First Order Variations

Let $t \mapsto x(t)$ be a solution to the O.D.E.

$$\dot{x} = g(t, x) \qquad\qquad x \in I\!R^n \tag{24}$$

Assume g measurable w.r.t. t and continuously differentiable w.r.t. x

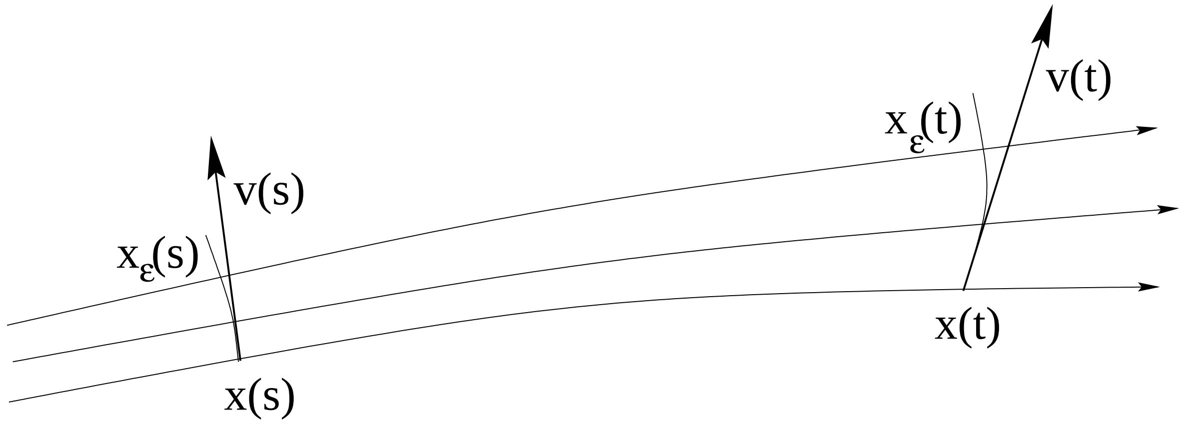
First order perturbation:

$$x_\varepsilon(t) = x(t) + \varepsilon v(t) + o(\varepsilon) \tag{25}$$

$o(\varepsilon)$= infinitesimal of higher order w.r.t. ε

If $t \mapsto x_\varepsilon(t)$ is another solution of (24), letting $\varepsilon \to 0$ we find a linearized evolution equation for the first order tangent vector v

$$\dot{v}(t) = A(t)\, v(t) \qquad\qquad A(t) \doteq D_x g\big(t,\, x(t)\big) \tag{26}$$



$$\text{adjoint system:} \qquad\qquad \dot{p}(t) = -p(t)A(t) \tag{27}$$

Here A is an $n \times n$ matrix, with entries $A_{ij} = \partial g_i / \partial x_j$,
$p \in I\!R^n$ is a **row vector** and $v \in I\!R^n$ is a **column vector**.

If $t \mapsto p(t)$ and $t \mapsto v(t)$ satisfy (27) and (26), then the product $t \mapsto p(t)v(t)$ is constant in time:

$$\frac{d}{dt}\big(p(t)\, v(t)\big) = \dot{p}(t)v(t) + p(t)\dot{v}(t) = -p(t)A(t)v(t) + p(t)A(t)v(t) = 0$$

Necessary Conditions for Optimality

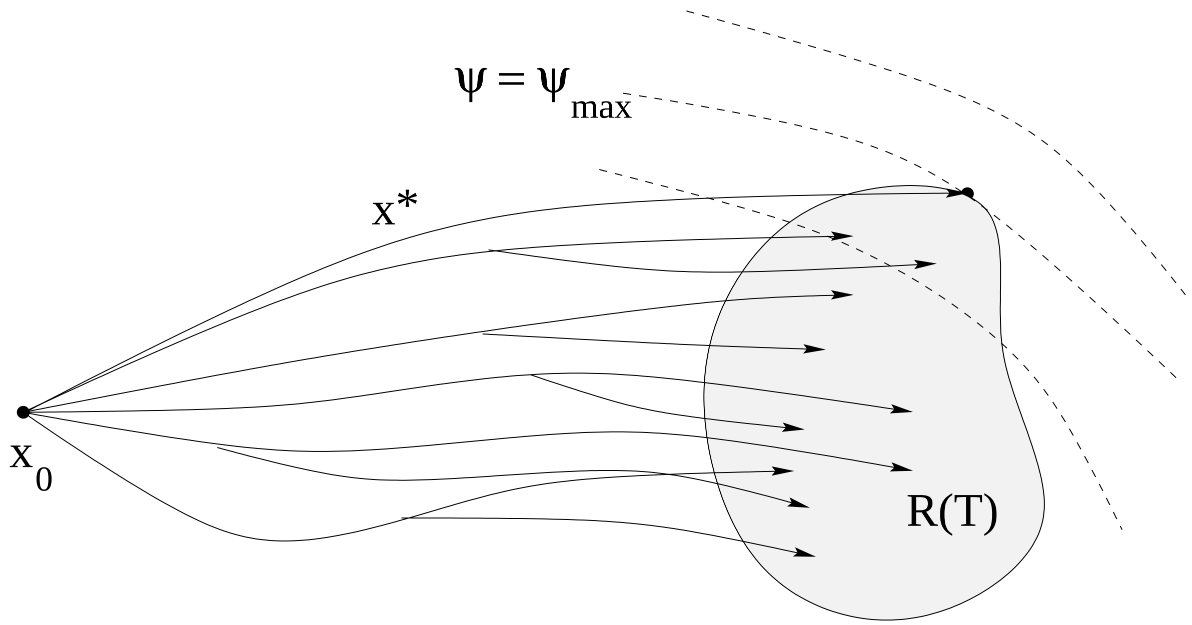
$$\dot{x} = f(x, u) \qquad t \in [0, T], \qquad\qquad x(0) = x_0 \tag{28}$$

Family of admissible control functions: $\mathcal{U} \doteq \{u : [0, T] \mapsto U\}$

For $u(\cdot) \in \mathcal{U}$, the trajectory of (28) is $t \mapsto x(t, u)$

Mayer problem, free terminal point:

$$\max_{u \in \mathcal{U}} \; \psi\big(x(T, u)\big) \tag{29}$$

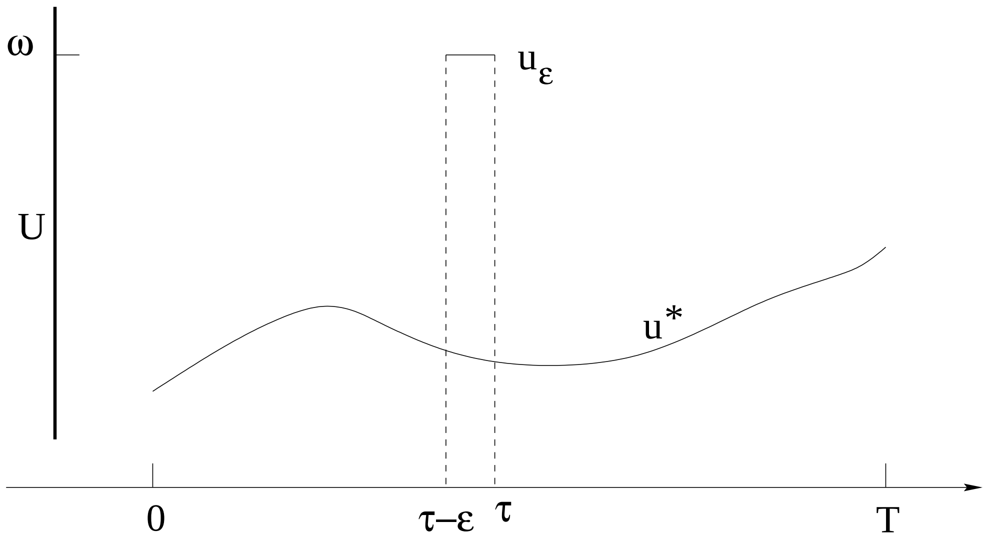


Assume: $t \mapsto u^*(t)$ is an optimal control
$t \mapsto x^*(t) = x\big(t, u^*(t)\big)$ is the corresponding optimal trajectory.

The value $\psi\big(x(T, u^*)\big)$ cannot be increased by any perturbation of the control $u^*(\cdot)$.

Fix a time $\tau \in]0, T]$, $\omega \in U$ and consider the *needle variation* $u_\varepsilon \in \mathcal{U}$

$$u_\varepsilon(t) = \begin{cases} \omega & \text{if} \quad t \in [\tau - \varepsilon,\, \tau] \\ u^*(t) & \text{if} \quad t \notin [\tau - \varepsilon,\, \tau] \end{cases} \tag{30}$$



Call $t \mapsto x_\varepsilon(t) = x(t, u_\varepsilon)$ the perturbed trajectory.

We shall compute the terminal point $x_\varepsilon(T) = x(T, u_\varepsilon)$ and check that the value of ψ is not increased by the perturbation.

Assuming that the optimal control u^* is continuous at time $t = \tau$, we have

$$v(\tau) \doteq \lim_{\varepsilon \to 0} \frac{x_\varepsilon(\tau) - x^*(\tau)}{\varepsilon} = f\big(x^*(\tau), \omega\big) - f\big(x^*(\tau), u^*(\tau)\big) \tag{31}$$

Indeed, $x_\varepsilon(\tau - \varepsilon) = x^*(\tau - \varepsilon)$ and on the small interval $[\tau - \varepsilon,\ \tau]$ we have

$$\dot{x}_\varepsilon \approx f\big(x^*(\tau),\ \omega\big), \qquad\qquad \dot{x}^* \approx f\big(x^*(\tau),\ u^*(\tau)\big)\,.$$

Since $u_\varepsilon = u^*$ on the remaining interval $t \in [\tau, T]$, the evolution of the tangent vector

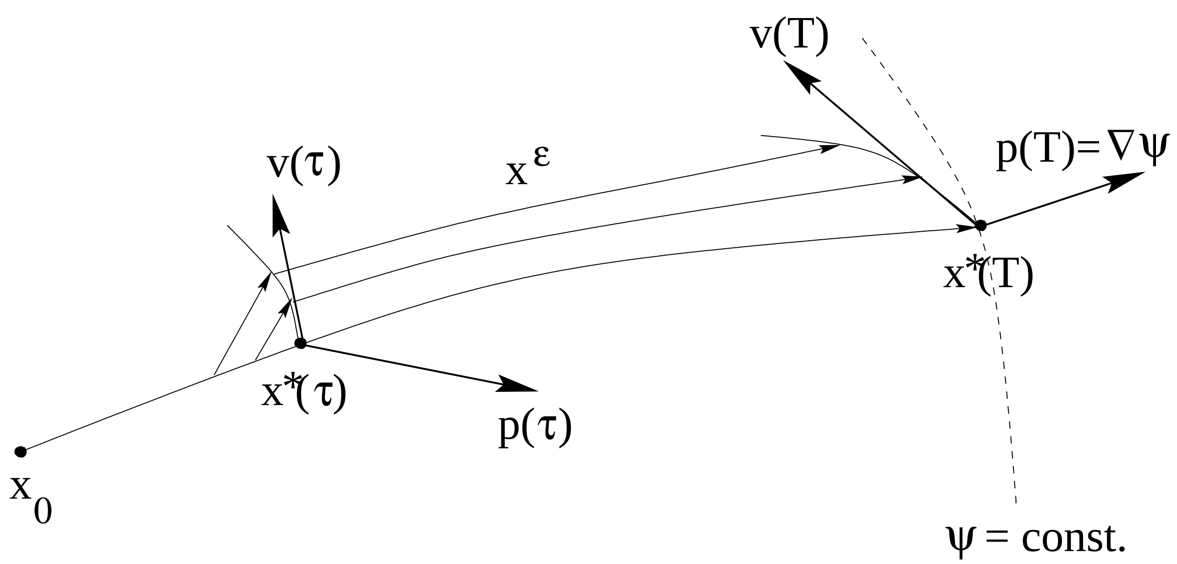
$$v(t) \doteq \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t) - x^*(t)}{\varepsilon} \qquad\qquad t \in [\tau, T]$$

is governed by the linear equation

$$\dot{v}(t) = A(t)\, v(t) \qquad\qquad A(t) \doteq D_x f\big(x^*(t), u^*(t)\big)\,. \tag{32}$$

By maximality, $\psi\big(x_\varepsilon(T)\big) \leq \psi\big(x^*(T)\big)$, therefore

$$\nabla\psi\big(x^*(T)\big) \cdot v(T) \leq 0\,. \tag{33}$$



Summing up: For every time τ and every control value $\omega \in U$, we can generate the vector

$$v(\tau) \doteq f\big(x^*(\tau),\, \omega\big) - f\big(x^*(\tau),\, u^*(\tau)\big)$$

and propagate it forward in time, by solving the linearized equation (32). The inequality (33) is then a necessary condition for optimality.

Instead of propagating the (infinitely many) vectors $v(\tau)$ forward in time, it is more convenient to propagate the single vector $\nabla\psi$ backward. We thus define the row vector $t \mapsto p(t)$ as the solution of

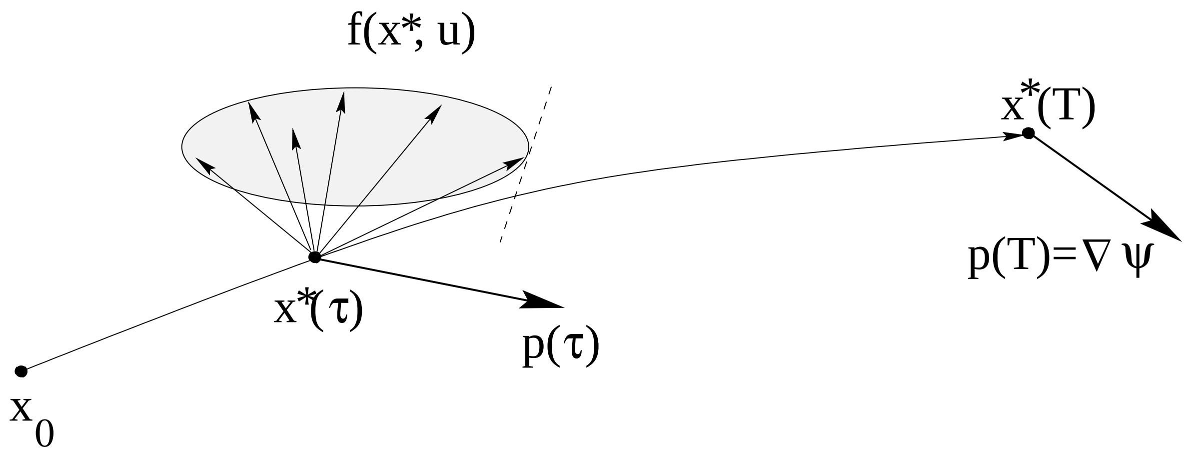
$$\dot p(t) = -p(t)\, A(t), \qquad\qquad p(T) = \nabla\psi\big(x^*(T)\big) \tag{34}$$

This yields $p(t)v(t) = p(T)v(T)$ for every t. In particular, (33) implies

$$p(\tau) \cdot \Big[f\big(x^*(\tau),\, \omega\big) - f\big(x^*(\tau),\, u^*(\tau)\big)\Big] = \nabla\psi\big(x^*(T)\big) \cdot v(T) \leq 0$$

$$p(\tau) \cdot \dot x^*(\tau) \;=\; p(\tau) \cdot f\big(x^*(\tau),\, u^*(\tau)\big) \;=\; \max_{\omega \in U}\; \Big\{ p(\tau) \cdot f\big(x^*(\tau),\, \omega\big)\Big\}\,.$$

For every time $\tau \in \,]0, T]$, the speed $\dot x^*(\tau)$ corresponding to the optimal control $u^*(\tau)$ is the one having has inner product with $p(\tau)$ as large as possible.



Pontryagin Maximum Principle (Mayer Problem, free terminal point)

Consider the control system

$$\dot{x} = f(x,u) \qquad u(t) \in U \qquad t \in [0,T] \tag{35}$$

with initial data

$$x(0) = x_0 \,. \tag{36}$$

Let $t \mapsto u^*(t)$ be an optimal control and $t \mapsto x^*(t) = x(t,u^*)$ be the optimal trajectory for the maximization problem

$$\max_{u\in\mathcal{U}} \ \psi\big(x(T,u)\big) \,. \tag{37}$$

Define the vector $t \mapsto p(t)$ as the solution to the linear adjoint system

$$\dot{p}(t) = -p(t)\,A(t), \qquad\qquad A(t) \doteq D_x f\big(x^*(t),\, u^*(t)\big) \tag{38}$$

with terminal condition

$$p(T) = \nabla\psi\big(x^*(T)\big). \tag{39}$$

Then, for almost every $\tau \in [0,T]$ the following maximality condition holds

$$p(\tau)\cdot f\big(x^*(\tau),\, u^*(\tau)\big) \;=\; \max_{\omega\in U}\ \Big\{p(\tau)\cdot f\big(x^*(\tau),\,\omega\big)\Big\} \tag{39}$$

Computing the Optimal Control

STEP 1: solve the pointwise maximixation problem (39), obtaining the optimal control u^* as a function of p, x, i.e.

$$u^*(x,p) = \underset{\omega \in U}{\operatorname{argmax}} \big\{ p \cdot f(x,\omega) \big\} \tag{40}$$

STEP 2: solve the two-point boundary value problem

$$\begin{cases} \dot{x} = f\big(x,\ u^*(x,p)\big) \\ \dot{p} = -p \cdot D_x f\big(x,\ u^*(x,p)\big) \end{cases} \qquad \begin{cases} x(0) = x_0 \\ p(T) = \nabla\psi\big(x(T)\big) \end{cases} \tag{41}$$

- In general, the function $u^* = u^*(p,x)$ in (40) is highly nonlinear. It may be multivalued or discontinuous.

- The two-point boundary value problem (41) can be solved by a *shooting method*: Guess an initial value $p(0) = p_0$ and solve the corresponding Cauchy problem. Try to adjust the value of p_0 so that the terminal values $x(T),\ p(T)$ satisfy the given conditions.

Example 1 (Linear pendulum).

$q(t)$ = position of a linearized pendulum, controlled by an external force with magnitude $u(t) \in [-1,\, 1]$.

$$\ddot{q}(t) + q(t) = u(t), \qquad q(0) = \dot{q}(0) = 0, \qquad u(t) \in [-1,\, 1]$$

We wish to maximize the terminal displacement $q(T)$.

Equivalent control system: $x_1 = q$, $x_2 = \dot{q}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u - x_1 \end{cases} \qquad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \end{cases}$$

$$\text{maximize } x_1(T) \qquad \text{over all controls } \; u : [0, T] \mapsto [-1, 1]$$

Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory. The linearized equation for a tangent vector is

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The corresponding adjoint vector $p = (p_1, p_2)$ satisfies

$$(\dot{p}_1, \dot{p}_2) = -(p_1, p_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (p_1, p_2)(T) = \nabla\psi\big(x^*(T)\big) = (1, 0) \tag{42}$$

because $\psi(x) \doteq x_1$.

In this special linear case, we can explicitly solve (42) without needing to know x^*, u^*. An easy computation yields

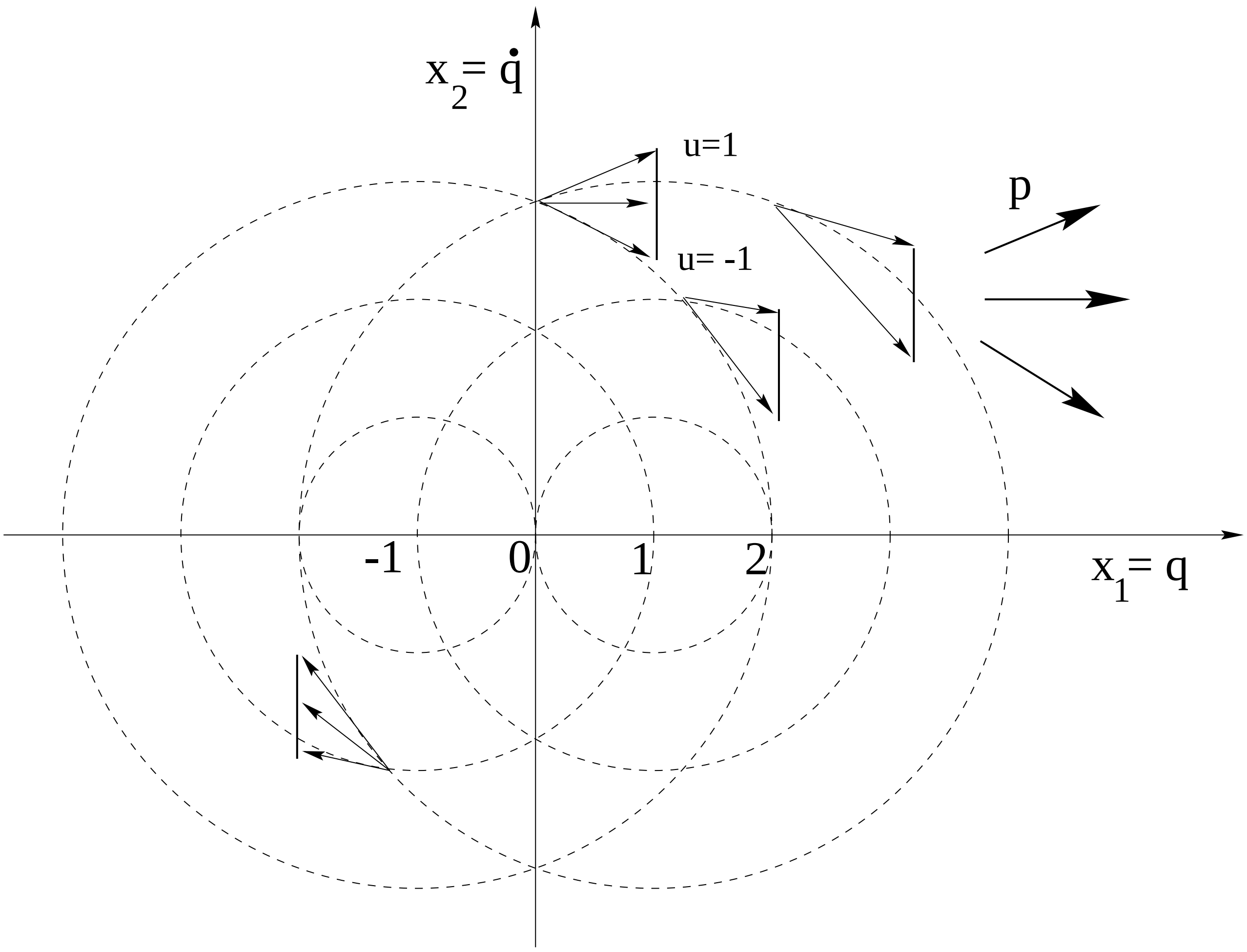
$$(p_1, p_2)(t) = \big(\cos(T - t), \;\; \sin(T - t)\big) \tag{43}$$

For each t, we must now choose the value $u^*(t) \in [-1,\, 1]$ so that

$$p_1 x_2 + p_2(-x_1 + u^*) = \max_{\omega \in [-1,1]} p_1 x_2 + p_2(-x_1 + \omega)$$

By (43), the optimal control is

$$u^*(t) = \operatorname{sign}\big(p_2(t)\big) = \operatorname{sign}\big(\sin(T - t)\big)$$



Example 2. Consider the problem on $I\!R^3$

$$\text{maximize } x_3(T) \qquad \text{over all controls } \ u:[0,T]\mapsto[-1,1]$$

for the system

$$\begin{cases} \dot x_1 = u \\ \dot x_2 = -x_1 \\ \dot x_3 = x_2 - x_1^2 \end{cases} \qquad\qquad \begin{cases} x_1(0)=0 \\ x_2(0)=0 \\ x_3(0)=0 \end{cases}$$

The adjoint equations take the form

$$(\dot p_1, \dot p_2, \dot p_3) = (p_2 + 2x_1 p_3, \quad -p_3, \quad 0) \qquad\qquad (p_1,p_2,p_3)(T) = (0,0,1) \tag{44}$$

Maximizing the inner product $p\cdot\dot x$ we obtain the optimality conditions for the control u^*

$$p_1 u^* + p_2\,(-x_1) + p_3\,(x_2 - x_1^2) = \max_{\omega\in[-1,1]} p_1\omega + p_2\,(-x_1) + p_3\,(x_2 - x_1^2) \tag{45}$$

$$\begin{cases} u^* = 1 & \text{if} \quad p_1 > 0 \\ u^* \in [-1,1] & \text{if} \quad p_1 = 0 \\ u^* = -1 & \text{if} \quad p_1 < 0 \end{cases}$$

Solving the terminal value problem (44) for p_2, p_3 we find

$$p_3(t) \equiv 1, \qquad\qquad p_2(t) = T - t$$

The function p_1 can now be found from the equations

$$\ddot{p}_1 = -1 + 2u^* = -1 + 2\,\mathrm{sign}(p_1), \qquad\qquad p_1(T) = 0, \quad \dot{p}_1(0) = p_2(0) = T$$

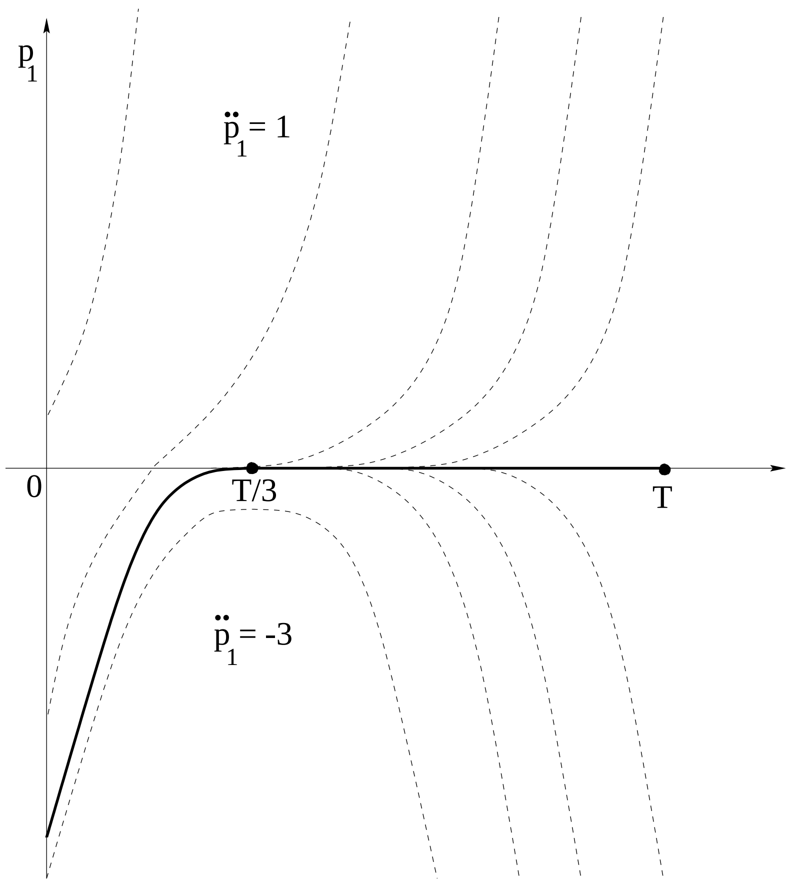
with the convention: $\mathrm{sign}(0) = [-1, 1]$. The only solution is found to be

$$p_1(t) = \begin{cases} -\dfrac{3}{2}\left(\dfrac{T}{3} - t\right)^2 & \text{if} \quad 0 \le t \le T/3 \\ \\ 0 & \text{if} \quad T/3 \le t \le T \end{cases}$$

The optimal control is

$$u^*(t) = \begin{cases} -1 & \text{if} \quad 0 \le t \le T/3 \\ 1/2 & \text{if} \quad T/3 \le t \le T \end{cases}$$

Observe that on the interval $[T/3,\ T]$ the optimal control is derived not from the maximality condition (45) but from the equation $\ddot{p}_1 = (-1 + 2u) \equiv 0$. An optimal control with this property is called **singular**.



Tangent Cones

$$\dot{x} = f(x, u) \qquad u \in U, \qquad\qquad x(0) = x_0$$

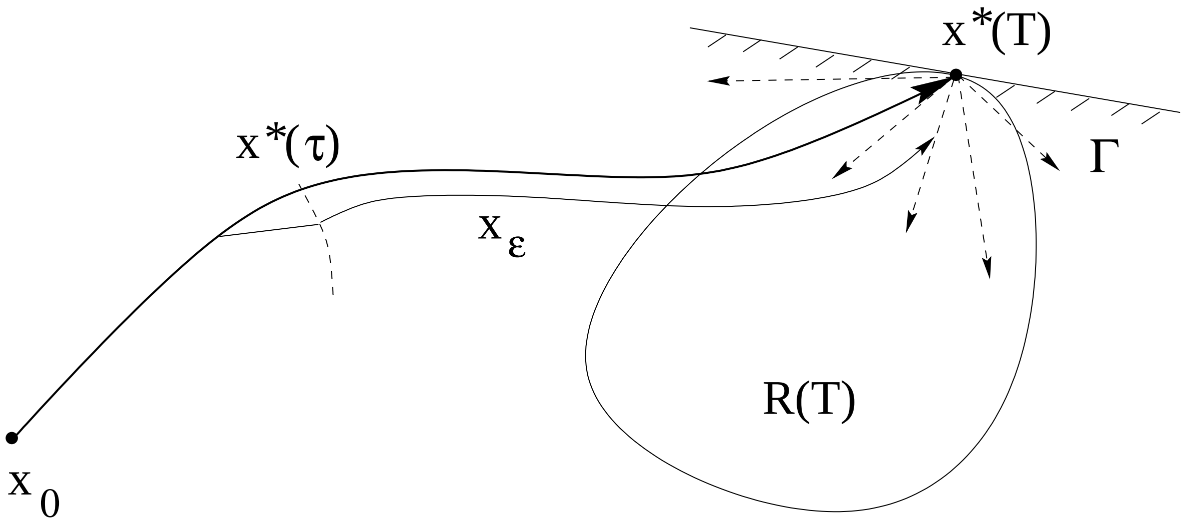
Let $t \mapsto x^*(t) = x(t, u^*)$ be a reference trajectory.
Given $\tau \in]0, T]$, $\omega \in U$, consider the family of *needle variations*

$$u_\varepsilon(t) = \begin{cases} \omega & \text{if} \quad t \in [\tau - \varepsilon,\, \tau] \\ u^*(t) & \text{if} \quad t \notin [\tau - \varepsilon,\, \tau] \end{cases} \tag{30}$$

Call $$v^{\tau,\omega}(T) \doteq \lim_{\varepsilon \to 0} \frac{x(T, u_\varepsilon) - x(T, u^*)}{\varepsilon}$$

the first order variation of the terminal point of the corresponding trajectory.

Define Γ as the smallest convex cone containing all vectors $v^{\tau,\omega}$
This is a **cone of feasible directions**, i.e. directions in which we can move the terminal point $x(T, u^*)$ by suitably perturbing the control u^*.



Consider a **terminal constraint** $x(T) \in S$, where

$$S \doteq \{x \in I\!R^n\,; \quad \phi_i(x) = 0, \qquad i = 1, \ldots, N\}.$$

Assume that the $N+1$ gradients $\nabla\psi,\ \nabla\phi_1, \ldots, \nabla\phi_N$ are linearly independent at the point x^*. Then the tangent space to S at x^* is

$$T_S = \{v \in I\!R^n\,; \quad \nabla\phi_i(x^*) \cdot v = 0 \quad i = 1, \ldots, N\}.$$

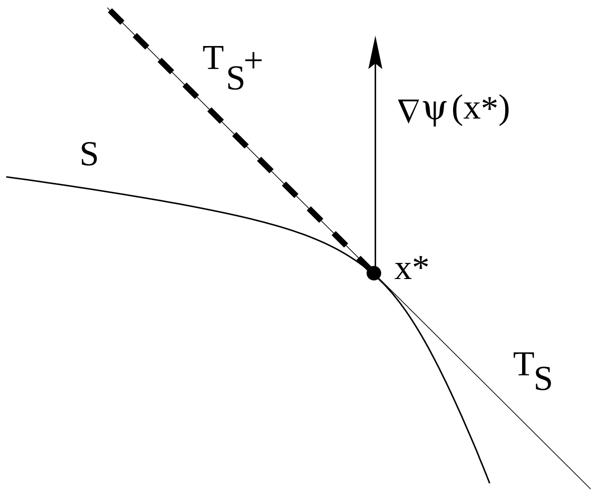
The tangent cone to the set

$$S^+ = \{x \in S\,; \quad \psi(x) \geq \psi(x^*)\}$$

is

$$T_{S^+} = \{v \in I\!R^n\,; \quad \nabla\psi(x^*) \cdot v \geq 0, \qquad \nabla\phi_i(x^*) \cdot v = 0 \quad i = 1, \ldots, N\}$$

When $x^* = x^*(T)$, we think of T_{S^+} as the **cone of profitable directions**, i.e. those directions in which we would like to move the terminal point, in order to increase the value of ψ and still satisfy the constraint $x(T) \in S$.



Lemma 1. A vector $p \in I\!R^n$ satisfies

$$p \cdot v \geq 0 \qquad \text{for all} \quad v \in T_{S^+} \tag{46}$$

if and only if it can be written as a linear combination

$$p = \lambda_0 \nabla\psi(x^*) + \sum_{i=1}^{N} \lambda_i \nabla\phi_i(x^*) \qquad \text{with} \quad \lambda_0 \geq 0. \tag{47}$$

Mayer Problem with terminal constraints

$$\text{maximixe} \quad \psi\big(x(T)\big)$$

for the control system

$$\dot{x} = f(x,u) \qquad u(t) \in U \qquad t \in [0,T]$$

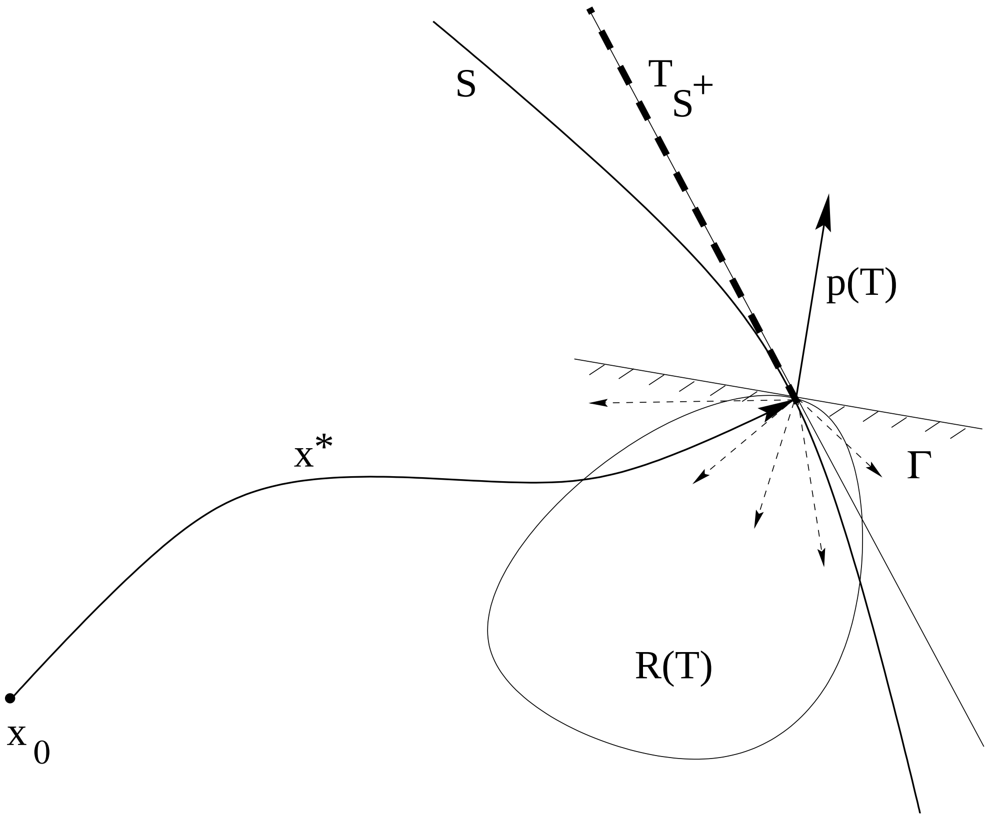
with initial and terminal constraints

$$x(0) = x_0, \qquad\qquad \phi_i\big(x(T)\big) = 0, \qquad i = 1,\ldots,N$$

PMP, geometric version. Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory, corresponding to the control $u^*(\cdot)$. Then the cones Γ and T_{S+} are weakly separated, i.e. there exists a non-zero vector $p(T)$ such that

$$p(T) \cdot v \geq 0 \qquad \text{for all} \quad v \in T_{S+} \tag{48}$$

$$p(T) \cdot v \leq 0 \qquad \text{for all} \quad v \in \Gamma \tag{49}$$



PMP, analytic version. Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory, corresponding to the control $u^*(\cdot)$. Then there exists a non-zero vector function $t \mapsto p(t)$ such that

$$p(T) = \lambda_0 \nabla\psi\big(x^*(T)\big) + \sum_{i=1}^{N} \lambda_i \nabla\phi_i\big(x^*(T)\big) \qquad \text{with} \;\; \lambda_0 \geq 0 \tag{50}$$

$$\dot p(t) = -p(t)\, D_x f\big(x^*(t),\, u^*(t)\big) \qquad\qquad t \in [0, T] \tag{51}$$

$$p(\tau) \cdot f\big(x^*(\tau),\, u^*(\tau)\big) \;\; = \;\; \max_{\omega \in U} \Big\{ p(\tau) \cdot f\big(x^*(\tau),\, \omega\big) \Big\} \qquad \text{for a.e. } \tau \in [0, T]. \tag{52}$$

Indeed, Lemma 1 states that (48) $\Longleftrightarrow$ (50)

We now show that (49)$\Longleftrightarrow$ (51)+(52).
Recall that every tangent vector $v^{\tau,\omega}$ satisfies the linear evolution equation

$$\dot v^{\tau,\omega}(t) = D_x f\big(x^*(t),\, u^*(t)\big)\, v^{\tau,\omega}(t)$$

If $t \mapsto p(t)$ satisfies (51), then the product $p(t) \cdot v^{\tau,\omega}(t)$ is constant. Hence

$$p(T) \cdot v^{\tau,\omega}(T) \leq 0$$

if and only if

$$p(\tau) \cdot v^{\tau,\omega}(\tau) \leq 0$$

if and only if

$$p(\tau) \cdot \Big[f\big(x^*(\tau),\, \omega\big) - f\big(x^*(\tau),\, u^*(\tau)\big) \Big] \leq 0$$

if and only if (52) holds.

High Order Conditions ?

The Pontryagin Maximum Principle is a *first order necessary condition.* For the Mayer problem

$$\text{maximize} \quad \psi\big(x(T,u)\big)$$

if $u^*(\cdot)$ is a given control, and if we can find a sequence of perturbed controls u_ε such that

$$\lim_{\varepsilon\to 0} \frac{\psi\big(x(T,u_\varepsilon)\big)-\psi\big(x(T,u^*)\big)}{\|u_\varepsilon-u^*\|_{\mathbf{L}^1}} > 0$$

then the PMP will detect the non-optimality of u^*. However, if the increment in the value of ψ is of second or higher order w.r.t. the variation $\|u_\varepsilon-u^*\|_{\mathbf{L}^1}$, then the conclusion of the PMP may still hold, even if u^* is not optimal.

Example 3.

$$\text{maximize} \quad \psi\big(x(T)\big) \doteq x_2(T)$$

for the system

$$(\dot x_1,\dot x_2)=(u,x_1^2), \qquad (x_1,x_2)(0)=(0,0), \qquad u(t)\in[-1,\,1].$$

The constant control $u^*(t)\equiv 0$ yields the constant trajectory $(x_1^*,x_2^*)(t)\equiv(0,0)$. The corresponding adjoint vector satisfying

$$(\dot p_1,\dot p_2)=(-2x_1p_2,\ 0)=(0,0), \qquad\qquad (p_1,p_2)(T)=(0,1)$$

is trivially found to be $(p_1,p_2)(t)\equiv(0,1)$. Hence the maximality condition

$$p_1u^*+p_2\,(x_1^*)^2=0=\max_{\omega\in[-1,1]} 0\cdot\omega+1\cdot 0$$

is satisfied for every $t\in[0,T]$. However, the control $u^*\equiv 0$ produces the worst possible outcome. Any other control $u\neq u^*$ would yield

$$x_2(T,u)=\int_0^T\left(\int_0^t u(s)\,ds\right)^2 dt > x_2(T,u^*)=0.$$

Notice that in this case the increase in ψ is only of second order w.r.t. the perturbation $u-u^*$

$$x_2(T,u)\le\int_0^T\left(\int_0^T \big|u(s)\big|\,ds\right)^2 dt\le T\cdot\|u\|_{\mathbf{L}^1}^2$$

Lagrange Minimization Problem, fixed terminal point

$$\text{minimize} \qquad \int_0^T L(t,x,u)\,dt \tag{53}$$

for the control system on $\mathbb{R}^n$

$$\dot{x} = f(t,x,u) \qquad\qquad u(t) \in U \tag{54}$$

with initial and terminal constraints

$$x(0) = x_0, \qquad\qquad x(T) = x^\sharp \tag{55}$$

PMP, Lagrange problem.

Let $t \mapsto x^*(t) = x(t,u^*)$ be an optimal trajectory, corresponding to the optimal control $u^*(\cdot)$. Then there exist a constant $\lambda_0 \geq 0$ and a row vector $t \mapsto p(t)$ (not both $= 0$) such that

$$\dot{p}(t) = -p(t)\,D_x f\big(t,\, x^*(t),\, u^*(t)\big) - \lambda_0 D_x L\big(t,\, x^*(t),\, u^*(t)\big) \tag{56}$$

$$\begin{aligned} p(t)\cdot f\big(t,\, x^*(t),\, u^*(t)\big) &+ \lambda_0\, L\big(t,\, x^*(t),\, u^*(t)\big) \\ &= \min_{\omega\in U} \Big\{ p(t)\cdot f\big(t,\, x^*(t),\, \omega\big) + \lambda_0\, L\big(t,\, x^*(t),\, \omega\big) \Big\}\,. \end{aligned} \tag{57}$$

This follows by applying the previous results to the Mayer problem

$$\text{minimize} \qquad x_{n+1}(T)$$

with

$$\dot{x}_{n+1} = L(t,x,u), \qquad\qquad x_{n+1}(0) = 0$$

Because of the terminal constraints $(x_1,\ldots,x_n)(T) = (x_1^\sharp,\ldots,x_n^\sharp)$, the only requirement on the terminal value $(p_1,\ldots,p_n,p_{n+1})(T)$ is

$$p_{n+1}(T) \geq 0$$

Observe that $\dot{p}_{n+1} = 0$, hence $p_{n+1}(t) \equiv \lambda_0$ for some constant $\lambda_0 \geq 0$.

Applications to the Calculus of Variations

Standard problem of the Calculus of Variations:

$$\text{minimize} \qquad \int_0^T L\big(t,\, x(t),\, \dot x(t)\big)\, dt \tag{58}$$

over all absolutely continuous functions $x : [0,T] \mapsto \mathbb{R}^n$ such that

$$x(0) = x_0, \qquad\qquad x(T) = x^\sharp \tag{55}$$

This corresponds to the optimal control problem (53), for the control system

$$\dot x = u, \qquad\qquad u \in \mathbb{R}^n \tag{59}$$

We assume that L is smooth, and that $x^*(\cdot)$ is an optimal solution. By the Pontryagin Maximum Principle (56)-(57), there exist a constant $\lambda_0 \geq 0$ and a row vector $t \mapsto p(t)$ (not both $= 0$) such that

$$\dot p(t) = -\lambda_0 \frac{\partial}{\partial x} L\big(t,\, x^*(t),\, \dot x^*(t)\big) \tag{60}$$

$$p(t) \cdot \dot x^*(t) + \lambda_0\, L\big(t,\, x^*(t),\, \dot x^*(t)\big) = \min_{\omega \in \mathbb{R}^n} \Big\{ p(t) \cdot \omega + \lambda_0\, L\big(t,\, x^*(t),\, \omega\big) \Big\} \tag{61}$$

If $\lambda_0 = 0$, then $p(t) \neq 0$. But in this case $\dot x^*$ cannot provide a minimum over the whole space $\mathbb{R}^n$. This contradiction shows that we must have $\lambda_0 > 0$.

Since λ_0, p are determined up to a positive scalar multiple, we can assume $\lambda_0 = 1$.

With this choice (61) implies

$$p(t) = -\frac{\partial}{\partial \dot x} L\big(t,\, x^*(t),\, \dot x^*(t)\big) \tag{62}$$

The evolution equation

$$\dot{p}(t) = -\frac{\partial}{\partial x} L\big(t,\, x^*(t),\, \dot{x}^*(t)\big) \tag{60}$$

now yields the famous **Euler-Lagrange equations**

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}} L\big(t,\, x^*(t),\, \dot{x}^*(t)\big)\right] = \frac{\partial}{\partial x} L\big(t,\, x^*(t),\, \dot{x}^*(t)\big). \tag{63}$$

Moreover, the minimality condition

$$p(t)\cdot \dot{x}^*(t) + L\big(t,\, x^*(t),\, \dot{x}^*(t)\big) = \min_{\omega\in \mathbb{R}^n} \Big\{ p(t)\cdot\omega + L\big(t,\, x^*(t),\, \omega\big)\Big\} \tag{61}$$

yields the **Weierstrass necessary conditions**

$$L(t,\, x^*(t),\, \omega) \geq L\big(t,\, x^*(t),\, \dot{x}^*(t)\big) + \frac{\partial L\big(t,\, x^*(t),\, \dot{x}^*(t)\big)}{\partial \dot{x}} \cdot \big(\omega - \dot{x}^*(t)\big) \tag{64}$$

for every $\omega \in \mathbb{R}^n$.