Solutions for TMA4170 Fourier Analysis

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Problem 1

Since $f_n \geq f_{n+1} \geq 0$, we know that $\lim_{n\to\infty} f_n(x) = f(x) \geq 0$ exists almost everywhere (but it might be equal to ∞ for many x). Furthermore, since $\int_{\mathbb{R}} f_n \, dx \to 0$ as $n \to \infty$, we first conclude that there exists an N such that $\int_{\mathbb{R}} f_N \, dx < \infty$, and hence f_N, and thus also f, is finite almost everywhere. Using Lebesgue's dominated convergence theorem (using f_N to dominate) we find that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dx = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, dx = \int_{\mathbb{R}} f \, dx,$$

from which we conclude that $f = 0$ almost everywhere.

Problem 2

a) Given

$$g'' + 2\alpha g' + g = f,$$

we perform a Fourier-transform which yields

$$((2\pi i\lambda)^2 + 4\pi i\lambda\alpha + 1)\hat{g}(\lambda) = \hat{f}(\lambda).$$

Thus

$$H(\lambda) = \frac{1}{(2\pi i\lambda)^2 + 4\pi i\lambda\alpha + 1}.$$

b) We find that

$$H(\lambda) = \begin{cases} \frac{1}{2\sqrt{\alpha^2-1}}\left(\frac{1}{2\pi i\lambda-(-\alpha+\sqrt{\alpha^2-1})} - \frac{1}{2\pi i\lambda-(-\alpha-\sqrt{\alpha^2-1})}\right) & \text{for } \alpha \neq 1, \\ \frac{1}{(2\pi i\lambda+1)^2} & \text{for } \alpha = 1. \end{cases}$$

c) There are three distinct cases:

(A) $\alpha > 1$. Here we find that the impulse response reads

$$\begin{aligned} h(t) &= \frac{1}{2\sqrt{\alpha^2-1}}\left(e^{(-\alpha+\sqrt{\alpha^2-1})t} - e^{-(\alpha+\sqrt{\alpha^2-1})t}\right)u(t) \\ &= \frac{e^{-\alpha t}}{\sqrt{\alpha^2-1}}\sinh(\sqrt{\alpha^2-1}\,t)u(t). \end{aligned}$$

Note that $-\alpha \pm \sqrt{\alpha^2-1} < 0$ when $\alpha > 1$.

(B) $\alpha = 1$. The two roots are coinciding, and the impulse response reads

$$h(t) = te^{-t}u(t).$$

(C) $0 < \alpha < 1$. Here we get two complex conjugate roots with solution

$$\begin{aligned} h(t) &= \frac{1}{2\sqrt{\alpha^2-1}}\big(e^{(-\alpha+\sqrt{\alpha^2-1})t} - e^{-(\alpha+\sqrt{\alpha^2-1})t}\big)u(t) \\ &= \frac{1}{2i\sqrt{1-\alpha^2}}\big(e^{(-\alpha+i\sqrt{1-\alpha^2})t} - e^{-(\alpha+i\sqrt{1-\alpha^2})t}\big)u(t) \\ &= \frac{1}{2i\sqrt{1-\alpha^2}}\big(e^{i\sqrt{1-\alpha^2}t} - e^{-i\sqrt{1-\alpha^2}t}\big)e^{-\alpha t}u(t) \\ &= \frac{1}{\sqrt{1-\alpha^2}}\sin(\sqrt{1-\alpha^2}\,t)e^{-\alpha t}u(t). \end{aligned}$$

Observe that $\mathrm{Re}(-\alpha \pm \sqrt{\alpha^2-1}) = -\alpha < 0$ if $0 < \alpha < 1$.
d) In all cases we have that

$$g = h * f,$$

and thus

$$|g(t)| \le \int |h(t-s)f(s)|\,ds \le \|f\|_\infty \int |h(t-s)|\,ds = \|h\|_1\ \|f\|_\infty$$

from which it follows

$$\|g\|_\infty \le \|h\|_1\ \|f\|_\infty\,.$$

e) The filter is both stable and realizable in all cases, cf. Theorem 24.5.2, as the real parts of all poles are strictly negative.

Problem 3

a) Since $|f(x)| = 1/|b+ix| = 1/\sqrt{(\mathrm{Re}\,b)^2 + (\mathrm{Im}\,b + x)^2}$, we see that $f \notin L^1(\mathbb{R})$, but $f \in L^2(\mathbb{R})$. Thus some care is needed when computing $\hat{f}$.

b) Consider $g_\beta(x) = e^{-\beta x}u(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\mathrm{Re}\,\beta > 0$. Then we can use the ordinary definition of the Fourier transform, and we find, for $\alpha \in \mathbb{R}$, that

$$\mathcal{F}(g_\beta(x-\alpha))(\xi) = e^{-2\pi i\alpha\xi}\hat{g}_\beta(\xi) = e^{-2\pi i\alpha\xi}\frac{1}{\beta + 2\pi i\xi}.$$

Choose $\alpha = -a$ and $\beta = 2\pi b$ to conclude that

$$f(\xi) = \frac{e^{2\pi i a\xi}}{b + i\xi} = 2\pi\mathcal{F}(g_{2\pi b}(x+a))(\xi).$$

Applying Proposition 22.2.1 we find that

$$\begin{aligned} (\mathcal{F}f)(x) &= \mathcal{F}\big(2\pi\mathcal{F}(g_{2\pi b}(x+a))\big) = \big(2\pi g_{2\pi b}(x+a)\big)_\sigma \\ &= 2\pi g_{2\pi b}(-x+a) = 2\pi e^{-2\pi b(a-x)}u(a-x). \end{aligned}$$

Problem 4

a) By standard derivation we find $f'(x) = 2xu(x) \in C(\mathbb{R})$ for all x, but $f''(x) = 2u(x)$ for $x \neq 0$, and $f^{(k)}(x) = 0$ for $k = 3, 4, \ldots$ and for $x \neq 0$. So we find that

$$
\begin{aligned}
(T_f)' &= T_{f'} = T_{2xu(x)}, \quad \text{(using Sec. 28.4.4)},\\
(T_f)'' &= (T_{2xu(x)})' = T_{2u}, \quad \text{(using Sec. 28.4.4)},\\
(T_f)^{(3)} &= (T_{2u})' = 2\delta_0, \quad \text{(using Sec. 28.4.4)},\\
(T_f)^{(k)} &= 2\delta_0^{(k-3)}, \quad k = 3, 4, \ldots, \quad \text{(using common sense)}.
\end{aligned}
$$

b) Using the definition we find

$$
\begin{aligned}
(gT)'(\phi) &= -(gT)(\phi') = -T(g\phi') = -T((g\phi)' - g'\phi)\\
&= T'(g\phi) + g'T(\phi) = (g'T + gT')(\phi), \quad \phi \in \mathcal{D}.
\end{aligned}
$$