Problem 1

(a) Since \( f(x) = \arctan(x) \in L^1_{\text{loc}} \) (it is even continuous), \( f \) defines a regular distribution in \( \mathcal{D}' \).

(b) Since \( f \) does not decay at infinity, it cannot be in any \( L^p(\mathbb{R}) \) for \( 1 \leq p < \infty \). However, since \( f \) is bounded, it is in \( L^\infty(\mathbb{R}) \).

(c) \( f \) is in \( S' \) since it is in \( L^\infty(\mathbb{R}) \) (Prop. 31.1.8).

(d) We find first that \( f'(x) = \frac{1}{1 + x^2} \in L^1(\mathbb{R}) \), and hence
\[
\hat{f}'(\xi) = \int e^{-2\pi i x \xi} \frac{1}{1 + x^2} dx = \int e^{i(-2\pi \xi) x} \frac{1}{1 + x^2} dx = \pi e^{-|\xi|} = \pi e^{-2\pi |\xi|}.
\]

(e) We have
\[
T_{\pi e^{-2\pi |\xi|}} = T_{\mathcal{F}(1/(1+x^2))} = \mathcal{F}(T_f') \quad \text{Prop. 31.2.4}
\]
\[
= \mathcal{F}((T_f)') \quad \text{Prop. 31.2.4}
\]
which implies
\[
\xi \mathcal{F}(T_f) = \frac{1}{2i} e^{-2\pi |\xi|}.
\]

(f) Since
\[
\frac{1}{2i\xi} e^{-2\pi |\xi|}
\]
is not in \( L^1_{\text{loc}} \), we have to rewrite the equation. If we subtract the equation
\[
\xi \cdot \left( \frac{1}{2i} \text{pv}(1/\xi) \right) = \frac{1}{2i}
\]
from
\[
\xi \mathcal{F}(T_f) = \frac{1}{2i} e^{-2\pi |\xi|}.
\]
we find
\[
\xi [\mathcal{F}(T_f) - \frac{1}{2i} \text{pv}(1/\xi)] = \frac{1}{2i} (e^{-2\pi |\xi|} - 1).
\]
Using the information provided in the text, we conclude that
\[
\mathcal{F}(T_f) - \frac{1}{2i} \text{pv}(1/\xi) = \frac{1}{2i\xi} (e^{-2\pi |\xi|} - 1) + \alpha \delta
\]
for some complex number \( \alpha \). Thus
\[
\hat{f} = \mathcal{F}(T_f) = \frac{1}{2i} \text{pv}(1/\xi) + \frac{1}{2i\xi} (e^{-2\pi |\xi|} - 1) + \alpha \delta.
\]
(g) Use that the left-hand side of

\[
\mathcal{F}(T_f) - \left[ \frac{1}{2i} \text{pv}(1/\xi) + \frac{1}{2i \xi} (e^{-2\pi|\xi|} - 1) \right] = \alpha \delta
\]

is odd while the right-hand side is even to conclude that \( \alpha = 0 \).

Alternatively, one can multiply both sides of the identity with the function \( 2\pi i \xi \) (a smooth function). The we find that

\[
e^{-2\pi|\xi|} = \xi \cdot \text{pv}(1/\xi) + (e^{-2\pi|\xi|} - 1) + \alpha \delta
\]

or

\[
\alpha \delta = 0,
\]

which implies \( \alpha = 0 \).

**Problem 2**

(a) The equation

\[
(2\pi i \lambda)^2 + 3(2\pi i \lambda) + 2 = 0
\]

has roots

\[
2\pi i \lambda = -2, \quad 2\pi i \lambda = -1.
\]

Using the notation in the book (p. 211ff) we find

\[
H(\lambda) = -\frac{1}{2\pi i \lambda - (-2)} + \frac{1}{2\pi i \lambda - (-1)}
\]

and hence the impulse response \( h \) equals

\[
h(t) = \left( -e^{-2t} + e^{-t} \right) u(t).
\]

(b) General theory (p. 217) gives that the filter is realizable and stable.

**Problem 3**

(a) See p. 331 in the textbook. The impulse response \( h = \delta' \).

(b) The filter is realizable but not stable.