TMA4170 Fourieranalyse
December 11, 2004

Solutions

Problem 1

(a) Since $f(x) = \arctan(x) \in L^1_{\text{loc}}$ (it is even continuous), f defines a regular distribution in $\mathcal{D}'$.

(b) Since f does not decay at infinity, it cannot be in any $L^p(\mathbb{R})$ for $1 \leq p < \infty$. However, since f is bounded, it is in $L^\infty(\mathbb{R})$.

(c) f is in $\mathcal{S}'$ since it is in $L^\infty(\mathbb{R})$ (Prop. 31.1.8).

(d) We find first that $f'(x) = 1/(1+x^2) \in L^1(\mathbb{R})$, and hence

$$\widehat{f'}(\xi) = \int \frac{e^{-2\pi i x\xi}}{1+x^2}dx = \int \frac{e^{i(-2\pi\xi)x}}{1+x^2}dx = \pi e^{-|-2\pi\xi|} = \pi e^{-2\pi|\xi|}.$$

(e) We have

$$\begin{aligned} T_{\pi e^{-2\pi|\xi|}} &= T_{\mathcal{F}(1/(1+x^2))} \overset{\text{Prop. 31.2.2}}{=} \widehat{T}_{1/(1+x^2)} \\ &= \mathcal{F}(T_{f'}) \overset{\text{Sect. 28.4.3}}{=} \mathcal{F}((T_f)') \overset{\text{Prop. 31.2.4}}{=} 2\pi i\xi\, \mathcal{F}(T_f) \end{aligned}$$

which implies

$$\xi\mathcal{F}(T_f) = \frac{1}{2i}e^{-2\pi|\xi|}.$$

(f) Since

$$\frac{1}{2i\xi}e^{-2\pi|\xi|}$$

is not in L^1_{loc}, we have to rewrite the equation. If we subtract the equation

$$\xi \cdot \big(\frac{1}{2i}\,\text{pv}(1/\xi)\big) = \frac{1}{2i}$$

from

$$\xi\mathcal{F}(T_f) = \frac{1}{2i}e^{-2\pi|\xi|}.$$

we find

$$\xi\big[\mathcal{F}(T_f) - \frac{1}{2i}\,\text{pv}(1/\xi)\big] = \frac{1}{2i}(e^{-2\pi|\xi|} - 1).$$

Using the information provided in the text, we conclude that

$$\mathcal{F}(T_f) - \frac{1}{2i}\,\text{pv}(1/\xi) = \frac{1}{2i\xi}(e^{-2\pi|\xi|} - 1) + \alpha\delta$$

for some complex number α. Thus

$$\hat{f} = \mathcal{F}(T_f) = \frac{1}{2i}\,\text{pv}(1/\xi) + \frac{1}{2i\xi}(e^{-2\pi|\xi|} - 1) + \alpha\delta.$$

(g) Use that the left-hand side of

$$\mathcal{F}(T_f) - \Big[\frac{1}{2i}\,\mathrm{pv}(1/\xi) + \frac{1}{2i\xi}(e^{-2\pi|\xi|} - 1)\Big] = \alpha\delta$$

is odd while the right-hand side is even to conclude that $\alpha = 0$. Alternatively, one can multiply both sides of the identity with the function $2\pi i\xi$ (a smooth function). The we find that

$$e^{-2\pi|\xi|} = \xi \cdot \mathrm{pv}(1/\xi) + (e^{-2\pi|\xi|} - 1) + \alpha\delta$$

or

$$\alpha\delta = 0,$$

which implies $\alpha = 0$.

Problem 2

(a) The equation

$$(2\pi i\lambda)^2 + 3(2\pi i\lambda) + 2 = 0$$

has roots

$$2\pi i\lambda = -2, \quad 2\pi i\lambda = -1.$$

Using the notation in the book (p. 211ff) we find

$$H(\lambda) = -\frac{1}{2\pi i\lambda - (-2)} + \frac{1}{2\pi i\lambda - (-1)}$$

and hence the impulse response h equals

$$h(t) = \big(-e^{-2t} + e^{-t}\big)u(t).$$

(b) General theory (p. 217) gives that the filter is realizable and stable.

Problem 3

(a) See p. 331 in the textbook. The impulse response $h = \delta'$.
(b) The filter is realizable but not stable.