Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourieranalyse

Exam, November 29, 2007, Time: 9:00–13:00

Contact during exam: Helge Holden, phone 92038625

Grades will be announced December 20, 2007

Aids: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Problem 1 Let $f_n \colon \mathbb{R} \to [0, \infty]$, $n \in \mathbb{N}$, be a sequence of measurable functions such that $f_n \geq f_{n+1}$ for $n \in \mathbb{N}$. Assume that $\int_{\mathbb{R}} f_n \, dx \to 0$ when $n \to \infty$. Show that $f_n \to 0$ almost everywhere as $n \to \infty$.

Problem 2

a) Consider the filter $A \colon \mathcal{S} \to \mathcal{S}$ (where $\mathcal{S}$ is the Schwartz space) given by $A(f) = g$ where

$$g'' + 2\alpha g' + g = f,$$

where $\alpha \in (0, \infty)$ is a parameter. Show that one can write

$$\hat{g}(\lambda) = H(\lambda)\hat{f}(\lambda),$$

and determine H.

b) Compute the partial fractions of H for all values of α.

c) Determine the impulse response $h = \mathcal{F}^{-1}H$ for all values of α.

d) Show that

$$\|g\|_\infty \leq \|h\|_1 \, \|f\|_\infty \, .$$

e) Is the filter realizable and stable for all values of α? Explain your answer.

Problem 3

a) Given the function

$$f(x) = \frac{e^{2\pi i a x}}{b + ix}$$

where $a \in \mathbb{R}$, $\operatorname{Re} b > 0$. Determine if $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$.

b) Determine the Fourier transform of f. Explain your answer.

Problem 4

a) Let T_f be the distribution defined by $f(x) = x^2 u(x)$ where u is the Heaviside funksjon (or the unit step function) given by $u(x) = 1$ for x positive og zero otherwise. Determine

$$(T_f)^{(n)}$$

for all $n \in \mathbb{N}$.

b) Let $g \in C^\infty$ and $T \in \mathcal{D}'$. Show that

$$(gT)' = g'T + gT'.$$