Bang-bang Property

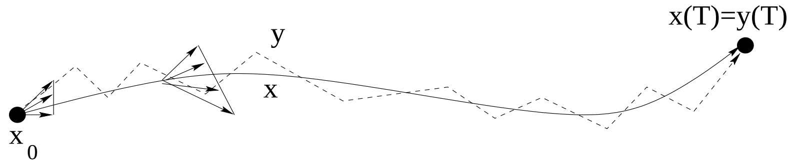
Assume that, for every solution of

$$\dot{x} \in \overline{co}\, G(x) \qquad t \in [0, T], \qquad\qquad x(0) = x_0$$

one can find a solution of

$$\dot{y} \in \text{ext}\, G(x) \qquad t \in [0, T], \qquad\qquad x(0) = x_0$$

such that $y(T) = x(T)$. Then we say that the multifunction G has the **bang-bang property**.



Theorem 6 (Bang-Bang for Linear Systems). Let $U \subset I\!R^m$ be compact. Then, for every solution of

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad\qquad x(0) = x_0, \qquad u(t) \in U, \quad t \in [0, T]$$

there exists a solution of

$$\dot{y}(t) = Ay(t) + Bu(t), \qquad\qquad y(0) = x_0, \qquad u(t) \in \text{ext}\, U, \quad t \in [0, T]$$

such that $y(T) = x(T)$

In general, the bang-bang property does not hold for nonlinear systems (Example 4)

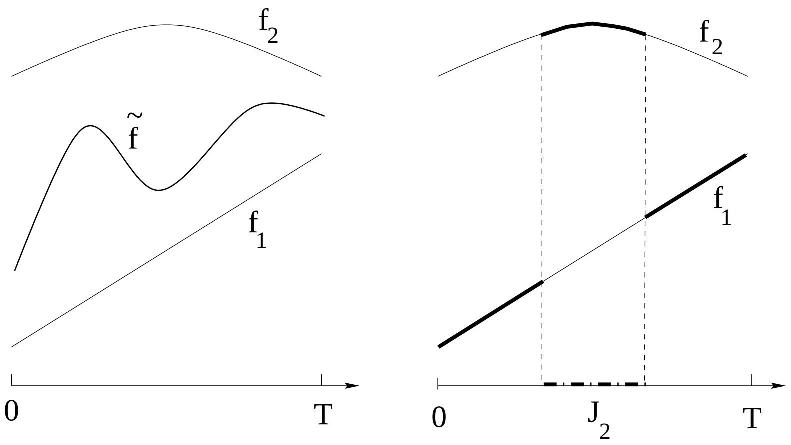
A Lyapunov-type Convexity Theorem

$$f_1, \ldots, f_N \in \mathbf{L}^1\big([0,T]\,;\ I\!\!R^n\big)$$

consider a pointwise convex combination of the f_i

$$\tilde f(t) \doteq \sum_{i=1}^N \theta_i(t) f_i(t)$$

$$\theta_1, \ldots, \theta_N : [0,T] \mapsto [0,1], \qquad\qquad \sum \theta_i(t) = 1 \quad \text{for all } t$$



Theorem 7. There exists a partition

$$[0,T] = \bigcup_{i=1}^N J_i, \qquad\qquad J_i \cap J_k = 0 \quad \text{if } i \neq k$$

such that

$$\int_0^T \tilde f(t)\, dt = \sum_{i=1}^N \int_{J_i} f_i(t)\, dt$$

Proof of Theorem 7. Consider the set of coefficients of convex combinations

$$\Gamma \doteq \left\{ (w_1, \dots w_N) ; \quad w_i(t) \in [0,1], \quad \sum_{i=1}^{N} w_i(t) = 1, \quad \int_0^T \sum_i w_i(t) f_i(t)\, dt = \int_0^T \tilde{f}(t)\, dt \right\}$$

1. Γ is non-empty, because $(\theta_1, \dots, \theta_N) \in \Gamma$

2. $\Gamma \subset \mathbf{L}^\infty$ is closed and convex, hence compact in a weak topology

3. By the Krein-Milman Theorem, Γ has an extreme point $(w_1^*, \dots, w_n^*)$

4. By extremality, the functions $w_i^* : [0,T] \mapsto [0,1]$ actually take values in $\{0,1\}$

5. Take $J_i = \big\{ t \in [0,T] ; \quad w_i^*(t) = 1 \big\}$

Proof of Theorem 6.

Consider any control $u : [0,T] \mapsto U \subset \mathbb{R}^m$. Then

$$u(t) = \sum_{i=0}^{m} \theta_i(t) u_i(t) \qquad\qquad u_i(t) \in \operatorname{ext} U$$

Using the control $u(\cdot)$, the terminal point is

$$x(T) = \int_0^T e^{(T-s)A} Bu(s)\, ds = \int_0^T \sum_{i=0}^{m} \theta_i(t) e^{(T-s)A} Bu_i(s)\, ds$$

Apply previous theorem with $f_i(s) = e^{(T-s)A} Bu_i(s)$. For a suitable partition $[0,T] = J_0 \cup J_1 \cup \cdots \cup J_m$, one has

$$x(T) = \sum_{i=0}^{m} \int_{J_i} e^{(T-s)A} Bu_i(s)\, ds$$

Hence the control

$$u^*(t) = u_i(t) \qquad \text{if } \ t \in J_i$$

takes values in $\operatorname{ext} U$ and reaches exactly the same terminal point $x(T)$.

Baire Category Approach

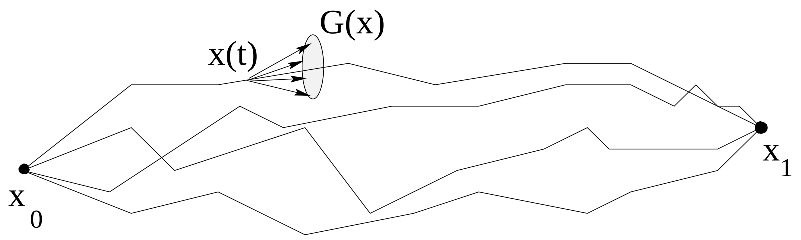
Baire Category Theorem. Let K be a complete metric space, $(K_n)_{n\geq 1}$ a sequence of open, dense subsets. Then $\bigcap_{n\geq 1} K_n$ is non-empty and dense in K.

Alternative proof of the Bang-Bang Theorem

We can assume that all sets $G(x)$ are convex.
Assume there exists at least one trajectory of

$$\dot{x}(t) \in G\big(x(t)\big), \qquad x(0) = x_0, \qquad x(T) = x_1$$

The set K of all such trajectories is then a non-empty, compact subset of $\mathcal{C}\big([0,T]\big)$. In particular, K is a complete metric space.



Under suitable assumptions on the multifunction G, the set K^{ext} of solutions of

$$\dot{x} \in \operatorname{ext} G(x), \qquad x(0) = x_0, \qquad x(T) = x_1$$

can be written as the intersection of countably many open dense subsets $K_n \subset K$

Hence by the Baire Cathegory Theorem, $K^{ext} \neq \emptyset$

This technique applies to a class of "concave" multifunctions G, introduced in (Bressan-Piccoli, *J.Diff.Equat.* 1995)

1. For each compact, convex set $\Omega \subset I\!R^n$, define the function $\varphi_\Omega : \Omega \mapsto [0, \infty[$ as

$$\varphi_\Omega(p) \doteq \sup \left\{ \left(\int_0^1 \big|X(s) - p\big|^2 \, ds \right)^{1/2} ; \quad X : [0,1] \mapsto \Omega, \quad \int_0^1 X(s)\, ds = p \right\}$$

Observe that $\varphi^2_\Omega(p)$ is the maximum variance among all random variables X taking values inside Ω, whose expected value is $E[X] = p$.

2. Each function φ_Ω is ≥ 0, concave, continuous and $\varphi_\Omega(p) = 0$ iff $p \in \text{ext}\,\Omega$

3. The functional

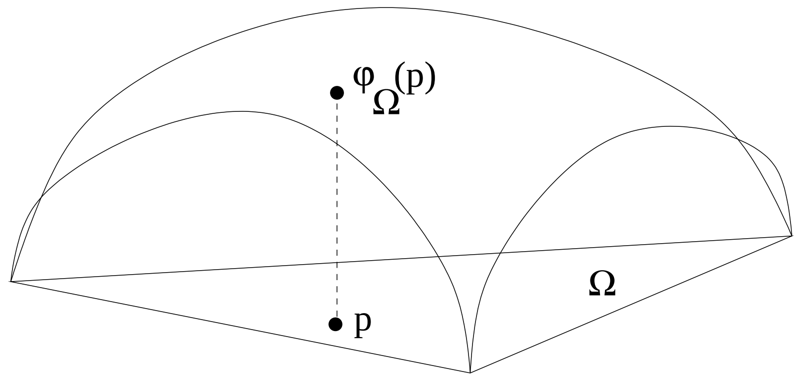
$$\Phi\big(x(\cdot)\big) \doteq \int_0^T \varphi_{G(x)}\big(\dot x(t)\big)\, dt$$

is well defined for all trajectories of $\dot x \in G(x)$
Moreover $\Phi\big(x(\cdot)\big) = 0$ iff $\dot x(t) \in \text{ext}\, G\big(x(t)\big)$ for a.e. $t \in [0, T]$

4. The sets $K_n \doteq \big\{x(\cdot) \in K\,; \quad \Phi(x) < 1/n\big\}$ are open and dense in K

5. By the Baire Cathegory Theorem, $K^{ext} = \bigcap_{n \geq 1} K_n \neq \emptyset$.

This shows that, (in a topological sense) "almost all" solutions of $\dot x \in G(x)$ are actually solutions of $\dot x \in \text{ext}\, G(x)$.



Chattering Controls

Given a control system on $\mathbb{R}^n$

$$\dot{x} = f(x,u) \qquad\qquad u \in U \tag{1}$$

for each x, the set of velocities

$$G(x) \doteq \big\{ f(x,u)\,; \quad u \in U \big\}$$

may not be convex. We seek a new control system

$$\dot{x} = \tilde{f}(x,\tilde{u}) \qquad\qquad \tilde{u} \in \widetilde{U} \tag{13}$$

such that

$$\widetilde{G}(x) \doteq \big\{ f(x,\tilde{u})\,; \quad \tilde{u} \in \widetilde{U} \big\} = \overline{\text{co}} G(x)$$

Since every $\omega \in \overline{\text{co}} G(x)$ is a convex combination of $n+1$ vectors in $G(x)$, the system (13) can be defined as follows:

$$\Delta \doteq \left\{ (\theta_0, \theta_1, \ldots, \theta_n)\,; \qquad \theta_i \in [0,1], \quad \sum_{i=0}^{n} \theta_i = 1 \right\}$$

$$\widetilde{U} \doteq U \times \cdots \times U \times \Delta$$

(n+1 times)

$$\tilde{f}(x,\tilde{u}) = \tilde{f}(x, u_1, \ldots, u_n, \theta_0, \ldots, \theta_n) = \sum_{i=0}^{n} \theta_i f(x, u_i) \tag{14}$$

The system (14) is called a **chattering system**
A control $\tilde{u} = (u_0, \ldots, u_n, \theta_0, \ldots, \theta_n) \in \widetilde{U}$ is called a **chattering control**

- The set of trajectories of a chattering system is always closed in $\mathcal{C}\big([0,T]\big)$
- The reachable set $\widetilde{R}(T)$ is compact.
- Every trajectory of (13) can be uniformly approximated by trajectories of the original system (1)

Lie Brackets

Given two smooth vector fields f, g, their **Lie bracket** is defined as

$$[f, g] \doteq (Dg) \cdot f - (Df) \cdot g$$

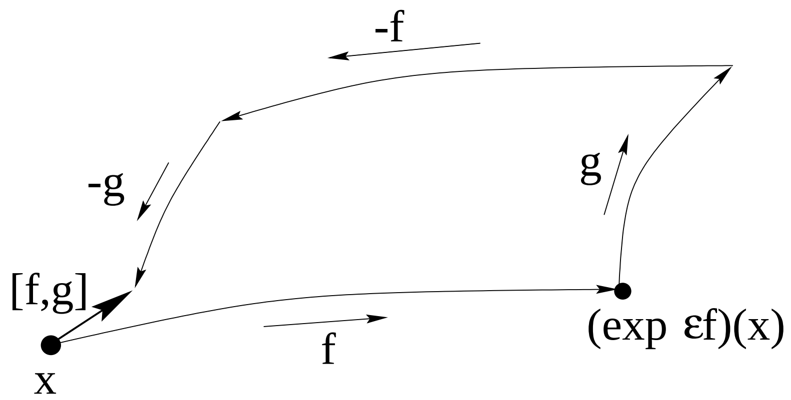
In other words, $[f, g]$ is the directional derivative of g in the direction of f minus the directional derivative of f in the direction of g.

Exponential notation for the flow map:
$\tau \mapsto (\exp \tau f)(x)$ denotes the solution of the Cauchy problem

$$\frac{dw}{d\tau} = f(w), \qquad w(0) = x$$

The Lie bracket can be equivalently characterized as

$$[f, g](x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Big[\big(\exp(-\varepsilon g) \big) \big(\exp(-\varepsilon f) \big) (\exp \varepsilon g)(\exp \varepsilon f)(x) \; - \; x \Big]$$



The **Lie algebra** generated by the vector fields $f_1, \ldots, f_n$ is the space of vector fields generated by the f_i and by all their iterated Lie brackets

$$[f_i, f_j], \qquad [f_i, [f_j, f_k]], \qquad \Big[[f_i, [f_j, f_k]], f_\ell \Big], \quad \ldots$$

Local Controllability

$$\dot{x} = \sum_{i=1}^{m} f_i(x)\, u_i \qquad\qquad u_i \in [-1,\, 1] \tag{15}$$

$$x(0) = x_0 \in \mathbb{R}^n$$

Theorem 8. If the linear span of all iterated Lie brackets of $f_1, \ldots, f_n$ at x_0 is the whole space $\mathbb{R}^n$, then the system (15) is locally controllable.
That means: for every $T > 0$ the reachable set $R(T)$ is a neighborhood of x_0.

More difficult case: a drift f_0 is present.

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)\, u_i \qquad\qquad u_i \in [-1,\, 1] \tag{16}$$

Theorem 9 (Sussmann-Jurdjevic). If the linear span of all iterated Lie brackets of $f_0, f_1, \ldots, f_n$ at x_0 is the whole space $\mathbb{R}^n$, then for every $T > 0$ the set of points reachable within time $\leq T$ has non-empty interior.

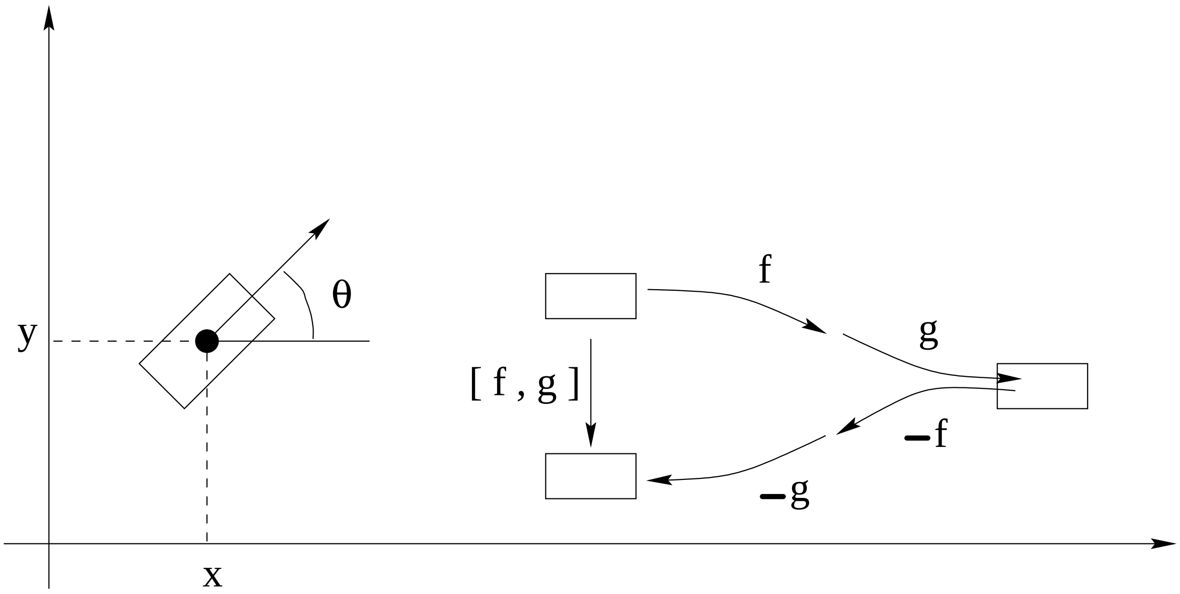
Local controllability for (16) is a hard problem!

Example 5. The position of a car is described by $P = (x, y, \theta)$. The first two variables x, y determine the location of the baricenter, while θ determines the orientation. If the car advances with unit speed steering to the right, its motion satisfies

$$\frac{dP}{dt} = f(P) \doteq (\cos\theta,\ \sin\theta,\ -1).$$

On the other hand, if the car steers to the left, its motion satisfies

$$\frac{dP}{dt} = g(P) \doteq (\cos\theta,\ \sin\theta,\ 1).$$



In a typical parking problem, one needs to shift the car toward one side, without changing its orientation. This is obtained by the manouver:

1. right-forward,
2. left-forward,
3. right-backward,
4. left-backward.

Indeed, the sequence of these four actions generates the Lie bracket

$$[f, g] = (Dg) \cdot f - (Df) \cdot g = 2\,(\sin\theta,\ -\cos\theta,\ 0)$$

which is precisely the desired motion.

The system on $I\!R^3$

$$\dot{P} = f(P)\, u_1 + g(P)\, u_2, \qquad\qquad u_1, u_2 \in [-1,\, 1]$$

is locally controllable. Indeed,

$$\operatorname{span}\big\{ f(P),\, g(P),\, [f,g](P) \big\} = I\!R^3$$

because it contains the three vectors $(\cos\theta,\ \sin\theta,\ -1)$, $(\cos\theta,\ \sin\theta,\ -1)$ and $(\sin\theta,\ -\cos\theta,\ 0)$

Example 6. The system on $I\!R^2$

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 \end{cases} \qquad\qquad u \in [-1,\, 1] \tag{17}$$

$$(x_1, x_2)(0) = (0, 0)$$

is not locally controllable. Indeed, for every trajectory

$$x_2(T) = \int_0^T x_1^2(s)\, ds \geq 0$$

We can write (17) in the form

$$\dot{x} = f_0(x) + f_1(x) u$$

$$f_0 = (0,\, x_1^2), \qquad\qquad f_1 = (1, 0)$$

In this case

$$[f_1, f_0] = (0, 2x_1), \qquad\qquad \big[f_1,\, [f_1, f_0]\big] = (0, 2)$$

hence the vectors f_1 and $\big[f_1,\, [f_1, f_0]\big]$ span $I\!R^2$.

For every $T > 0$ the reachable set $R(T)$ has non-empty interior.

Optimal Control Problems

$$\dot{x} = f(x,u) \qquad u \in U, \qquad t \in [0,T] \tag{1}$$

$$x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

Among all trajectories of (1)-(2), we seek one which is optimal w.r.t. some cost criterion:

Mayer Problem

$$\text{minimize} \qquad J = \psi\big(x(T)\big) \tag{18}$$

Lagrange Problem

$$\text{minimize} \qquad J = \int_0^T \varphi\big(t,\, x(t),\, u(t)\big)\, dt \tag{19}$$

Bolza Problem

$$\text{minimize} \qquad J = \int_0^T \varphi\big(t,\, x(t),\, u(t)\big)\, dt + \psi\big(x(T)\big) \tag{20}$$

$\varphi(x,u) =$ *running cost* per unit time
$\psi(x) =$ *terminal cost*

The minimization is always performed among all control functions $u : [0,T] \mapsto U$

One may also add a terminal constraint, requiring

$$x(T) \in S \tag{21}$$

for some set $S \subset \mathbb{R}^n$

Basic assumptions: The cost functions φ, ψ are continuous. The set S is closed.

Equivalence of various formulations

1. A Mayer Problem can be written as a Lagrange Problem, taking

$$\varphi(x,u) \doteq \nabla\psi(x) \cdot f(x,u)$$

We are then minimizing

$$\int_0^T \nabla\psi\big(x(t)\big) \cdot f\big(x(t),\, u(t)\big)\, dt = \int_0^T \nabla\psi\big(x(t)\big) \cdot \dot x(t)\, dt$$
$$= \int_0^T \left(\frac{d}{dt}\psi\big(x(t)\big)\right) dt = \psi\big(x(T)\big) - \psi\big(x_0\big)$$

2. A Lagrange Problem can be written as a Mayer Problem, introducing a new variable x_{n+1} with

$$x_{n+1}(0) = 0, \qquad\qquad \dot x_{n+1} = \varphi\big(t,\, x(t),\, u(t)\big)$$

and defining

$$\psi(x) \doteq x_{n+1}$$

This yields the terminal cost

$$\psi\big(x(T)\big) = x_{n+1}(T) = \int_0^T \varphi\big(t,\, x(t),\, u(t)\big)\, dt$$

Relations with the Calculus of Variations

Standard Problem in the Calculus of Variations

$$\text{minimize} \qquad \int_0^T L\big(t,\, x(t),\, \dot x(t)\big)\, dt \tag{22}$$

among all absolutely continuous functions $x : [0,T] \mapsto \mathbb{R}^n$ with

$$x(0) = x_0, \qquad\qquad x(T) = x_1$$

Optimal Control Problem

$$\text{minimize} \qquad \int_0^T \varphi\big(t,\, x(t),\, u(t)\big)\, dt \tag{19}$$

$$\dot x = f(x,u) \qquad\qquad u \in U, \qquad t \in [0,T] \tag{1}$$

$$x(0) = x_0, \qquad\qquad x(T) = x_1 \tag{2}$$

1. We can write (22) as an optimal control problem by setting

$$\dot x = u, \qquad\qquad u \in U \doteq \mathbb{R}^n$$
$$\varphi(t,x,u) \doteq L(t,x,u)$$

2. We can write (19) in the form (22) by setting

$$L(t,x,p) \doteq \min\big\{\varphi(t,x,u)\,;\quad u \in U, \quad f(x,u) = p\big\}$$
$$L(t,x,p) = \infty \qquad\qquad \text{if} \quad p \notin \big\{f(x,u)\,; u \in U\big\}$$

$L(t,x,p)$ is the minimum running cost, among all controls that yield the speed $\dot x(t) = p$.

Existence of Optimal Controls

Mayer problem:

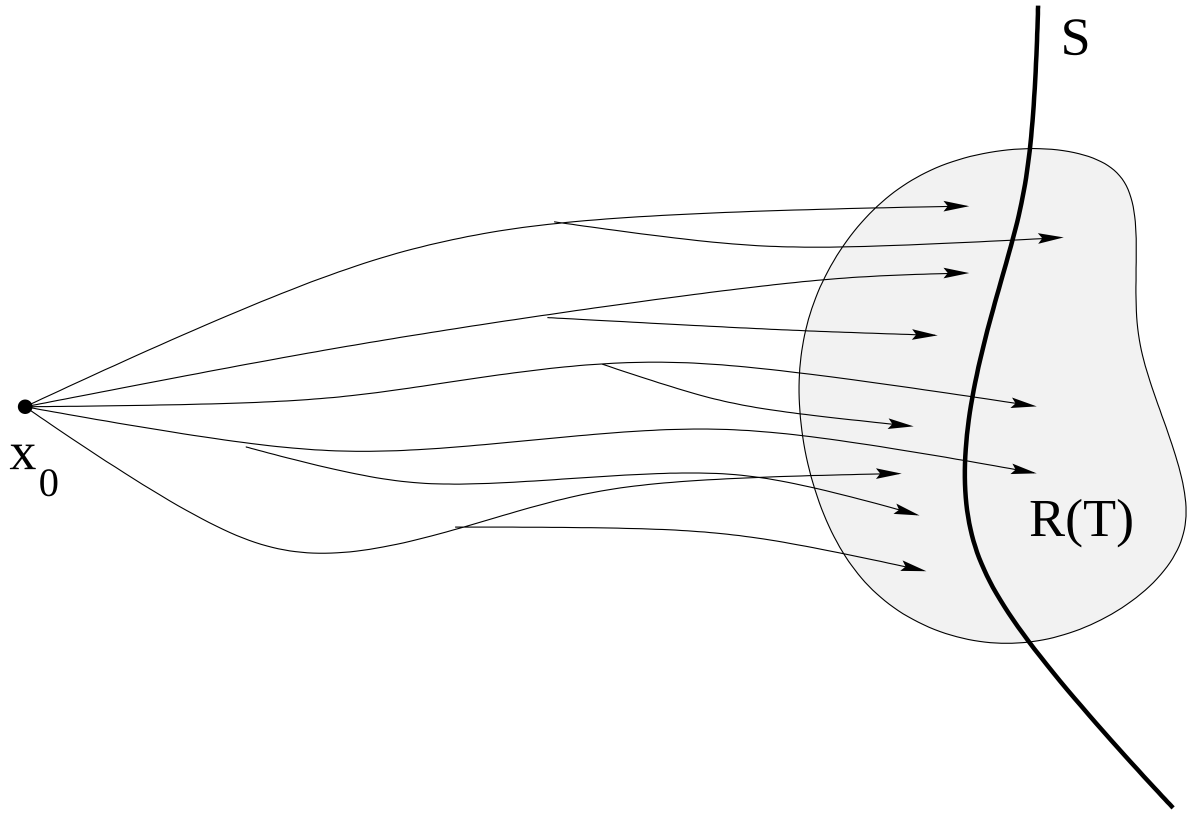
$$\text{minimize} \qquad J = \psi\big(x(T)\big) \tag{18}$$

for the system

$$\dot{x} = f(x,u), \qquad u \in U \tag{1}$$

with constraints

$$x(0) = x_0, \qquad x(T) \in S \tag{2}$$



Theorem 10 (Existence for Mayer Problem).
Assume that, for each x, the set of velocities

$$G(x) \doteq \{f(x,u)\,; \quad u \in U\}$$

is convex. If there exists at least one trajectory $x(\cdot)$ that satisfies (1)-(2), then the Mayer problem (18) admits an optimal solution.

Proof. By Theorem 4, the reachable set $R(T)$ is compact.

By the assumptions, $R(T) \cap S$ is non-empty and compact.

The continuous function ψ admits a global minimum on the compact set $R(T) \cap S$. This yields the optimal solution.

Bolza Problem:

$$\text{minimize} \qquad J = \int_0^T \varphi\big(t,\, x(t),\, u(t)\big)\, dt + \psi\big(x(T)\big) \tag{20}$$

for the system (1) with initial and terminal conditions (2).

Theorem 11 (Existence for the Bolza Problem). Assume that, for every t, x, the set

$$G^+(t,x) \doteq \big\{(y, y_{n+1}) \in I\!R^{n+1}\,; \quad y = f(x,u), \quad y_{n+1} \geq \varphi(t,x,u) \quad \text{for some } u \in U\big\}$$

is convex. If there exists at least one trajectory $x(\cdot)$ that satisfies (1)-(2), then the Bolza problem (20) admits an optimal solution.

Remark: For the problem:

$$\text{minimize} \qquad \int_0^T L\big(t,\, x(t),\, u(t)\big)\, dt \tag{22}$$

subject to

$$\dot{x} = u, \qquad x(0) = x_0, \qquad x(T) = x_1$$

corresponding to the standard problem in the Calculus of Variations, the set

$$\begin{aligned} G^+(t,x) &\doteq \Big\{(y, y_{n+1}) \in I\!R^{n+1}\,; \quad y = f(x,u), \ \ y_{n+1} \geq \varphi(t,x,u) \quad \text{for some } u \in U\Big\} \\ &= \Big\{(u, y_{n+1}) \in I\!R^{n+1}\,; \quad y_{n+1} \geq L(t,x,u)\Big\} \end{aligned}$$

is precisely the epigraph of the function $u \mapsto L(t,x,u)$

This is a convex set iff $L(t,x,\dot{x})$ is convex as a function of $\dot{x}$