Viscosity Solutions of Hamilton-Jacobi Equations

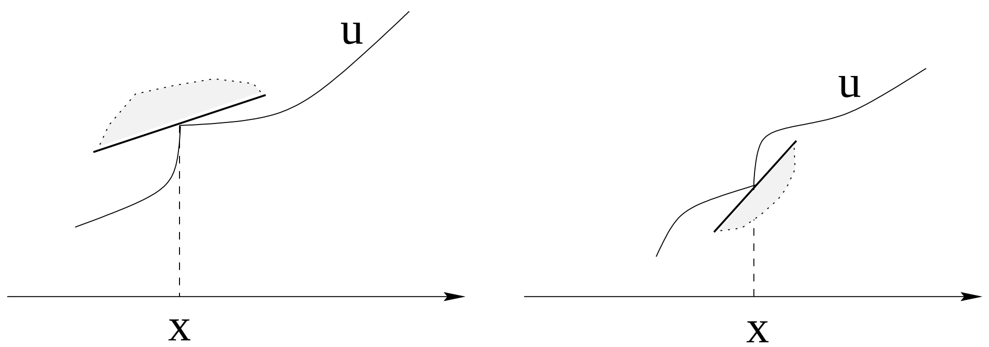
One-sided Differentials

The set of **super-differentials** of u at a point x is

$$D^+u(x) \doteq \left\{ p \in \mathbb{R}^n\,; \quad \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}$$

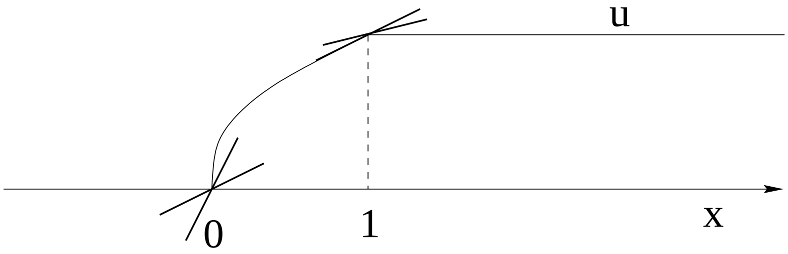
The set of **sub-differentials** of u at x is

$$D^-u(x) \doteq \left\{ p \in \mathbb{R}^n\,; \quad \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}$$



Example 1. Consider the function

$$u(x) \doteq \begin{cases} 0 & \text{if} \quad x < 0 \\ \sqrt{x} & \text{if} \quad x \in [0,1] \\ 1 & \text{if} \quad x > 1 \end{cases}$$



In this case we have

$$D^+u(0) = \emptyset, \qquad\qquad D^-u(0) = [0, \infty[$$

$$D^+u(x) = D^-u(x) = \left\{ \frac{1}{2\sqrt{x}} \right\} \qquad\qquad x \in]0,1[$$

$$D^+u(1) = \big[0,\ 1/2\big], \qquad\qquad D^-u(1) = \emptyset$$

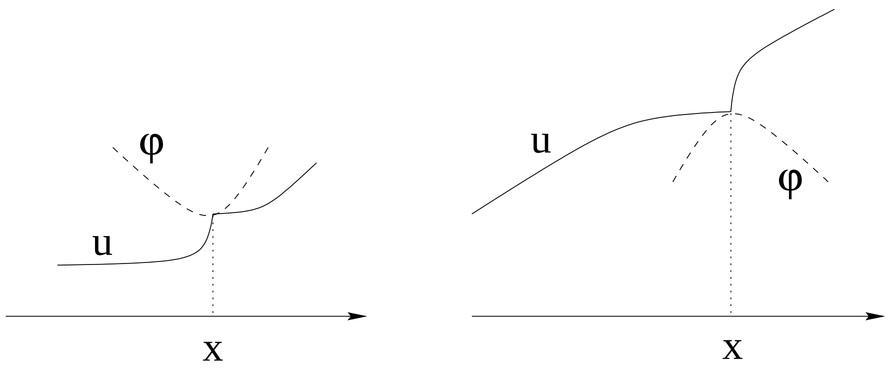
Characterization of super- and sub-differentials

Lemma 1. *Let $u \in \mathcal{C}(\Omega)$. Then*

(i) $p \in D^+u(x)$ *if and only if there exists a function* $\varphi \in \mathcal{C}^1(\Omega)$ *such that* $\nabla\varphi(x) = p$ *and* $u - \varphi$ *has a local maximum at* x.

(ii) $p \in D^-u(x)$ *if and only if there exists a function* $\varphi \in \mathcal{C}^1(\Omega)$ *such that* $\nabla\varphi(x) = p$ *and* $u - \varphi$ *has a local minimum at* x.

By adding a constant, it is not restrictive to assume that $\varphi(x) = u(x)$. In this case, we are saying that $p \in D^+u(x)$ iff there exists a smooth function $\varphi \geq u$ with $\nabla\varphi(x) = p, \quad \varphi(x) = u(x)$.



Proof. Assume that $p \in D^+u(x)$. Then we can find a continuous, increasing function $\sigma : [0, \infty[\mapsto I\!R$, with $\sigma(0) = 0$, such that

$$u(y) - u(x) - p \cdot (y - x) \leq \sigma\big(|y - x|\big)|y - x| = o(1) \cdot (y - x)$$

We then define

$$\rho(r) \doteq \int_0^r \sigma(t)\, dt$$

$$\varphi(y) \doteq u(x) + p \cdot (y - x) + \rho\big(2|y - x|\big)$$

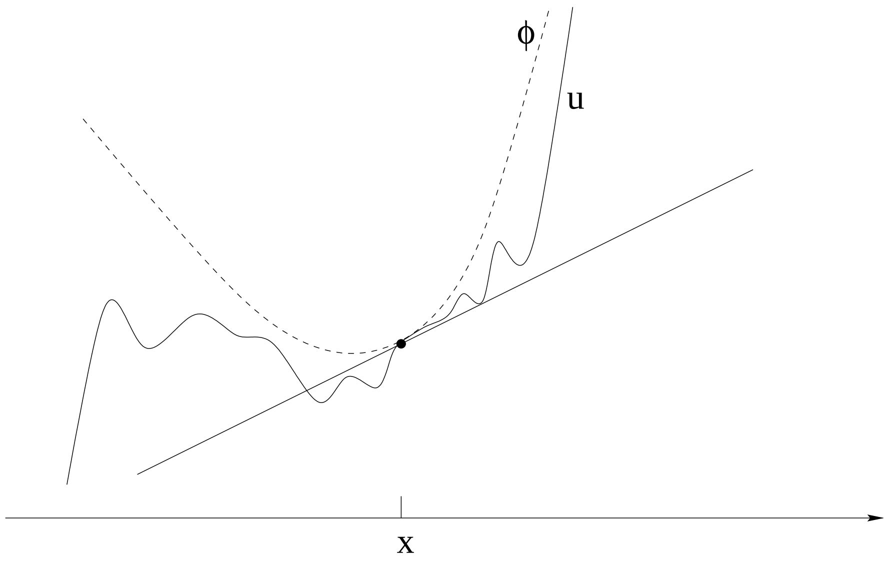
and check that $\varphi \in \mathcal{C}^1$ and $u - \varphi$ has a local maximum at x.

Conversely, if $\varphi \in \mathcal{C}^1$ and $u - \varphi$ has a local maximum at x, then

$$u(y) - u(x) \leq \varphi(y) - \varphi(x)$$

for every y in a neighborhood of x, and hence

$$\limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq \limsup_{y \to x} \frac{\varphi(y) - \varphi(x) - p \cdot (y - x)}{|y - x|} = 0$$



Remark: By possibly replacing φ with $\tilde{\varphi}(y) = \varphi(y) \pm |y - x|^2$, a **strict** local maximum or local minimum at the point x can be achieved.

Properties of super- and sub-differentials

Lemma 2. *Let* $u \in \mathcal{C}(\Omega)$*. Then*

(i) If u *is differentiable at* x*, then*

$$D^+u(x) = D^-u(x) = \{\nabla u(x)\}$$

(ii) If the sets $D^+u(x)$ *and* $D^-u(x)$ *are both non-empty, then* u *is differentiable at* x*.*

(iii) The sets of points where a one-sided differential exists:

$$\Omega^+ \doteq \{x \in \Omega;\ \ D^+u(x) \neq \emptyset\}, \qquad \Omega^- \doteq \{x \in \Omega;\ \ D^-u(x) \neq \emptyset\}$$

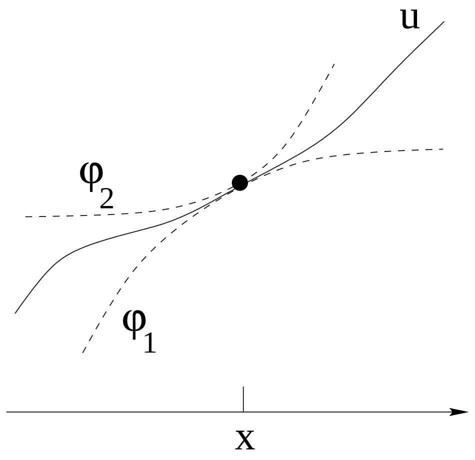
are both non-empty. Indeed, they are dense in Ω*.*

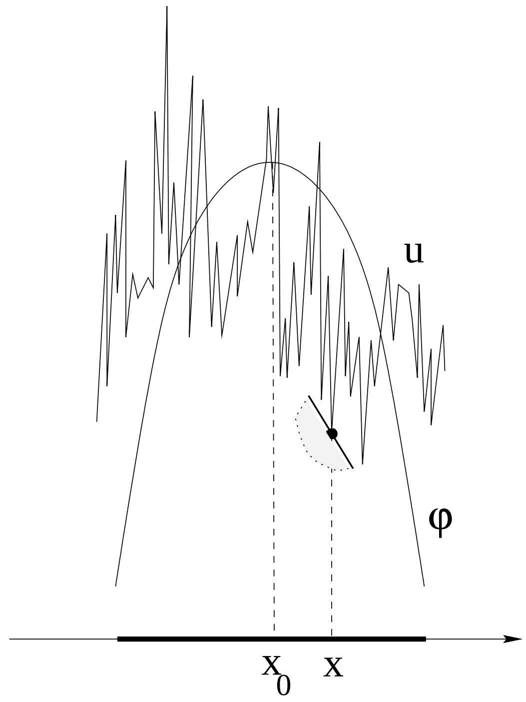
Proof of (i). Assume u is differentiable at x.

Trivially, $\nabla u(x) \in D^{\pm}u(x)$.

On the other hand, if $\varphi \in \mathcal{C}^1(\Omega)$ is such that $u - \varphi$ has a local maximum at x, then $\nabla\varphi(x) = \nabla u(x)$.

Hence $D^+u(x)$ cannot contain any vector other than $\nabla u(x)$.





Proof of (ii). Assume that the sets $D^+u(x)$ and $D^-u(x)$ are both non-empty. Then we can find $\varphi_1, \varphi_2 \in \mathcal{C}^1(\Omega)$ such that, for y near x,

$$\varphi_1(x) = u(x) = \varphi_2(x), \qquad \varphi_1(y) \le u(y) \le \varphi_2(y)$$

By comparison, this implies that u is differentiable at x and $\nabla u(x) = \nabla\varphi_1(x) = \nabla\varphi_2(x)$.

Proof of (iii). Fix any ball $B(x_0, \rho) \subset \Omega$.

By choosing $\varepsilon > 0$ sufficiently small, the smooth function

$$\varphi(x) \doteq u(x_0) - \frac{|x - x_0|^2}{2\varepsilon}$$

is strictly negative on the boundary of the ball, where $|x - x_0| = \rho$.

Since $u(x_0) = \varphi(x_0)$, the function $u - \varphi$ attains a local minimum at an interior point $x \in B(x_0, \rho)$.

By Lemma 1, the sub-differential of u at x is non-empty: $\nabla\varphi(x) = (x - x_0)/\varepsilon \in D^-u(x)$.

Viscosity Solutions

$$F\big(x,\, u(x),\, \nabla u(x)\big) = 0 \tag{HJ}$$

Definition 1. Let $F : \Omega \times I\!\!R \times I\!\!R^n \mapsto I\!\!R$ be continuous.

$u \in \mathcal{C}(\Omega)$ is a **viscosity subsolution** of (HJ) if

$$F\big(x, u(x), p\big) \le 0 \qquad \text{for every } x \in \Omega,\ p \in D^+u(x).$$

$u \in \mathcal{C}(\Omega)$ is a **viscosity supersolution** of (HJ) if

$$F\big(x, u(x), p\big) \ge 0 \qquad \text{for every } x \in \Omega,\ p \in D^-u(x).$$

We say that u is a **viscosity solution** of (HJ) if it is both a supersolution and a subsolution in the viscosity sense.

Evolution equation: $$u_t + H\big(t, x, u, \nabla u\big) = 0. \tag{E}$$

Definition 2. $u \in \mathcal{C}(\Omega)$ is a **viscosity subsolution** of (E) if, for every $\mathcal{C}^1$ function $\varphi = \varphi(t, x)$ such that $u - \varphi$ has a local maximum at (t, x), there holds

$$\varphi_t(t, x) + H(t, x, u, \nabla\varphi) \le 0.$$

$u \in \mathcal{C}(\Omega)$ is a **viscosity supersolution** of (E) if, for every $\mathcal{C}^1$ function $\varphi = \varphi(t, x)$ such that $u - \varphi$ has a local minimum at (t, x), there holds

$$\varphi_t(t, x) + H(t, x, u, \nabla\varphi) \ge 0.$$

• The definition of subsolution imposes conditions on u only on the set Ω^+ of points x where the super-differential $D^+u(x)$ is non-empty. Even if u is merely continuous, say nowhere differentiable, this set is non-empty and dense in Ω.

• If u is a $\mathcal{C}^1$ function that satisfies

$$F\big(x,\, u(x),\, \nabla u(x)\big) = 0 \tag{HJ}$$

at every $x \in \Omega$, then u is also a solution in the viscosity sense.

• Viceversa, if u is a viscosity solution, then the equality (HJ) must hold at every point x where u is differentiable.
If u is Lipschitz continuous, then by Rademacher's theorem it is a.e. differentiable. Hence (HJ) holds a.e. in Ω.

Example 2. The function $u(x) = |x|$ is a viscosity solution of

$$F(x, u, u_x) \doteq 1 - |u_x| = 0 \tag{1}$$

defined on the whole real line. Indeed, u is differentiable and satisfies the equation at all points $x \neq 0$. Moreover, we have

$$D^+u(0) = \emptyset, \qquad\qquad D^-u(0) = [-1,\ 1].$$

To show that u is a subsolution, there is nothing else to check. To show that u is a supersolution, take any $p \in [-1,\, 1]$. Then $1 - |p| \geq 0$, as required.

Notice that $u(x) = |x|$ is NOT a viscosity solution of the equation

$$|u_x| - 1 = 0 \tag{2}$$

Indeed, at $x = 0$, taking $p = 0 \in D^-u(0)$ we find $|0| - 1 < 0$. Therefore, $u(x) = |x|$ is a viscosity subsolution of (2), but not a supersolution.

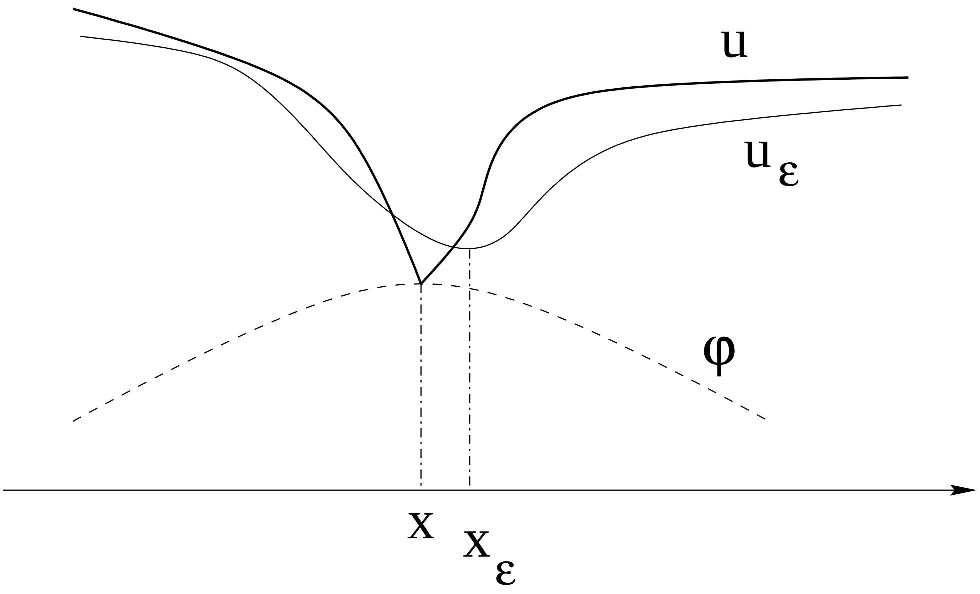
Vanishing Viscosity Limits

Lemma 3. *Let u_ε be a sequence of smooth solutions to the viscous equation*

$$F\big(x,\, u_\varepsilon(x),\, \nabla u_\varepsilon(x)\big) = \varepsilon\, \Delta u_\varepsilon\,. \tag{3}$$

Assume that, as $\varepsilon \to 0+$, we have the convergence $u_\varepsilon \to u$ uniformly on an open set $\Omega \subseteq \mathbb{R}^n$. Then u is a viscosity solution of

$$F\big(x,\, u(x),\, \nabla u(x)\big) = 0 \tag{HJ}$$



Proof. Fix $x \in \Omega$ and assume $p \in D^- u(x)$.
We need to show that $F(x,\, u(x),\, p) \geq 0$.

1. By Lemma 1 and Remark 1, there exists $\varphi \in \mathcal{C}^1$ with $\nabla\varphi(x) = p$, $\varphi(x) = u(x)$ and $\varphi(y) < u(y)$ for all $y \neq x$. For any $\delta > 0$ we can then find $0 < \rho \leq \delta$ and a function $\psi \in \mathcal{C}^2$ such that

$$\big|\nabla\varphi(y) - \nabla\varphi(x)\big| \leq \delta \qquad \text{if} \quad |y - x| \leq \rho$$

$$\|\psi - \varphi\|_{\mathcal{C}^1} \leq \delta$$

and such that each function $u_\varepsilon - \psi$ has a local minimum inside the ball $B(x;\, \rho)$.

2. Let $u_\varepsilon - \psi$ have a local minimum at $x_\varepsilon \in B(x,\rho)$. Then

$$\nabla\psi(x_\varepsilon) = \nabla u(x_\varepsilon), \qquad\qquad \Delta u(x_\varepsilon) \geq \Delta\psi(x_\varepsilon).$$

$$F\big(x,\, u_\varepsilon(x_\varepsilon),\;\; \nabla\psi(x_\varepsilon)\big) \geq \varepsilon\,\Delta\psi(x_\varepsilon) \tag{4}$$

3. Extract a convergent subsequence $x_\varepsilon \to \tilde{x} \in B(x,\rho)$.

Letting $\varepsilon \to 0+$ in (4) we have

$$F\big(x,\, u(\tilde{x}),\;\; \nabla\psi(\tilde{x})\big) \geq 0. \tag{5}$$

Choosing $\delta > 0$ small we can make

$$\big|u(\tilde{x}) - u(x)\big| \qquad\qquad \big|\nabla\psi(\tilde{x}) - p\big|$$

as small as we like.

By continuity, (5) yields $F(x,\, u(x),\, p) \geq 0$.

Hence u is a supersolution. The other half of the proof is similar.

Comparison Theorems

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ are, respectively, viscosity sub- and supersolutions of*

$$u + H(x, \nabla u) = 0 \qquad\qquad x \in \Omega, \tag{HJ}$$

and

$$u_1 \le u_2 \qquad\qquad on \;\; \partial\Omega. \tag{6}$$

Moreover, assume that the function $H : \Omega \times \mathbb{R}^n \mapsto \mathbb{R}$ satisfies

$$\big|H(x,p) - H(y,p)\big| \le \omega\Big(|x-y|\big(1+|p|\big)\Big), \tag{7}$$

where $\omega : [0, \infty[\mapsto [0, \infty[$ is continuous and non-decreasing, with $\omega(0) = 0$. Then

$$u_1 \le u_2 \qquad\qquad \text{on } \;\; \overline{\Omega}. \tag{8}$$

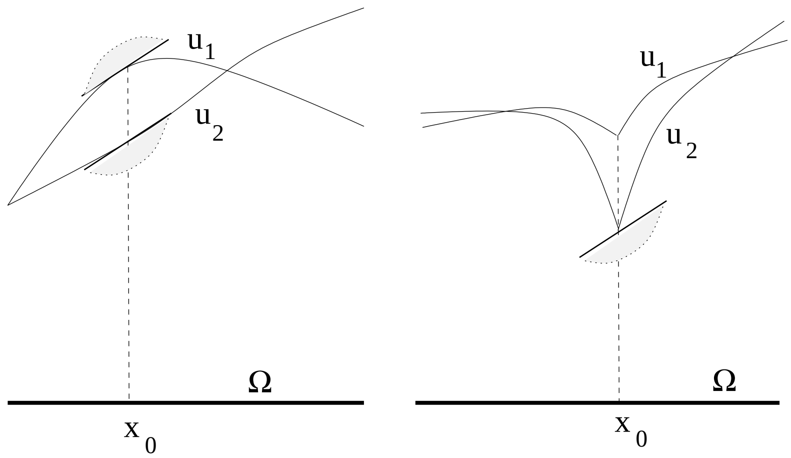
Proof. Easy case: u_1, u_2 smooth.

If (8) fails, then $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$.
Therefore $p \doteq \nabla u_1(x_0) = \nabla u_2(x_0)$.

By definition of sub- and supersolution:

$$\begin{aligned} u_1(x_0) + H(x_0, p) \le 0, \\ u_2(x_0) + H(x_0, p) \ge 0. \end{aligned} \tag{9}$$

Hence $u_1(x_0) - u_2(x_0) \le 0$ reaching a contradiction.



In the non-smooth case, we can reach again a contradiction provided that we can find a point x_0 such that

(i) $u_1(x_0) > u_2(x_0)$,

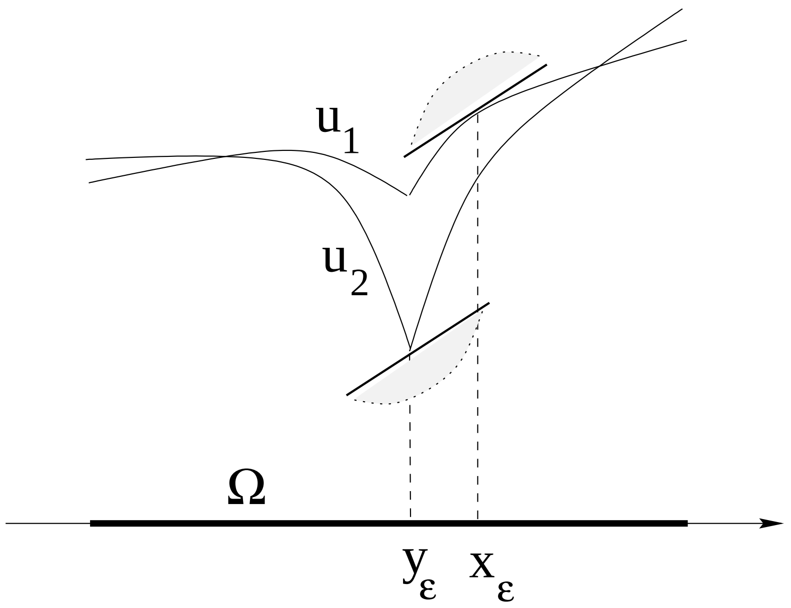
(ii) some vector p lies at the same time in the upper differential $D^+u_1(x_0)$ and in the lower differential $D^-u_2(x_0)$.

A natural candidate for x_0 is a point where $u_1 - u_2$ attains a global maximum. However, one of the sets $D^+u_1(x_0)$ or $D^-u_2(x_0)$ may be empty!

To proceed further, the key observation is that we don't need to compare values of u_1 and u_2 at exactly the same point. Indeed, to reach a contradiction, it suffices to find nearby points x_ε and y_ε such that

(i') $u_1(x_\varepsilon) > u_2(y_\varepsilon)$,

(ii') some vector p lies at the same time in the upper differential $D^+u_1(x_\varepsilon)$ and in the lower differential $D^-u_2(y_\varepsilon)$.



To find suitable points $x_\varepsilon, y_\varepsilon$:

Look at the function of two variables

$$\Phi_\varepsilon(x,y) \doteq u_1(x) - u_2(y) - \frac{|x-y|^2}{2\varepsilon} \tag{10}$$

This clearly admits a global maximum over the compact set $\overline{\Omega} \times \overline{\Omega}$.

If $u_1 > u_2$ at some point x_0, this maximum will be strictly positive.

Taking $\varepsilon > 0$ sufficiently small, the boundary conditions imply that the maximum is attained at some interior point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

The points $x_\varepsilon, y_\varepsilon$ must be close to each other, otherwise the penalization term in (10) will be very large and negative.

The function of one single variable

$$x \mapsto u_1(x) - \left(u_2(y_\varepsilon) + \frac{|x-y_\varepsilon|^2}{2\varepsilon} \right) = u_1(x) - \varphi_1(x) \tag{11}$$

attains its maximum at the point x_ε. Hence by Lemma 1

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = \nabla\varphi_1(x_\varepsilon) \in D^+u_1(x_\varepsilon).$$

The function of one single variable

$$y \mapsto u_2(y) - \left(u_1(x_\varepsilon) - \frac{|x_\varepsilon-y|^2}{2\varepsilon} \right) = u_2(y) - \varphi_2(y) \tag{12}$$

attains its minimum at the point y_ε. Hence

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = \nabla\varphi_2(y_\varepsilon) \in D^-u_2(y_\varepsilon).$$

We have thus discovered two points x_ε, y_ε and a vector $p = (x_\varepsilon - y_\varepsilon)/\varepsilon$ which satisfy the conditions (i')-(ii').

Proof of Theorem 1.

1. If the conclusion fails, then there exists $x_0 \in \Omega$ such that

$$u_1(x_0) - u_2(x_0) = \max_{x\in\overline{\Omega}} \left\{u_1(x) - u_2(x)\right\} \doteq \delta > 0. \tag{11}$$

For $\varepsilon > 0$, call $(x_\varepsilon, y_\varepsilon)$ a point where the function

$$\Phi_\varepsilon(x,y) \doteq u_1(x) - u_2(y) - \frac{|x-y|^2}{2\varepsilon} \tag{10}$$

attains its global maximum on the compact set $\overline{\Omega} \times \overline{\Omega}$. Clearly,

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \delta > 0. \tag{12}$$

2. Call M an upper bound for all values $\big|u_1(x)\big|$, $\big|u_2(x)\big|$, as $x \in \overline{\Omega}$. Then

$$\Phi_\varepsilon(x,y) \leq 2M - \frac{|x-y|^2}{2\varepsilon},$$

$$\Phi_\varepsilon(x,y) \leq 0 \qquad \text{if} \quad |x-y|^2 \geq M\varepsilon.$$

Hence

$$|x_\varepsilon - y_\varepsilon| \leq \sqrt{M\varepsilon}\,. \tag{13}$$

3. Since $u_1 \leq u_2$ on the boundary $\partial\Omega$, for $\varepsilon > 0$ small, the maximum in (10) must be attained at some interior point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

4. Consider the smooth functions of one single variable

$$\varphi_1(x) \doteq u_2(y_\varepsilon) + \frac{|x-y_\varepsilon|^2}{2\varepsilon}, \qquad\qquad \varphi_2(y) \doteq u_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}$$

Since x_ε provides a local maximum for $u_1 - \varphi_1$ and y_ε provides a local minimum for $u_2 - \varphi_2$, we have

$$p \doteq \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \in D^+u_1(x_\varepsilon) \cap D^-u_2(y_\varepsilon). \tag{14}$$

From the definition of viscosity sub- and supersolution we now obtain

$$\begin{aligned} u_1(x_\varepsilon) + H(x_\varepsilon, p) &\leq 0,\\ u_2(y_\varepsilon) + H(y_\varepsilon, p) &\geq 0. \end{aligned} \tag{15}$$

5. Observing that

$$\delta \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leq u_1(x_\varepsilon) - u_2(x_\varepsilon) + \big|u_2(x_\varepsilon) - u_2(y_\varepsilon)\big| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}\,,$$

we see that

$$\big|u_2(x_\varepsilon) - u_2(y_\varepsilon)\big| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \geq 0$$

and hence, by the uniform continuity of u_2,

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \qquad\qquad \text{as } \ \varepsilon \to 0. \tag{16}$$

6. Subtracting the second from the first inequality in (15) we obtain

$$\begin{aligned} \delta &\leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \\ &\leq u_1(x_\varepsilon) - u_2(y_\varepsilon) \\ &\leq \big|H(x_\varepsilon, p) - H(y_\varepsilon, p)\big| \\ &\leq \omega\Big((|x_\varepsilon - y_\varepsilon| \cdot \big(1 + |x_\varepsilon - y_\varepsilon|\varepsilon^{-1}\big)\Big). \end{aligned} \tag{17}$$

This yields a contradiction, Indeed, by (13) and (16) the right hand side of (17) can be made arbitrarily small by letting $\varepsilon \to 0$.

Corollary 1. *Let $\Omega \subset I\!\!R^n$ be a bounded open set. Let the Hamiltonian function H satisfy the equicontinuity assumption (7). Then the boundary value problem*

$$u + H(x, \nabla u) = 0 \qquad\qquad x \in \Omega$$

$$u = \psi \qquad\qquad x \in \partial\Omega$$

admits at most one viscosity solution.

Uniqueness for the Cauchy Problem

$$u_t + H(t, x, \nabla u) = 0 \qquad (t,x) \in]0,T[\, \times I\!R^n, \tag{18}$$

$$u(0,x) = g(x) \qquad x \in I\!R^n. \tag{19}$$

(H1) H is uniformly continuous on $[0,T] \times I\!R^n \times K$, for every compact set $K \subset I\!R^n$.

(H2) There exists a continuous, non-decreasing function $\omega : [0,\infty[\, \times [0,\infty[$ with $\omega(0) = 0$ such that

$$\Big|H(t,x,p) - H(t,y,p)\Big| \le \omega\Big(|x-y|\big(1+|p|\big)\Big).$$

Theorem 2. *Let the function $H : [0,T] \times I\!R^n \times I\!R^n$ satisfy the equicontinuity assumptions (H1)-(H2). Let u_1, u_2 be uniformly continuous sub- and super-solutions of (18) respectively. If $u_1(0,x) \le u_2(0,x)$ for all $x \in I\!R^n$, then*

$$u_1(t,x) \le u_2(t,x) \qquad \text{for all } (t,x) \in [0,T] \times I\!R^n.$$

Corollary 2. *Let the function H satisfy the assumptions (H1)-(H2). Then the Cauchy problem*

$$u_t + H(t, x, \nabla u) = 0 \qquad (t,x) \in]0,T[\, \times I\!R^n$$

$$u(0,x) = g(x) \qquad x \in I\!R^n$$

admits at most one uniformly continuous viscosity solution $u : [0,T] \times I\!R^n \mapsto I\!R$.

Sketch of the proof of Theorem 2.

Assume, on the contrary, that $u_1 > u_2$ at some point (t, x).
Then for some $\lambda > 0$ we still have

$$u_1(t, x) - \lambda t > u_2(t, x)$$

Assume that we can find a global maximum:

$$u_1(t_0,\, x_0) - \lambda t_0 - u_2(t_0,\, x_0) = \sup_{t\in[0,T],\ x\in \mathbb{R}^n} u_1(t, x) - \lambda t - u_2(t, x)$$

Clearly, $t_0 > 0$. If u_1, u_2 are smooth, then

$$\nabla u_1(t_0,\, x_0) = \nabla u_2(t_0,\, x_0), \qquad\qquad \partial_t\, u_1(t_0,\, x_0) - \lambda \geq \partial_t\, u_2(t_0,\, x_0) \tag{20}$$

On the other hand, by assumptions we have

$$\begin{aligned} \partial_t\, u_1(t_0,\, x_0) + H(t_0, x_0, u_1, \nabla u_1) &\leq 0 \\ \partial_t\, u_2(t_0,\, x_0) + H(t_0, x_0, u_1, \nabla u_1) &\geq 0 \end{aligned} \tag{21}$$

Together (20) and (21) yield a contradiction.

Toward the general case - two technical difficulties:

1. The supremum of $u_1 - u_2 - \lambda t$ may only be approached as $|x| \to \infty$.

2. The functions u_1, u_2 may not admit upper or lower differentials at the point where the maximum is attained.

Key idea: look at the function of double variables

$$\Phi_\varepsilon(t, x, s, y) \doteq u_1(t, x) - u_2(s, y) - \lambda(t + s) - \varepsilon\big(|x|^2 + |y|^2\big) - \frac{1}{\varepsilon}\big(|x - y|^2 + |t - s|^2\big)$$

The first penalization term guarantees that a global maximum is attained.

The second penalization term guarantees that, at the point $(t_\varepsilon,\, x_\varepsilon,\, s_\varepsilon,\, y_\varepsilon)$ where the maximum is attained, one has $t_\varepsilon \approx s_\varepsilon$ and $x_\varepsilon \approx y_\varepsilon$.

Extensions

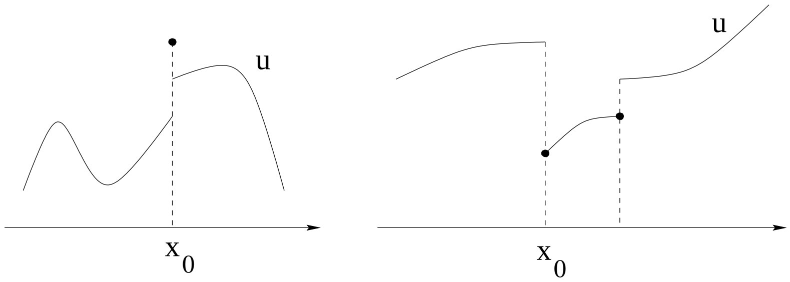
- **Discontinuous solutions**
- **Second order equations**

$u : \Omega \mapsto I\!R$ is **upper semicontinuous** at x_0 if

$$\limsup_{x \to x_0} u(x) \leq u(x_0),$$

$u : \Omega \mapsto I\!R$ is **lower semicontinuous** at x_0 if

$$\liminf_{x \to x_0} u(x) \geq u(x_0),$$



A nonlinear operator $F = F(x,\, u,\, Du,\, D^2u)$ is (possibly degenerate) **elliptic** if

$$F(x, u, p, X) \leq F(x,\, u,\, p,\, X + Y) \qquad \text{whenever } Y \geq 0$$

(i.e. whenever the symmetric matrix Y is non-negative definite)

General nonlinear elliptic equation:

$$F(x,\, u,\, Du,\, D^2u) = 0 \tag{22}$$

An upper semicontinuous function $u : \Omega \mapsto I\!\!R$ is a **viscosity subsolution** of the degenerate elliptic equation (22) if, for every $\varphi \in \mathcal{C}^2(\Omega)$, at each point x_0 where $u - \varphi$ has a local maximum there holds

$$F\big(x_0,\, u(x_0),\, D\varphi(x_0),\, D^2\varphi(x_0)\big) \le 0.$$

A lower semicontinuous function $u : \Omega \mapsto I\!\!R$ is a **viscosity supersolution** of the degenerate elliptic equation (22) if, for every $\varphi \in \mathcal{C}^2(\Omega)$, at each point x_0 where $u - \varphi$ has a local minimum there holds

$$F\big(x_0,\, u(x_0),\, D\varphi(x_0),\, D^2\varphi(x_0)\big) \ge 0.$$

Comparison and uniqueness results:

R. Jensen, *Arch. Rat. Mech. Anal.* (1988)

H. Ishii, *Comm. Pure Appl. Math.* (1989)

Systems of Hamilton-Jacobi Equations

$$w_t + H(x,\, Dw) = 0 \qquad\qquad w : \Omega \mapsto \mathbb{R}^n$$

1. Systems of conservation laws in one space dimension

$$u_t + f(u)_x = 0 \tag{23}$$

$u = (u_1, \ldots, u_n)$ **conserved quantities**

$f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ **fluxes**

Since (23) is in divergence form, we can take w such that

$$w_x = u \qquad\qquad w_t = f(u)$$

This yields the H-J system of equations

$$w_t + f(w_x) = 0$$

2. Non-cooperative m-persons differential games

$$\dot{x} = \sum_{i=1}^{m} f_i(x, u_i). \tag{23}$$

$t \mapsto u_i(t) \in U_i$ is the control chosen by the i-th player, whose aim is to minimize his cost functional

$$J_i \doteq \int_t^T h_i\big(x(s), u_i(s)\big)\, ds + g_i\big(x(T)\big). \tag{24}$$

Let $V_i(t,x)$ be the expected cost for the i-th player (if he behaves optimally), if the system starts at state x at time t.

If such value function $V = (V_1, \ldots, V_m)$ exists, it should provide a solution to the system of Hamilton-Jacobi equations

$$\partial_t V_i + H_i(x, \nabla V_1, \cdots, \nabla V_m) = 0, \tag{25}$$

with terminal data

$$V_i(T,x) = g_i(x). \tag{26}$$

The Hamiltonian functions H_i are defined as follows.
Assume that, for every given $x \in I\!R^n$ and $p_j = \nabla V_j \in I\!R^n$, there exist an optimal choice $u_j^*(x, p_j) \in U_j$ for the j-th player, such that

$$f_j\big(x,\, u_j^*(x,p_j)\big) \cdot p_j + h_j\big(x,\, u_j^*(x,p_j)\big) \;=\; \min_\omega \big\{ f_j(x,\omega) \cdot p_j + h_j(x,\, \omega) \big\}$$

Then the Hamiltonian functions in (25) are

$$H_i(x,\, p_1, \ldots, p_m) \doteq \sum_{j=1}^{m} f_j\big(x,\, u_j^*(x,p_j)\big) \cdot p_i + h_i\big(x,\, u_i^*(x,p_i)\big)$$

(No general existence, uniqueness theory is yet available for these systems)