Solutions for TMA4170 Fourier Analysis

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Problem 1

a) Two possible approaches:
(i) Since $\mathcal{F}(e^{-\alpha|x|}) = 2\alpha/(\alpha^2 + (2\pi\xi)^2)$ (p. 159), we infer that

$$\hat{f}_\alpha(\xi) = e^{-\alpha|\xi|}.$$

(ii) An alternative is to write

$$f_\alpha(x) = \frac{1}{\alpha + 2\pi i x} + \frac{1}{\alpha - 2\pi i x}$$

and then use that (p. 166)

$$\mathcal{F}(\frac{1}{\alpha + 2\pi i x}) = e^{\alpha\xi}u(-\xi), \quad \mathcal{F}(\frac{1}{\alpha - 2\pi i x}) = e^{-\alpha\xi}u(\xi).$$

Adding the two terms yields the same result.

b) The convolution theorem (Prop. 23.1.2) yields

$$\widehat{f_\alpha * f_\beta} = \hat{f}_\alpha \hat{f}_\beta = e^{-\alpha|\xi|}e^{-\alpha|\xi|} = e^{-(\alpha+\beta)|\xi|} = \hat{f}_{\alpha+\beta},$$

which shows that

$$f_\alpha * f_\beta = f_{\alpha+\beta},$$

Problem 2

a) It is clearly linear. As for the continuity, we have for $\phi_m \in \mathcal{S}$, $\phi_m \to 0$ in $\mathcal{S}$ that

$$|\delta_c(\phi_m)| = |\phi_m(c)| \leq \|\phi_m\|_\infty \to 0.$$

Hence $\delta_c \to 0$ in $\mathcal{S}'$. As for the Fourier transform we find

$$\hat{\delta}_c(\phi) = \delta_c(\hat{\phi}) = \hat{\phi}(c) = \int e^{-2\pi i c x}\phi(x)dx,$$

thus

$$\hat{\delta}_c = e^{-2\pi i c x}.$$

b) Linearity is clear. If $\phi_m \in \mathcal{S}$, $\phi_m \to 0$ in $\mathcal{S}$ we have

$$\begin{aligned} |D_a(\phi_m)| &\leq \sum_n |\phi_m(an)| \leq \sum_n \frac{1}{1+(na)^2}|(1+(na)^2)\phi_m(an)| \\ &\leq \left\|(1+x^2)\phi_m(x)\right\|_\infty \sum_n \frac{1}{1+(na)^2} \to 0 \end{aligned}$$

as $m \to \infty$ since $\left\|(1+x^2)\phi_m(x)\right\|_\infty \to 0$ by assumption, and $\sum_n \frac{1}{1+(na)^2}$ converges.

c) Pointwise we have that

$$g'(x) = 1/a, \quad x \in \mathbb{R} \setminus \{na \mid n \in \mathbb{Z}\}.$$

At points na, $n \in \mathbb{Z}$ the function g makes a jump of minus one. Thus we find (cf. Section 28.4.4) that

$$(T_g)' = T_{g'} - \sum_n \delta_{na}.$$

or

$$g' = \frac{1}{a} - \sum_{n\in\mathbb{Z}} \delta_{na}$$

in the sense of distributions.

d) Using the standard formula

$$g(x) = \sum_n c_n e^{2\pi inx/a}$$

where

$$c_n = \frac{1}{a}\int_0^a g(x)e^{-2\pi inx/a}dx$$

we find

$$c_0 = \frac{1}{2}, \qquad c_n = \frac{i}{2\pi n}, \quad n \neq 0.$$

Convergence is pointwise to $g(x)$ for all x except at points na, $n \in \mathbb{Z}$ by using Dirichlet's theorem (Theorem 5.2.4). At points na, $n \in \mathbb{Z}$ Dirichlet's theorem gives convergence to $1/2$. The Fourier series converges in $L_p^2(0, a)$ from Theorem 16.3.9.

e) From Proposition 29.3.2 we infer that the partial sums of the Fourier series converge to g in the sense of distributions. From Theorem 29.1.3 we conclude that the partial sums of the pointwise derivates converge to the distributional derivative of g. From this we infer using **d)** that

$$g' = -\frac{1}{a}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{2\pi inx/a}$$

in the sense of distributions.

f) If we combine **a)** and **b)** we find

$$\widehat{D}_a = \sum_n e^{-2\pi inax} = \sum_n e^{2\pi inx/(1/a)}.$$

On the other hand, if we combine **c)** and **e)** we find

$$g' - \frac{1}{a} = -\frac{1}{a}\sum_{n\in\mathbb{Z}} e^{2\pi inx/a} = -\sum_n \delta_{na}.$$

By replacing a by $1/a$ in the last result we can write this as

$$\frac{1}{1/a}\sum_{n\in\mathbb{Z}} e^{2\pi inx/(1/a)} = \sum_n \delta_{n/a}.$$

This yields

$$\widehat{D}_a = \sum_n e^{2\pi inx/(1/a)} = \frac{1}{a}\sum_n \delta_{n/a} = \frac{1}{a}D_{1/a}.$$

Observe that this implies that $\widehat{D}_1 = D_1$.

Problem 3

a) Given

$$g'' + 2g' + \beta g = f,$$

we perform a Fourier transform which yields

$$((2\pi i\lambda)^2 + 4\pi i\lambda + \beta)\hat{g}(\lambda) = \hat{f}(\lambda).$$

Thus

$$H(\lambda) = \frac{1}{(2\pi i\lambda)^2 + 4\pi i\lambda + \beta}.$$

b) We find that

$$H(\lambda) = \begin{cases} \frac{1}{2\sqrt{1-\beta}}\left(\frac{1}{2\pi i\lambda-(-1+\sqrt{1-\beta})} - \frac{1}{2\pi i\lambda-(-1-\sqrt{1-\beta})}\right) & \text{for } \beta \neq 1, \\ \frac{1}{(2\pi i\lambda+1)^2} & \text{for } \beta = 1. \end{cases}$$

c) There are four distinct cases:

(A) $0 < \beta < 1$. Here we find that the impulse response reads

$$\begin{aligned} h(t) &= \frac{1}{2\sqrt{1-\beta}}\big(e^{(-1+\sqrt{1-\beta})t} - e^{-(1+\sqrt{1-\beta})t}\big)u(t) \\ &= \frac{e^{-t}}{\sqrt{1-\beta}}\sinh(\sqrt{1-\beta}\,t)u(t). \end{aligned}$$

Note that $-1 \pm \sqrt{1-\beta} < 0$ when $0 < \beta < 1$.

(B) $\beta = 1$. The two roots are coinciding, and the impulse response reads

$$h(t) = te^{-t}u(t).$$

(C) $\beta > 1$. Here we get two complex conjugate roots with solution

$$\begin{aligned} h(t) &= \frac{1}{2\sqrt{1-\beta}}\big(e^{(-1+\sqrt{1-\beta})t} - e^{-(1+\sqrt{1-\beta})t}\big)u(t) \\ &= \frac{1}{2i\sqrt{\beta-1}}\big(e^{(-1+i\sqrt{\beta-1})t} - e^{-(1+i\sqrt{\beta-1})t}\big)u(t) \\ &= \frac{1}{2i\sqrt{\beta-1}}\big(e^{i\sqrt{\beta-1}t} - e^{-i\sqrt{\beta-1}t}\big)e^{-t}u(t) \\ &= \frac{1}{\sqrt{\beta-1}}\sin(\sqrt{\beta-1}\,t)e^{-t}u(t). \end{aligned}$$

Observe that $\mathrm{Re}(-1 \pm \sqrt{1-\beta}) = -1 < 0$ if $\beta > 1$.
(D) $\beta < 0$. Here we have real roots with opposite sign.

$$\begin{aligned} h(t) &= \frac{1}{2\sqrt{1-\beta}} \left(e^{(-1+\sqrt{1-\beta})t} u(-t) - e^{-(1+\sqrt{1-\beta})t} u(t) \right) \\ &= -\frac{\mathrm{sign}(t)}{\sqrt{1-\beta}} e^{-t-\sqrt{1-\beta}|t|}. \end{aligned}$$

d) In all cases except $\beta = 0$ we have that

$$g = h * f,$$

and thus

$$|g(t)| \leq \int |h(t-s) f(s)| \, ds \leq \|f\|_\infty \int |h(t-s)| \, ds = \|h\|_1 \, \|f\|_\infty$$

from which it follows

$$\|g\|_\infty \leq \|h\|_1 \, \|f\|_\infty \, .$$

e) The filter is realizable when $\beta > 0$, cf. Theorem 24.5.2, as the real parts of all poles are strictly negative in that case. The filter is stable for all $\beta \neq 0$ because no poles are on the imaginary axis, cf. Theorem 24.4.2.