_{xxx})_{H^{s-3}} - \frac{3}{2}(F,\partial^2(v_x)^2)_{H^{s-3}}$$

$$\leq \frac{3}{2} \left\| F \right\|_{H^{s-3}} \left\| \partial_x^{j+2} (v_x)^2 \right\|_{H^{s-3}}.$$

Since H^s is an algebra

$$\left\| \partial^{j+2} (v_x)^2 \right\|_{H^{s-3}} \leq \left\| (v_x)^2 \right\|_{H^{s-1}} \leq K \left\| v_x \right\|_{H^{s-1}}^2 \leq K_\alpha^2.$$

Also $F(t, t_n) = 0$ for $t \in [t_n, t_{n+1}]$, and we conclude that

$$\left\| F(t, \tau) \right\|_{H^{s-3}} \leq K_\alpha \Delta t, \quad (t, \tau) \in \Omega_{\Delta t}.$$

As for the error function $w = v(t, t) - u(t)$, we have the following estimates. Let $E(t) = \left\| w(t) \right\|_{H^{s-3}}$. By taking the inner product with w in (14)

$$\frac{1}{2} \frac{d}{dt} E^2(t) = (w, (uw)_x)_{H^{s-3}} - (w, w_{xxx})_{H^{s-3}} + (w, F)_{H^{s-3}} + (w, ww_x)_{H^{s-3}}$$

(17)
$$= (w, (uw)_x)_{H^{s-3}} + (w, ww_x)_{H^{s-3}} + (w, F)_{H^{s-3}}$$

$$= \sum_{j=0}^{s-3} \int \left(\partial^j w \partial^j \left(uw_x + u_x w + ww_x \right) \right) dx + (w, F)_{H^{s-3}}.$$

The first integrand on the right is expanded by Leibniz' rule. We get

$$\sum_{j=0}^{s-3} \sum_{k=0}^{j} \binom{j}{k} \int \partial^j w \partial^{k+1} w \partial^{j-k} u + \partial^j w \partial^k w \partial^{j+1-k} u + \partial^j w \partial^k w \partial^{j+1-k} w \, dx.$$

For $0 \leq k \leq j < s - 3$, we can estimate

$$\left| \int \partial^j w \, \partial^{k+1(k)} w \, \partial^{j-k(+1)} u \, dx \right| \leq \left\| w \right\|_{H^{s-3}}^2 \left\| \partial^{j-k(+1)} u \right\|_{L^\infty}$$

$$\leq K \left\| w \right\|_{H^{s-3}}^2 \left\| u \right\|_{H^s},$$

$$\left| \int \partial^j w \, \partial^k w \, \partial^{j+1-k} w \, dx \right| \leq \left\| \partial^j w \right\|_{L^\infty} \left\| w \right\|_{H^{s-3}}^2$$

$$\leq K \left(\left\| u \right\|_{H^s} + \left\| v \right\|_{H^s} \right) \left\| w \right\|_{H^{s-3}}^2.$$

For $j = s - 3$, we can use the same strategy for those terms with fewer than $s - 2$ derivatives on w. The term with $s - 2$ derivatives on w can be estimated as

$$\left| \int \partial^{s-3} w \, \partial^{s-2} w \, u \, dx \right| = \frac{1}{2} \left| \int \left(\partial^{s-3} w \right)^2 \, \partial u \, dx \right| \leq K \left\| u \right\|_{H^s} \left\| w \right\|_{H^{s-3}}^2,$$

$$\left| \int \partial^{s-3} w \, w \partial^{s-2} w \, dx \right| = \frac{1}{2} \left| \int \left(\partial^{s-3} w \right)^2 \, \partial w \, dx \right|$$

$$\leq K \left(\left\| u \right\|_{H^s} + \left\| v \right\|_{H^s} \right) \left\| w \right\|_{H^{s-3}}^2.$$

The last term in (17) is overestimated by $\left\| F \right\|_{H^{s-3}} \left\| w \right\|_{H^{s-3}}$, and we get

$$\frac{d}{dt} E^2(t) \leq K_\alpha E^2(t) + K E(t) \left\| F \right\|_{H^{s-3}} \leq K_\alpha E^2(t) + K \Delta t,$$

which implies

$$\frac{d}{dt} E(t) \leq K_\alpha E(t) + K \Delta t.$$

Since $E(0) = 0$, Gronwall's inequality yields

$$\left\| w(t) \right\|_{H^{s-3}} = E(t) \leq K_\alpha \Delta t.$$

Recall that $\left\| v(t, \tau) \right\|_{H^{s-3}} = \left\| v(t, t) \right\|_{H^{s-3}}$ because

$$\partial_\tau \left\| v(t, \tau) \right\|_{H^{s-3}}^2 = -2 \left(v, v_{xxx} \right)_{H^{s-3}} = 0.$$

Now u is bounded in H^s, and we infer that

$$\left\| v(t, \tau) \right\|_{H^{s-3}} \leq K + K_\alpha \Delta t.$$

Since $s \geq 5$,

$$s - 3 \geq \hat{k} = \frac{s-1}{2},$$

and we get

$$\|v(t,\tau)\|_{H^{\hat{k}}} \leq K + K_\alpha \Delta t.$$

First choose $\alpha \geq 4K$, then choose Δt so small that $K_\alpha \Delta t \leq \alpha/4$, so that $K + K_\alpha \Delta t \leq \alpha/4 + \alpha/4 = \alpha/2$. Hence, by the bootstrap argument we have proved the following theorem:

Theorem 2.4. *Fix $T > 0$. Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 5$. Then for Δt sufficiently small we have*

$$\|v(t,t) - u(t)\|_{H^{s-3}(\mathbb{R})} \leq K\Delta t, \quad t \in [0, T],$$

where K depends on s, T and u_0 only.

Remark 2.5. *Instead of defining v by (4) for the KdV equation, we could also interchange the order of the Airy operator A and the Burgers operator B in the definition of v. The same procedure as described above would apply, and Theorem 2.4 would remain valid. This remark is important for the Strang splitting to be discussed next.*

3. Strang splitting

To achieve higher-order convergence it is common to consider the so-called Strang splitting. Now we approximate the solution by using two Godunov splittings, each with a time step of $\Delta t/2$, and in alternating order. Explicitly, we define

$$
\begin{aligned}
v(0,0) &= u_0, \\
v_t(t, t_n) &= B(v(t, t_n)), \quad t \in (t_n, t_{n+1/2}], \\
v_\tau(t, \tau) &= A(v(t, \tau)), \quad (t, \tau) \in [t_n, t_{n+1/2}] \times (t_n, t_{n+1/2}], \\
v_\tau(t_{n+1/2}, \tau) &= A(v(t_{n+1/2}, \tau)), \quad \tau \in (t_{n+1/2}, t_{n+1}], \\
v_t(t, \tau) &= B(v(t, \tau)), \quad (t, \tau) \in (t_{n+1/2}, t_{n+1}] \times [t_{n+1/2}, t_{n+1}],
\end{aligned}
$$

(18)

for $n = 0, \ldots, \lfloor T/\Delta t \rfloor$. We consider this function for (t, τ) in the domain

$$\widetilde{\Omega}_{\Delta t} = \bigcup_{n=0}^{\lfloor T/\Delta t \rfloor} \left(\left[t_n, t_{n+1/2}\right]^2 \cup \left[t_{n+1/2}, t_{n+1}\right]^2 \right).$$

The aim is now to show that

$$\|v(t,t) - u(t)\| = \mathcal{O}\left(\Delta t^2\right)$$

in an appropriate norm. Here the abstract analysis of Section 2.1 applies, and we find:

(i): On the domains $[t_n, t_{n+1/2}] \times [t_n, t_{n+1/2}]$ we have

$$w_t - dC(u)[w] = F(t) + \int_0^1 (1-\alpha) d^{(2)} C(u + \alpha w)[w]^2 d\alpha,$$

(19)

$$F(t, \tau) = v_t(t, \tau) - B(v(t, \tau)).$$

The forcing term satisfies the following time development

$$
\begin{aligned}
F_\tau &= dA(v)[F] + [A, B](v), \\
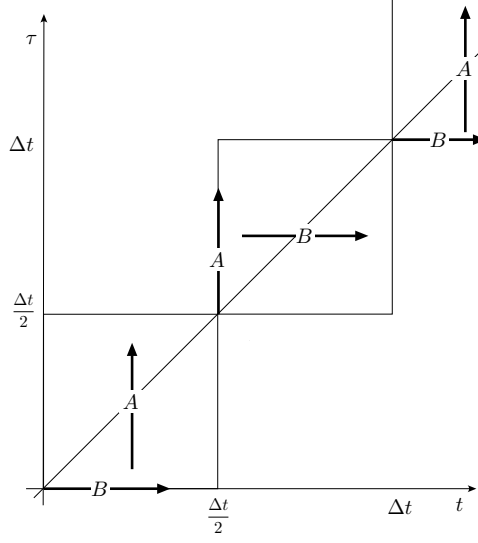F_t &= v_{tt} - dB(v)[F] - dB(v)[B(v)].
\end{aligned}
$$

FIGURE 3. A schematic view of Strang splitting.

(ii): On the domains $[t_{n+1/2}, t_{n+1}] \times [t_{n+1/2}, t_{n+1}]$ we have (here we write w_t rather than w_τ since w is a function of one variable only)

$$w_t - dC(u)[w] = G(t) + \int_0^1 (1-\alpha)d^{(2)}C(u+\alpha w)[w]^2 d\alpha,$$

(20)
$$G(t,\tau) = v_\tau(t,\tau) - A(v(t,\tau)).$$

The forcing term satisfies the following time development

$$G_t = dB(v)[G] + [B,A](v),$$
$$G_\tau = v_{\tau\tau} - dA(v)[G] - dA(v)[A(v)].$$

We extend F and G to all of $\widetilde{\Omega}_{\Delta t}$ using the same definitions, (19) and (20), respectively. Observe that this implies that $F = 0$ on $[t_{n+1/2}, t_{n+1}] \times [t_{n+1/2}, t_{n+1}]$, while $G = 0$ on $[t_n, t_{n+1/2}] \times [t_n, t_{n+1/2}]$. The total forcing term is defined by

$$H(t,\tau) = F(t,\tau) + G(t,\tau).$$

3.1. **Ordinary differential equations.** One can consider the case of ordinary differential equations, as we did in Subsection 2.2 for Godunov splitting, but for reasons of brevity we will only revisit the example in Remark 2.2.

Remark 3.1. *We find, using the definitions* (18), *that*

(21)
$$v(t,\tau) = \begin{cases} \frac{v(t_n,t_n)e^{-(\tau-t_n)}}{1-v(t_n,t_n)(t-t_n)}, & \text{for } t,\tau \in [t_n, t_{n+1/2}], \\ \frac{v(t_{n+1/2}, t_{n+1/2})}{e^{\tau-t_{n+1/2}} - v(t_{n+1/2}, t_{n+1/2})(t-t_{n+1/2})}, & \text{for } t,\tau \in [t_{n+1/2}, t_{n+1}], \end{cases}$$

for $n = 0, \ldots, \lfloor T/\Delta t \rfloor$. *By induction we determine*

(22)
$$v(t_n, t_n) = \frac{u_0(1 - e^{-\Delta t})}{(1 - e^{-\Delta t})e^{t_n} + u_0 \Delta t(e^{t_n} - 1)(e^{\Delta t} + 1)/2}.$$
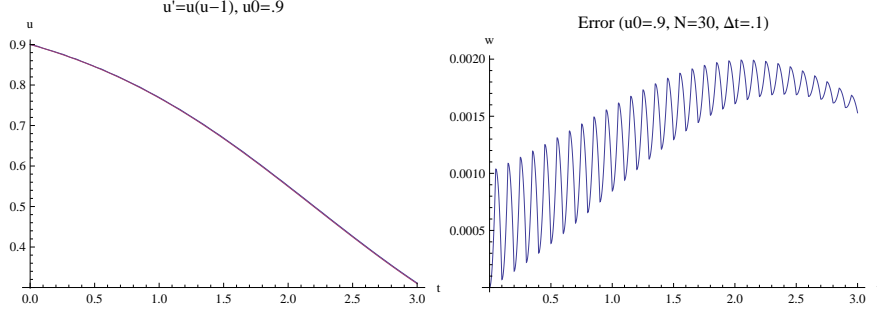
FIGURE 4. (Left) The exact (blue) and the approximate (red) solution. (Right) The error $v(t, t) - u(t)$.

In this case we compute, when we for convenience write $\alpha = u_0(1 - e^{t_n})$,

$$|w(t_n)| = |v(t_n, t_n) - u(t_n)|$$

$$= \left| \frac{u_0(1 - e^{-\Delta t})}{(1 - e^{-\Delta t})e^{t_n} + u_0 \Delta t(e^{t_n} - 1)(e^{\Delta t} + 1)/2} - \frac{u_0}{e^{t_n} + u_0(1 - e^{t_n})} \right|$$

$$(23) \qquad = |u_0| \left| \frac{1}{e^{t_n} + \alpha \Delta t(1 + e^{\Delta t})/(2(1 - e^{-\Delta t}))} - \frac{1}{e^{t_n} + \alpha} \right|$$

$$\leq \frac{|u_0|^2 \, e^{t_n}(e^{t_n} - 1)}{|(e^{t_n} + \alpha \Delta t(1 + e^{\Delta t})/(2(1 - e^{-\Delta t})))(e^{t_n} + \alpha)|} \left| 1 - \frac{\Delta t(1 + e^{\Delta t})}{2(1 - e^{-\Delta t})} \right|$$

$$\leq \mathcal{O}(\Delta t^2).$$

The example is illustrated in Figure 4. Observe the strong oscillations in the error, it is these oscillations which prevent the error from growing too large.

3.2. **The KdV equation.** For the KdV equation, we use $B(v) = vv_x$ and $A(v) = -v_{xxx}$, and the above analysis yields:

    (i): On domains $[t_n, t_{n+1/2}] \times [t_n, t_{n+1/2}]$ we have

$$(24) \qquad w_t - (uw)_x + w_{xxx} = F(t) + ww_x,$$
$$F(t, \tau) = v_t - vv_x.$$

    The forcing term satisfies the following time development

$$(25) \qquad F_\tau = -F_{xxx} - \frac{3}{2}\partial_x^2(v_x)^2,$$
$$F_t = v_{tt} - (vF)_x - (2vv_x^2 + v^2 v_{xx}),$$

    since $dB(v)[B(v)] = v(vv_x)_x + v_x(vv_x) = 2vv_x^2 + v^2 v_{xx}$.

    (ii): On domains $[t_{n+1/2}, t_{n+1}] \times [t_{n+1/2}, t_{n+1}]$ we have

$$(26) \qquad w_t - (uw)_x + w_{xxx} = G(t) + ww_x,$$
$$G(t, \tau) = v_\tau(t, \tau) + v_{xxx}.$$

    The forcing term satisfies the following time development

$$(27) \qquad G_t = (vG)_x + \frac{3}{2}\partial_x^2(v_x)^2,$$
$$G_\tau = v_{\tau\tau} + G_{xxx} - v^{(6)},$$

    since $dA(v)[A(v)] = (v_{xxx})_{xxx}$.

To start the bootstrapping procedure we fix an odd integer s and a positive constant α, whose values will be determined in the course of the argument. Now assume that

$$\|v(t,\tau)\|_{H^{\hat{k}}} \leq \alpha, \quad (t,\tau) \in \widetilde{\Omega}_{\Delta t},$$

where $\hat{k} = (s-1)/2$. As a consequence, we have

$$\|v(t,\tau)\|_{H^s} \leq K_\alpha, \quad (t,\tau) \in \widetilde{\Omega}_{\Delta t},$$

cf. the proof of Lemma 2.3, which can be easily adapted to Strang splitting.

We need to introduce the function

$$(28) \qquad z(t) = w(t) + w(t + \frac{\Delta t}{2}) = w(t) + \tilde{w}(t).$$

In the following we will write $\tilde{\phi}(t) = \phi(t + \frac{\Delta t}{2})$ for any function ϕ. Straightforward calculations yield that z satisfies

$$(29) \qquad z_t - (\frac{1}{2}z^2 + uz - z_{xx})_x = H + \widetilde{H} + \big(\tilde{w}(\tilde{u} - u) - w\tilde{w}\big)_x.$$

By our techniques, we must work in H^{s-9} (so at least $s \geq 9$), therefore set

$$E(t) = \|z(t)\|_{H^{s-9}(\mathbb{R})}.$$

By taking the H^{s-9} inner product with z in (29) we get

$$\frac{1}{2}\frac{d}{dt}E^2(t) = (z, (\frac{1}{2}z^2 + uz - z_{xx})_x)_{H^{s-9}} + (z, H + \widetilde{H})_{H^{s-9}} + (z, (\tilde{w}(\tilde{u} - u) - w\tilde{w})_x)_{H^{s-9}}$$

$$= (z, zz_x + u_x z + uz_x)_{H^{s-9}} + (z, H + \widetilde{H})_{H^{s-9}} - (z_x, \tilde{w}(\tilde{u} - u) - w\tilde{w})_{H^{s-9}}$$

$$\leq (z, zz_x + u_x z + uz_x)_{H^{s-9}} + E(t)\left\|H + \widetilde{H}\right\|_{H^{s-9}} + E(t)\|\tilde{w}(\tilde{u} - u) - w\tilde{w}\|_{H^{s-9}}$$

$$\leq K_\alpha E^2(t) + KE(t)\big(\left\|H + \widetilde{H}\right\|_{H^{s-9}} + \|\tilde{w}(\tilde{u} - u) - w\tilde{w}\|_{H^{s-9}}\big).$$

Thus by Gronwall's inequality

$$(30) \qquad \begin{aligned} E(t) &\leq E(0) + \int_0^t e^{K_\alpha(t-\sigma)}\big(\left\|H(\sigma,\sigma) + \widetilde{H}(\sigma,\sigma)\right\|_{H^{s-9}} \\ &\qquad + \|\tilde{w}(\sigma)(\tilde{u}(\sigma) - u(\sigma)) - w(\sigma)\tilde{w}(\sigma)\|_{H^{s-9}}\big)\,d\sigma \\ &\leq E(0) + e^{K_\alpha t}\int_0^t \big(\left\|H(\sigma,\sigma) + \widetilde{H}(\sigma,\sigma)\right\|_{H^{s-9}} \\ &\qquad + \|\tilde{w}(\sigma)(\tilde{u}(\sigma) - u(\sigma)) - w(\sigma)\tilde{w}(\sigma)\|_{H^{s-9}}\big)\,d\sigma. \end{aligned}$$

Next we turn to the detailed estimate of each of the terms in (30). We start with most involved one, the forcing term, which can be estimated as follows. We consider the term F first. Since $F(t, t_n) = 0$, we easily see that $F_t(t_n, t_n) = 0$; thus (cf. (25)) $(F_t + F_\tau)(t_n, t_n) = -\frac{3}{2}\partial_x^2(v_x)^2(t_n, t_n)$. Thus

$$\begin{aligned} F(t,t) &= (v_t - vv_x)(t,t) \\ &= F(t_n, t_n) + (F_t + F_\tau)(t_n, t_n)(t - t_n) \\ &\quad + \frac{(t - t_n)^2}{2}\int_0^1 (F_{tt} + 2F_{t\tau} + F_{\tau\tau})(\sigma(t - t_n) + t_n, \sigma(t - t_n) + t_n)d\sigma \\ &= -\frac{3}{2}\partial_x^2(v_x)^2(t_n, t_n)(t - t_n) \\ &\quad + \frac{(t - t_n)^2}{2}\int_0^1 (F_{tt} + 2F_{t\tau} + F_{\tau\tau})(\sigma(t - t_n) + t_n, \sigma(t - t_n) + t_n)d\sigma. \end{aligned}$$

As for the second derivatives, we find

$$F_{\tau\tau} = -F_{\tau xxx} - \frac{3}{2}\partial_x^2 \partial_\tau (v_x)^2$$

$$(31) \qquad = \partial_x^6 F + \frac{3}{2}\partial_x^5 (v_x)^2 - 3\partial_x^2 (v_x v^{(6)}),$$

$$F_{\tau t} = -v_{ttxxx} - (v_{xxx}v_x + vv_{xxxx})_t,$$

$$F_{tt} = v_{ttt} - v_{tt}v_x - 2v_t v_{xt} - vv_{xtt}.$$

Similarly we find for the forcing term G the following estimates. Since $G(t_{n+1/2}, \tau) = 0$, we easily see that $G_\tau(t_{n+1/2}, t_{n+1/2}) = 0$; thus (cf. (27)) $(G_t + G_\tau)(t_{n+1/2}, t_{n+1/2}) = \frac{3}{2}\partial_x^2 (v_x)^2 (t_{n+1/2}, t_{n+1/2})$. Thus

$$G(t + \frac{\Delta t}{2}, t + \frac{\Delta t}{2}) = (v_\tau + v_{xxx})(t + \frac{\Delta t}{2}, t + \frac{\Delta t}{2})$$

$$= G(t_{n+1/2}, t_{n+1/2}) + (G_t + G_\tau)(t_{n+1/2}, t_{n+1/2})(t - t_n)$$

$$+ \frac{(t - t_n)^2}{2} \int_0^1 (G_{tt} + 2G_{t\tau} + G_{\tau\tau})(\sigma(t - t_n) + t_n, \sigma(t - t_n) + t_n)d\sigma$$

$$= \frac{3}{2}\partial_x^2 (v_x)^2 (t_{n+1/2}, t_{n+1/2})(t - t_n)$$

$$+ \frac{(t - t_n)^2}{2} \int_0^1 (G_{tt} + 2G_{t\tau} + G_{\tau\tau})(\sigma(t - t_n) + t_n, \sigma(t - t_n) + t_n)d\sigma.$$

The second derivatives $G_{tt} + 2G_{t\tau} + G_{\tau\tau}$ will have to be considered similarly to those for F. These read

$$G_{tt} = 2v_x^2 G + 4vv_x G_x + v^2 G_{xx} + Gvv_{xx}$$

$$+ 6v_{xx}(v_x v)_{xx} + vv_x v_{xxx} + v(vv_x)_{xxx},$$

$$G_{\tau t} = v_{\tau\tau}v_x + 2v_\tau v_{\tau x} + vv_{\tau\tau x} + \partial^4(Gv) + \frac{3}{2}\partial^5(v_x^2) - \partial^6(vv_x),$$

$$G_{\tau\tau} = v_{\tau\tau\tau} + v_{\tau\tau xxx}.$$

Lemma 3.2. *We have the estimate*

$$\|\Phi(t, \tau)\|_{H^{s-9}}^2 + \|\Psi(\bar{t}, \bar{\tau})\|_{H^{s-9}}^2 \le K_\alpha, \quad (t, \tau), (\bar{t}, \bar{\tau}) \in \widetilde{\Omega}_{\Delta t},$$

where $\Phi = (F_{tt} + 2F_{t\tau} + F_{\tau\tau})$ and $\Psi = (G_{tt} + 2G_{t\tau} + G_{\tau\tau})$.

Proof. We have that $\|F\|_{H^{s-9}} \le K_\alpha \Delta t$, we shall get a similar estimate for $\|G\|_{H^{s-9}}$. For $t_n \le \tau \le t_{n+1/2}$, we have that

$$G_t = v_x G + vG_x + \frac{3}{2}\partial^2(v_x^2),$$

and $G(t, t_n) = 0$. Taking the H^{s-9} inner product with G we get

$$\frac{1}{2}\frac{d}{dt}\|G\|_{H^{s-9}}^2 = \sum_{k=0}^{s-9} \int \partial^k(v_x G)\partial^k G + \partial^k(vG_x)\partial^k G + \frac{3}{2}\partial^{k+2}(v_x^2)\partial^k G\, dx.$$

The first term expands by the Leibniz rule; a typical term in this expansion reads (here $0 \le j \le k \le s - 9$),

$$\int \partial^{j+1}v\partial^{k-j}G\partial^k G\, dx \le \|\partial^{j+1}v\|_{L^\infty} \|G\|_{H^{s-9}}^2.$$

Similarly the second term can be expanded and estimated, except the term containing $\partial^{k+1}G$, which is estimated as

$$\int v\partial^{k+1}G\partial^k G\, dx = \frac{1}{2}\int v\partial(\partial^k G)^2\, dx = -\frac{1}{2}\int \partial v(\partial^k G)^2\, dx$$

$$\leq \frac{1}{2} \left\| \partial v \right\|_{L^\infty} \left\| G \right\|_{H^{s-9}}^2 .$$

We bound the last term as

$$\left| \int \partial^{k+2} \left(v_x^2 \right) \partial^k G \, dx \right| \leq K \left\| v \right\|_{H^s}^2 \left\| G \right\|_{H^{s-9}} .$$

Summing up, we get

$$\frac{d}{dt} \left\| G \right\|_{H^{s-9}}^2 \leq K_\alpha \left\| G \right\|_{H^{s-9}}^2 + K_\alpha \left\| G \right\|_{H^{s-9}} .$$

Using Gronwall's inequality and that $G(t_n, \tau) = 0$ we get

$$\left\| G \right\|_{H^{s-9}} \leq K_\alpha \Delta t.$$

We also need estimates for v_t, v_{tt} and v_{ttt} where $t, \tau \in [t_n, t_{n+1/2}]$. In this set, $v_\tau = -v_{xxx}$, and this evolution preserves the H^k norm. For $\tau = t_n$,

$$v_t = v v_x,$$
$$v_{tt} = v_t v_x + v v_{tx},$$
$$v_{ttt} = v_{tt} v_x + 2 v_t v_{tx} + v v_{ttx}.$$

Therefore

$$\left\| v_t \right\|_{H^k} \leq \left\| v \right\|_{H^k} \left\| v \right\|_{H^{k+1}} \leq K \left\| v \right\|_{H^{k+1}}^2 ,$$
$$\left\| v_{tt} \right\|_{H^k} \leq K \left\| v \right\|_{H^{k+2}}^3 ,$$
$$\left\| v_{ttt} \right\|_{H^k} \leq K \left\| v \right\|_{H^{k+3}}^4 .$$

Next we turn to estimates of v_τ, $v_{\tau\tau}$ and $v_{\tau\tau\tau}$ in the set $t, \tau \in [t_{n+1/2}, t_{n+1}]$. Here $v_t = v v_x$, and setting $\omega = v_\tau$, $\theta = v_{\tau\tau}$ and $\gamma = v_{\tau\tau\tau}$, we get

$$\omega_t = v_x \omega + v \omega_x, \qquad\qquad \omega(t_{n+1/2}, \tau) = -\partial^3 v,$$
$$\theta_t = 2 \omega \omega_x + v_x \theta + v \theta_x, \qquad\qquad \theta(t_{n+1/2}, \tau) = \partial^6 v,$$
$$\gamma_t = 3 \theta \omega_x + 3 \theta_x \omega + v_x \gamma + v \gamma_x, \qquad\qquad \gamma(t_{n+1/2}, \tau) = -\partial^9 v.$$

Starting with ω, for $k < s - 9$ we get

$$\frac{1}{2} \frac{d}{dt} \left\| \omega \right\|_{H^k}^2 = \sum_{j=0}^k \int \partial^j \left(v_x \omega \right) \partial^j \omega + \partial^j \left(v \omega_x \right) \partial^j \omega \, dx.$$

Using Leibniz' rule, all terms except the ultimate one will be of the type

$$\left| \int \partial^\ell v \partial^{j-\ell} \omega \partial^j \, dx \right| \leq \left\| v \right\|_{H^s} \left\| \omega \right\|_{H^k}^2 .$$

The ultimate term (with one too many derivatives on ω) is as usual estimated as

$$\left| \int v \partial^{k+1} \omega \partial^k \omega \, dx \right| = \frac{1}{2} \left| \int \partial v \left(\partial^k \omega \right)^2 dx \right| \leq \left\| v \right\|_{H^s} \left\| \omega \right\|_{H^k}^2 .$$

Gronwall's inequality then yields

$$\left\| \omega \right\|_{H^k} (t) \leq e^{K_\alpha \Delta t} \left\| \omega \right\|_{H^k} (t_{n+1/2}) \leq K_\alpha \left\| v \right\|_{H^{k+3}} (t_{n+1/2}, \tau) .$$

Reasoning similarly for θ, we find that

$$\left\| \theta \right\|_{H^k} (t) \leq e^{K_\alpha \Delta t} \left(\max_{s \in [t_{n+1/2}, t_{n+1}]} \left\| \omega \right\|_{H^{k+1}}^2 (s) + \left\| \theta \right\|_{H^k} (t_{n+1/2}) \right)$$
$$\leq K_\alpha \left(\left\| v \right\|_{H^{k+4}}^2 + \left\| v \right\|_{H^{k+6}} \right) (t_{n+1/2}, \tau) .$$

Finally, the estimate for γ reads

$$\left\| \gamma \right\|_{H^k} \leq K_\alpha \left(\left\| \theta \right\|_{H^{k+1}} \left\| \omega \right\|_{H^{k+1}} + \left\| \gamma \right\|_{H^k} \right) (t_{n+1/2}, \tau)$$
$$\leq K_\alpha \left(\left\| v \right\|_{H^{k+4}}^2 + \left\| v \right\|_{H^{k+5}}^4 + \left\| v \right\|_{H^{k+7}}^2 + \left\| v \right\|_{H^{k+9}} \right) (t_{n+1/2}, \tau) .$$

Summing up

$$\|v_\tau\|_{H^k} \le K_\alpha \|v\|_{H^{k+3}}, \ \|v_{\tau\tau}\|_{H_k} \le K_\alpha \|v\|_{H^{k+6}}, \ \|v_{\tau\tau\tau}\|_{H_k} \le K_\alpha \|v\|_{H^{k+9}}.$$

Now

$$\|\Phi\|_{H^{s-9}} \le \|F_{\tau\tau}\|_{H^{s-9}} + 2\|F_{t\tau}\|_{H^{s-9}} + \|F_{tt}\|_{H^{s-9}}.$$

Working in the square $[t_n, t_{n+1/2}]^2$, from (31),

$$\|F_{\tau\tau}\|_{H^{s-9}} \le \left\|\partial^6 F\right\|_{H^{s-9}} + \frac{3}{2}\left\|\partial^5 (\partial v)^2\right\|_{H^{s-9}} + 3\left\|\partial^2 \left(\partial v \partial^6 v\right)\right\|_{H^{s-9}}$$

$$(32) \qquad \le \|F\|_{H^{s-3}} + K_\alpha \|v\|_{H^{s-3}}^2 + K_\alpha \|v\|_{H^{s-1}}^2$$

$$\le K_\alpha,$$

and

$$\|F_{t\tau}\|_{H^{s-9}} \le \|v_{tt}\|_{H^{s-6}} + K_\alpha \Big(\|v\|_{H^{s-8}} \|v_t\|_{H^{s-6}} + \|v_t\|_{H^{s-9}} \|v\|_{H^{s-6}}$$

$$+ \|v\|_{H^{s-9}} \|v_t\|_{H^{s-5}} + \|v_t\|_{H^{s-9}} \|v\|_{H^{s-5}} \Big)$$

$$(33) \qquad \le K_\alpha \Big(\|v\|_{H^{s-4}} + \|v\|_{H^{s-8}} \|v\|_{H^{s-5}}$$

$$+ \|v\|_{H^{s-7}} \|v\|_{H^{s-7}} + \|v\|_{H^{s-9}} \|v\|_{H^{s-4}}$$

$$+ \|v\|_{H^{s-8}} \|v\|_{H^{s-5}} \Big)$$

$$\le K_\alpha,$$

and also

$$\|F_{tt}\|_{H^{s-9}} \le \|v_{ttt}\|_{H^{s-9}} + \|v_{tt} v_x\|_{H^{s-9}} + 2\|v_t v_{xx}\|_{H^{s-9}} + \|v v_{txx}\|_{H^{s-9}}$$

$$\le K_\alpha \Big(\|v\|_{H^{s-6}} + \|v\|_{H^{s-7}} \|v\|_{H^{s-8}} + \|v\|_{H^{s-9}} \|v\|_{H^{s-6}} \Big)$$

$$(34) \qquad \le K_\alpha.$$

Hence $\|\Phi\|_{H^{s-9}} \le K_\alpha$.

In the second square $[t_{n+1/2}, t_n]^2$, we write

$$\|\Psi\|_{H^{s-9}} \le \|G_{tt}\|_{H^{s-9}} + 2\|G_{t\tau}\|_{H^{s-9}} + \|G_{\tau\tau}\|_{H^{s-9}}.$$

Each term above is estimated individually as

$$\|G_{tt}\|_{H^{s-9}} \le 2\left\|v_x^2 G\right\|_{H^{s-9}} + 4\|v v_x G_x\|_{H^{s-9}} + \left\|v^2 G_{xx}\right\|_{H^{s-9}}$$

$$+ 3\left\|v_{xx} \left(v^2\right)_{xx}\right\|_{H^{s-9}} + \|v v_{xx} v_{xxx}\|_{H^{s-9}} + \left\|v \left(v v_x\right)_{xxx}\right\|_{H^{s-9}}$$

$$\le K_\alpha \Big(\|v\|_{H^{s-8}} \|G\|_{H^{s-9}} + \|v\|_{H^{s-8}} \|G\|_{H^{s-8}} + \|v\|_{H^{s-9}} \|G\|_{H^{s-7}}$$

$$+ \|v\|_{H^{s-7}}^2 + \|v\|_{H^{s-6}}^3 + \|v\|_{H^{s-5}}^3 \Big)$$

$$(35) \qquad \le K_\alpha,$$

$$\|G_{t\tau}\|_{H^{s-9}} \le \|v_{\tau\tau} v_x\|_{H^{s-9}} + 2\|v_\tau v_{xx}\|_{H^{s-9}} + \|v v_{\tau\tau x}\|_{H^{s-9}} + \left\|\partial^4 (Gv)\right\|_{H^{s-9}}$$

$$+ \frac{3}{2}\left\|\partial^5 (v_x^2)\right\|_{H^{s-9}} + \left\|\partial^6 (v v_x)\right\|_{H^{s-9}}$$

$$\le K_\alpha \Big(\|v\|_{H^{s-3}} \|v\|_{H^{s-8}} + \|v\|_{H^{s-6}} \|v\|_{H^{s-8}} + \|v\|_{H^{s-9}} \|v\|_{H^{s-1}}$$

$$+ \|G\|_{H^{s-5}} \|v\|_{H^{s-5}} + \|v\|_{H^{s-3}}^2 + \|v\|_{H^{s-3}} \|v\|_{H^{s-2}} \Big)$$

$$(36) \qquad \le K_\alpha,$$

$$\|G_{\tau\tau}\|_{H^{s-9}} \le \|v_{\tau\tau\tau}\|_{H^{s-9}} + \|v_{\tau\tau xxx}\|_{H^{s-9}}$$

$$\le K_\alpha \Big(\|v\|_{H^s} + \|v\|_{H^s} \Big)$$

(37) $\leq K_\alpha.$

Collecting (34), (33), (32), (35), (36) and (37) finishes the proof of the lemma.   □

Lemma 3.3. *The map*

$$[0,T] \ni t \mapsto \partial^2 (v_x^2)(t,t) \in H^{s-9},$$

is Lipschitz continuous with Lipschitz constant at most K_α.

Proof. Set $w(t,\tau) = \partial^2 (v_x^2)(t,\tau)$. Then we have

$$w_t = 4v_{xx}v_{txx} + 2v_{txxx}v_x + 2v_{xxx}v_{tx},$$
$$w_\tau = 4v_{xx}v_{\tau xx} + 2v_{\tau xxx}v_x + 2v_{xxx}v_{\tau x}.$$

In the square $[t_n, t_{n+1/2}]^2$ we have $v_\tau = -v_{xxx}$, thus

$$w_\tau = -4v_{xx}\partial^5 v - 2\partial^6 v v_x - 2v_{xxx}\partial^4 v.$$

Hence in this square

$$\|w_t + w_\tau\|_{H^{s-9}} \leq K \Big(\|v\|_{H^{s-7}} \|v_t\|_{H^{s-7}} + \|v_t\|_{H^{s-6}} \|v\|_{H^{s-8}}$$
$$+ \|v\|_{H^{s-6}} \|v_t\|_{H^{s-8}} + \|v\|_{H^{s-7}} \|v\|_{H^{s-4}}$$
$$+ \|v\|_{H^{s-3}} \|v\|_{H^{s-1}} + \|v\|_{H^{s-6}} \|v\|_{H^{s-5}} \Big)$$
$$\leq K_\alpha.$$

In the second square $[t_{n+1/2}, t_{n+1}]^2$ we have $w_t = vv_x$, and

$$w_t = 4v_{xx} (vv_x)_{xx} + 2 (vv_x)_{xxx} v_x + 2v_{xxx} (vv_x)_x.$$

Therefore in this square

$$\|w_t + w_\tau\|_{H^{s-9}} \leq K \Big(\|v\|_{H^{s-7}} \|vv_x\|_{H^{s-7}} + \|vv_x\|_{H^{s-6}} \|v\|_{H^{s-9}}$$
$$+ \|v\|_{H^{s-6}} \|vv_x\|_{H^{s-8}} + \|v\|_{H^{s-8}} \|v_\tau\|_{H^{s-8}}$$
$$+ \|v\|_{H^{s-8}} \|v_\tau\|_{H^{s-6}} + \|v\|_{H^{s-6}} \|v_\tau\|_{H^{s-8}} \Big)$$
$$\leq K \Big(\|v\|_{H^{s-7}}^2 \|v\|_{H^{s-6}} + \|v\|_{H^{s-7}} \|v\|_{H^{s-6}} \|v\|_{H^{s-9}}$$
$$+ \|v\|_{H^{s-6}} \|v\|_{H^{s-8}} \|v\|_{H^{s-7}} + \|v\|_{H^{s-8}} \|v\|_{H^{s-5}}$$
$$+ \|v\|_{H^{s-8}} \|v\|_{H^{s-3}} + \|v\|_{H^{s-6}} \|v\|_{H^{s-5}} \Big)$$
$$\leq K_\alpha.$$

□

From this lemma it follows that

$$\Gamma(t) = -\partial^2 (v_x)^2(t,t) + \partial^2 (v_x)^2 \left(t + \frac{\Delta t}{2}, t + \frac{\Delta t}{2} \right)$$

satisfies

$$\|\Gamma(t)\|_{H^{s-9}} \leq K_\alpha \Delta t, \quad t \in [0,T].$$

The following lemma will be convenient:

Lemma 3.4. *For $t \in [t_m, t_{m+1/2}]$ we have*

(38) $$\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(t) + G\left(t + \Delta t/2\right) \right) \right)^2 dx \leq K_\alpha \Delta t^4,$$

(39) $$\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(t) + G\left(t - \Delta t/2\right) \right) \right)^2 dx \leq K_\alpha \Delta t^4.$$

Proof. We show (38), (39) is proved similarly. By a Taylor expansion, for $t \in [t_m, t_{m+1/2}]$

$$\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(t) + G\left(t + \Delta t/2\right)\right)\right)^2 dx$$

$$= \sum_{j=0}^{s-9} \int \left[\partial^j \left(-\frac{3}{2} \partial^2 \left(v_x\right)^2 (t_m)(t - t_m) + \frac{(t - t_m)^2}{2} \int_0^1 \Phi\left(\sigma(t - t_m) + t_m\right) d\sigma \right.\right.$$

$$\left.\left. + \frac{3}{2} \partial^2 \left(v_x\right)^2 \left(t_m + \Delta t/2\right)(t - t_m) + \frac{(t - t_m)^2}{2} \int_0^1 \Psi\left(\sigma(t - t_m) + t_m\right) d\sigma \right)\right]^2 dx$$

$$= \sum_{j=0}^{s-9} \int \left[\partial^j \left(-\frac{3}{2} \partial^2 \left(v_x\right)^2 (t_m) + \frac{3}{2} \partial^2 \left(v_x\right)^2 \left(t_m + \Delta t/2\right)(t - t_m) \right.\right.$$

$$\left.\left. + \frac{(t - t_m)^2}{2} \int_0^1 \left(\Phi\left(\sigma(t - t_m) + t_m\right) + \Psi\left(\sigma(t - t_m) + t_m\right)\right) d\sigma \right)\right]^2 dx$$

$$\leq 2 \frac{9}{4} \sum_{j=0}^{s-9} \int \left(\partial^j \left(-\partial^2 \left(v_x\right)^2 (t_m) + \partial^2 \left(v_x\right)^2 \left(t_m + \Delta t/2\right)\right)(t - t_m)\right)^2 dx$$

$$+ 2 \sum_{j=0}^{s-9} \int \left(\partial^j \int_0^1 \left(\Phi\left(\sigma(t - t_m) + t_m\right) + \Psi\left(\sigma(t - t_m) + t_m\right)\right) d\sigma \, (t - t_m)^2 \right)^2 dx$$

$$= \frac{9}{2} \left\| \Gamma(t) \right\|_{H^{s-9}}^2 (t - t_m)^2$$

$$+ 2 \sum_{j=0}^{s-9} \int \left(\partial^j \int_0^1 \left(\Phi\left(\sigma(t - t_m) + t_m\right) + \Psi\left(\sigma(t - t_m) + t_m\right)\right) d\sigma \right)^2 dx \, (t - t_m)^4$$

$$\leq K_\alpha \Delta t^2 (t - t_m)^2$$

$$+ 4 \int_0^1 \left(\left\| \Phi\left(\sigma(t - t_m) + t_m\right) \right\|_{H^{s-9}}^2 + \left\| \Psi\left(\sigma(t - t_m) + t_m\right) \right\|_{H^{s-9}}^2 \right) d\sigma \, (t - t_m)^4$$

$$\leq K_\alpha \Delta t^2 (t - t_m)^2 + K_\alpha (t - t_m)^4$$

$$\leq K_\alpha \Delta t^4.$$

$$\square$$

Combining the above results, we find that for $t \in [t_{n-1}, t_n)$,

$$\int_0^t \left\| H(\sigma) + \widetilde{H}(\sigma) \right\|_{H^{s-9}} d\sigma \leq \int_0^{t_n} \left\| H(\sigma) + \widetilde{H}(\sigma) \right\|_{H^{s-9}} d\sigma$$

$$= \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma) + G(\sigma)\right) + \partial^j \left(F(\sigma + \frac{\Delta t}{2}) + G(\sigma + \frac{\Delta t}{2})\right)\right)^2 dx \right)^{1/2} d\sigma$$

$$= \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma) + G(\sigma + \frac{\Delta t}{2})\right) + \partial^j \left(F(\sigma + \frac{\Delta t}{2}) + G(\sigma)\right)\right)^2 dx \right)^{1/2} d\sigma$$

$$\leq \sqrt{2} \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \left(\sum_{j=0}^{s-9} \int \left(\left(\partial^j \left(F(\sigma) + G(\sigma + \frac{\Delta t}{2})\right)\right)^2 \right.\right.$$

$$\left.\left. + \left(\partial^j \left(F(\sigma + \frac{\Delta t}{2}) + G(\sigma)\right)\right)^2 dx \right)^{1/2} d\sigma$$

$$\leq \sqrt{2} \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \left[\left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma) + G(\sigma + \frac{\Delta t}{2}) \right) \right)^2 dx \right)^{1/2} \right.$$

$$\left. + \left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma + \frac{\Delta t}{2}) + G(\sigma) \right) \right)^2 dx \right)^{1/2} \right] d\sigma$$

$$= \sqrt{2} \sum_{m} \int_{t_m}^{t_{m+1/2}} \left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma) + G(\sigma + \Delta t/2) \right) \right)^2 dx \right)^{1/2} d\sigma$$

$$+ \int_{t_{m+1}}^{t_{m+3/2}} \left(\sum_{j=0}^{s-9} \int \left(\partial^j \left(F(\sigma) + G(\sigma - \Delta t/2) \right) \right)^2 dx \right)^{1/2} d\sigma$$

$$\leq K_\alpha \Delta t^2 \left(\sum_{m} \int_{t_m}^{t_{m+1/2}} d\sigma + \sum_{m} \int_{t_{m+1}}^{t_{m+3/2}} d\sigma \right)$$

$$\leq K_\alpha \Delta t^2,$$

where we have used Lemma 3.4. This finishes the estimate for the forcing term.

Next we turn to the estimate of the term $\int_0^t \|\tilde{w}(\sigma)(\tilde{u}(\sigma) - u(\sigma)) - w(\sigma)\tilde{w}(\sigma)\|_{H^{s-9}} d\sigma$ in (30). Here we can use the estimates from the Godunov splitting to infer that

$$\|w(\sigma)\|_{H^{s-9}} + \|\tilde{w}(\sigma)\|_{H^{s-9}} \leq K_\alpha \Delta t, \quad \sigma \in [0, T].$$

From the KdV equation we infer immediately

$$\|\tilde{u}(\sigma) - u(\sigma)\|_{H^{s-9}} \leq \int_0^{\Delta t/2} \|(u u_x - u_{xxx})(\sigma + \tau)\|_{H^{s-9}} d\tau \leq K \Delta t, \quad \sigma \in [0, T].$$

Thus

$$\int_0^t \|\tilde{w}(\sigma)(\tilde{u}(\sigma) - u(\sigma)) - w(\sigma)\tilde{w}(\sigma)\|_{H^{s-9}} d\sigma$$

$$\leq K \int_0^t \left(\|\tilde{w}(\sigma)\|_{H^{s-9}} \|(\tilde{u} - u)(\sigma)\|_{H^{s-9}} + \|w(\sigma)\|_{H^{s-9}} \|\tilde{w}(\sigma)\|_{H^{s-9}} \right) d\sigma$$

$$(40) \quad \leq K_\alpha \Delta t^2.$$

The last term to estimate in (30) is $E(0) = w(0) + \tilde{w}(0) = w(\Delta t/2)$. For $t \leq \Delta t/2$, we find that

$$w(t) = v(t, t) - u(t) = t \int_0^1 \left(B(v(st, 0)) + A(v(t, st)) - C(u(st)) \right) ds$$

$$= t \int_0^1 \left(B(u(st)) + \int_0^1 dB(u(st) + \sigma(v(st, 0) - u(st)))[v(st, 0) - u(st)] d\sigma \right.$$

$$+ A(u(st)) + \int_0^1 dA(u(st) + \sigma(v(t, st) - u(st)))[v(t, st) - u(st)] d\sigma$$

$$\left. - (A + B)(u(st)) \right) ds$$

$$= t \int_0^1 \int_0^1 \left(dB(u(st) + \sigma(v(st, 0) - u(st)))[v(st, 0) - u(st)] \right.$$

$$\left. + dA(u(st) + \sigma(v(t, st) - u(st)))[v(t, st) - u(st)] \right) ds \, d\sigma$$

$$= t \int_0^1 \int_0^1 \left(dB(u(st) + \sigma(v(st, 0) - u(st))) \left[\int_0^{st} \frac{d}{d\tau}(v(\tau, 0) - u(\tau)) d\tau \right] \right.$$

$$\left. + dA(u(st) + \sigma(v(t, st) - u(st))) \left[\int_0^{st} \frac{d}{d\tau}(v(t, \tau) - u(\tau)) d\tau + \int_0^t \frac{d}{d\tau} v(\tau, 0) d\tau \right] \right) ds \, d\sigma$$

$$= t \int_0^1 \int_0^1 \Bigg[\int_0^{st} \Big(dB(u(st) + \sigma(v(st,0) - u(st)))[B(v(\tau,0)) - (A+B)(u(\tau))]$$

$$+ dA(u(st) + \sigma(v(t,st) - u(st)))[B(v(t,\tau)) - (A+B)(u(\tau))] \Big) d\tau$$

$$+ \int_0^t dA(u(st) + \sigma(v(t,st) - u(st)))[B(v(\tau,0))]d\tau \Bigg] ds\, d\sigma$$

$$= t \int_0^1 \int_0^1 \Bigg[\int_0^{st} \Big((u(st) + \sigma(v(st,0) - u(st)))\big(v(\tau,0)v(\tau,0)_x - u(\tau)u(\tau)_x + u(\tau)_{xxx}\big) \Big)_x$$

$$+ \Big(v(t,\tau)v(t,\tau)_x - u(\tau)u(\tau)_x + u(\tau)_{xxx} \Big)_{xxx} d\tau$$

$$+ \int_0^t \Big(v(\tau,0)v(\tau,0)_x \Big)_{xxx} d\tau \Bigg] ds\, d\sigma.$$

Taking the H^{s-9} norm above, using the triangle inequality, the bounds on v and u and the Cauchy–Schwarz inequality, we find that each of the above integrands are bounded by K_α. Thus

(41)
$$\|w(t)\|_{H^{s-9}} \le t \int_0^1 \int_0^1 \Big[\int_0^{st} K_\alpha d\tau + \int_0^t K_\alpha d\tau \Big] ds\, d\sigma$$
$$\le K_\alpha t^2.$$

Hence we infer that

(42)
$$E(0) = \|w(\Delta t/2)\|_{H^{s-9}} \le K_\alpha \Delta t^2.$$

Collecting the estimates from (40), (41), and (42), we find that (30) reads

(43)
$$E(t) = \|z(t)\|_{H^{s-9}} \le K_\alpha \Delta t^2.$$

By the triangle inequality

(44)
$$\|w(t)\|_{H^{s-9}} \le 2\|w(t)\|_{H^{s-9}} \le \|z(t)\|_{H^{s-9}} + \|w(t) - \tilde{w}(t)\|_{H^{s-9}}.$$

To estimate the last term on the right-hand side we write:

(45)
$$\|\tilde{w}(t) - w(t)\|_{H^{s-9}} \le \big\|w(t_n) - w(t_{n+1/2})\big\|_{H^{s-9}} + \|w(t) - w(t_n)\|_{H^{s-9}}$$
$$+ \big\|w(t + \Delta t/2) - w(t_{n+1/2})\big\|_{H^{s-9}},$$

for $t \in [t_n, t_{n+1/2}]$. (Similar expressions hold when $t \in [t_{n+1/2}, t_{n+1}]$.) We note that

$$w(t_{n+1/2}) = \Phi_A(\frac{\Delta t}{2})\Phi_B(\frac{\Delta t}{2})v(t_n, t_n) - \Phi_C(\frac{\Delta t}{2})u(t_n),$$

(46)
$$w(t) = \Phi_A(t - t_n)\Phi_B(t - t_n)v(t_n, t_n) - \Phi_C(t - t_n)u(t_n),$$

$$w(t + \frac{\Delta t}{2}) = \Phi_A(t - t_n)\Phi_B(t - t_n)v(t_{n+1/2}, t_{n+1/2}) - \Phi_C(\frac{\Delta t}{2})u(t_{n+1/2}),$$

when $t \in [t_n, t_{n+1/2}]$. Each of the expressions on the right-hand side of (45) needs to be estimated:

$$w(t_{n+1/2}) - w(t_n) = \Phi_A\big(\frac{\Delta t}{2}; \Phi_B(\frac{\Delta t}{2}; v(t_n, t_n))\big) - \Phi_C\big(\frac{\Delta t}{2}; u(t_n)\big) - (v(t_n, t_n) - u(t_n))$$

$$= \Phi_A\big(\frac{\Delta t}{2}; \Phi_B(\frac{\Delta t}{2}; v(t_n, t_n))\big) - \Phi_C\big(\frac{\Delta t}{2}; v(t_n, t_n)\big)$$

$$+ \Phi_C\big(\frac{\Delta t}{2}; v(t_n, t_n)\big) - \Phi_C\big(\frac{\Delta t}{2}; u(t_n)\big) - (v(t_n, t_n) - u(t_n))$$

$$= \Big(\Phi_A\big(\frac{\Delta t}{2}; \Phi_B(\frac{\Delta t}{2}; v(t_n, t_n))\big) - \Phi_C\big(\frac{\Delta t}{2}; v(t_n, t_n)\big) \Big)$$

(47)
$$+ \Big(\Phi_C\big(\frac{\Delta t}{2}; \cdot\big) - I \Big) \circ v(t_n, t_n) - \Big(\Phi_C\big(\frac{\Delta t}{2}; \cdot\big) - I \Big) \circ u(t_n).$$

First we find that
$$\left\| \Phi_A(\frac{\Delta t}{2}; \Phi_B(\frac{\Delta t}{2}; v(t_n, t_n))) - \Phi_C(\frac{\Delta t}{2}; v(t_n, t_n)) \right\|_{H^{s-9}} \le K_\alpha \Delta t^2$$
by using (41) (with $v(t_n, t_n)$ as initial data).

Introduce the function $V = V(x, t)$ satisfying
$$V_t = V V_x - V_{xxx}, \quad V|_{t=t_n} = v(t_n, t_n).$$
Then the very last line of (47) can be written as
$$\left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ v(t_n, t_n) - \left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ u(t_n)$$
$$= \int_{t_n}^{t_{n+1/2}} \left(V_t(\sigma) - u_t(\sigma) \right) d\sigma$$
$$= \int_{t_n}^{t_{n+1/2}} \left(\frac{1}{2}(V(\sigma)^2)_x - \frac{1}{2}(u(\sigma)^2)_x - V_{xxx}(\sigma) + u_{xxx}(\sigma) \right) d\sigma$$
$$= \int_{t_n}^{t_{n+1/2}} \left(\frac{1}{2}((V(\sigma) + u(\sigma))(V(\sigma) - u(\sigma)))_x - (V(\sigma) - u(\sigma))_{xxx} \right) d\sigma.$$
Taking the H^{s-9} norm we find
$$\left\| \left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ v(t_n, t_n) - \left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ u(t_n) \right\|_{H^{s-9}}$$
$$\le \int_{t_n}^{t_{n+1/2}} \left(\frac{1}{2} \| (V(\sigma) + u(\sigma))(V(\sigma) - u(\sigma)) \|_{H^{s-8}} + \| V(\sigma) - u(\sigma) \|_{H^{s-6}} \right) d\sigma$$
$$\le \int_{t_n}^{t_{n+1/2}} \left(K \left(\| V(\sigma) \|_{H^{s-8}} + \| u(\sigma) \|_{H^{s-8}} \right) \| V(\sigma) - u(\sigma) \|_{H^{s-8}} + \| V(\sigma) - u(\sigma) \|_{H^{s-6}} \right) d\sigma$$
$$\le K_\alpha \int_{t_n}^{t_{n+1/2}} \| V(\sigma) - u(\sigma) \|_{H^{s-6}} d\sigma.$$
By the H^k stability of the KdV equation
$$\| V(\sigma) - u(\sigma) \|_{H^{s-6}} \le K \| v(t_n, t_n) - u(t_n) \|_{H^{s-6}} \le K_\alpha \Delta t,$$
since by the arguments of Section 2.3 we have the estimate $\| w(t_n) \|_{H^{s-6}} \le \| w(t_n) \|_{H^{s-3}} \le K_\alpha \Delta t$. Therefore
$$\left\| \left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ v(t_n, t_n) - \left(\Phi_C(\frac{\Delta t}{2}; \cdot) - I \right) \circ u(t_n) \right\|_{H^{s-9}} \le K_\alpha \Delta t^2.$$
Thus we have shown that
$$(48) \qquad\qquad \| w(t_{n+1/2}) - w(t_n) \|_{H^{s-9}} \le K_\alpha \Delta t^2.$$
The other terms on the right hand side of (45) can be estimated in the same manner, using the expressions (46). Thus we conclude that (cf. (44))
$$(49) \qquad\qquad \| w(t) \|_{H^{s-9}} \le K_\alpha \Delta t^2.$$

If $t, \tau \in [t_n, t_{n+1/2}]$, we have $\| v(t, \tau) \|_{H^{\hat{k}}} = \| v(t, t) \|_{H^{\hat{k}}}$, and if $t, \tau \in [t_{n+1/2}, t_n]$, then an estimate analogous to (16) shows that
$$\left| \| v(t, \tau) \|_{H^{\hat{k}}} - \| v(t, t) \|_{H^{\hat{k}}} \right| \le K_\alpha \left| t - \tau \right|.$$
The rest of the argument follows the procedure for the Godunov splitting. Now $s - 9 \ge \hat{k}$ is the same as $s \ge 17$. Assuming this, we get
$$\| v(t, \tau) \|_{H^{\hat{k}}} \le \| v(t, t) \|_{H^{\hat{k}}} + \left| \| v(t, \tau) \|_{H^{\hat{k}}} - \| v(t, t) \|_{H^{\hat{k}}} \right|$$
$$\le K + K_\alpha \Delta t^2 + K_\alpha \Delta t.$$

Choosing α such that $K \leq \alpha/4$, and then Δt such that $K_\alpha \Delta t(\Delta t + 1) \leq \alpha/4$ implies that $\|v(t,\tau)\|_{H^{\hat{k}}} \leq \alpha/2$. Hence by the bootstrap lemma and (44), the following holds:

Theorem 3.5. *Fix $T > 0$. Let $u_0 \in H^s$ for some $s \geq 17$. Then for Δt sufficiently small we have*

$$\|v(t,t) - u(t)\|_{H^{s-9}} \leq K\Delta t^2, \quad t \in [0,T],$$

where the constant K depends on u_0, s and T only.

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