Sufficient Conditions for Optimality

$$\dot{x} = f(x, u) \qquad t \in [0, T], \qquad\qquad x(0) = x_0 \tag{1}$$

Family of admissible control functions: $\mathcal{U} \doteq \big\{u : [0, T] \mapsto U\big\}$

For $u(\cdot) \in \mathcal{U}$, the trajectory of (1) is $t \mapsto x(t, u)$

Terminal constraint

$$x(T) \in S \tag{2}$$

Mayer problem, free terminal point:

$$\max_{u \in \mathcal{U}} \ \psi\big(x(T, u)\big) \tag{3}$$

Sufficient Condition: Existence + PMP.

Assume that the optimization problem (1)–(3) admits an optimal solution.
Let $\{u_1, u_2, \ldots, u_N\}$ be the set of all control functions that staisfy the Pontryagin Maximum Principle. If

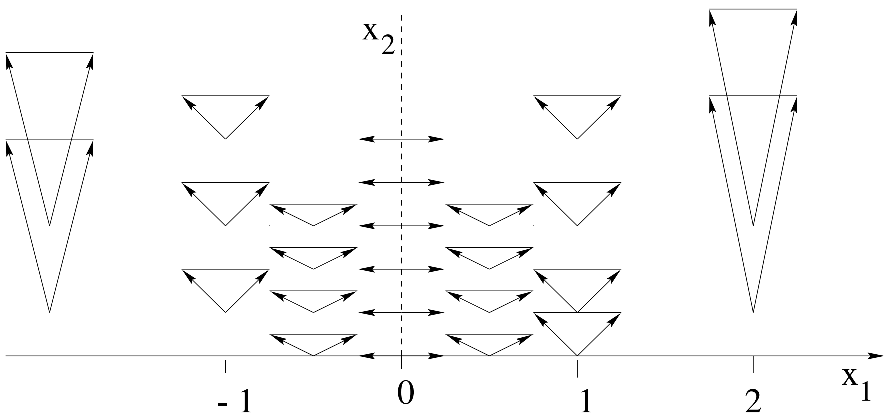
$$\psi\big(x(T, u_k)\big) = \min_{1 \le j \le N} \psi\big(x(T, u_j)\big)$$

Then the control u_k is optimal.

Example 1.

$$(\dot{x}_1, \dot{x}_2) = (u, x_1^2) \qquad u \in [-1,1], \qquad (x_1, x_2)(0) = (0,0)$$

$$\max_{u \in \mathcal{U}} \; x_2(T) \qquad \text{with constraint} \qquad x_1(T) = 0$$



If $t \mapsto x^*(t) = x(t, u^*)$ is an optimal trajectory, the PMP yields

$$(\dot{p}_1, \dot{p}_2) = (-2x_1 p_2,\, 0), \qquad p_2(T) \geq 0$$

$$p_1 u^* + p_2 x_1^2 = \max_{\omega \in [-1,1]} p_1 \omega + p_2 x_1^2 \qquad \Longrightarrow \qquad u^* = \operatorname{sign} p_1$$

$$p_2(t) \equiv \bar{p}_2 \geq 0$$

$$p_2 \equiv 0 \qquad \Longrightarrow \qquad p_1 \equiv \bar{p}_1 \qquad u^*(t) \equiv 0, \qquad x^*(t) \equiv (0,0)$$

$$p_2 \equiv 1 \qquad \Longrightarrow \qquad \dot{p}_1 = -2x_1$$

$$\ddot{p}_1 = -2\,\mathrm{sign}\, p_1, \qquad \dot{p}_1(0) = 0$$

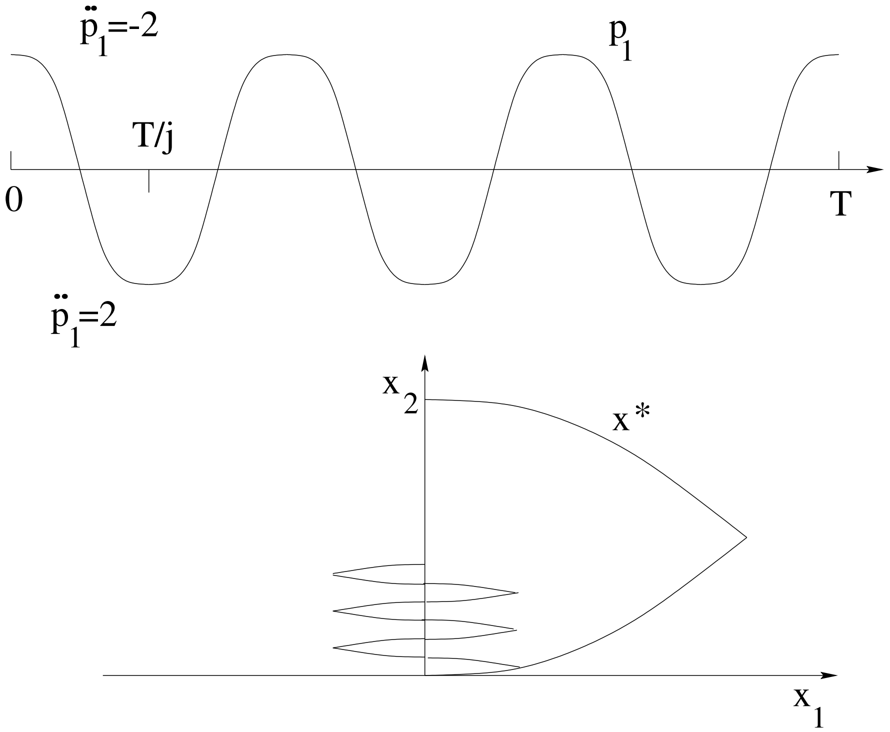
$$\dot{x}_1 = \mathrm{sign}\, p_1 \qquad \dot{x}_2 = x_1^2$$

Countably many controls u_j satisfy the PMP.

Optimal controls:

$$u^*(t) = \begin{cases} 1 & \text{if} \quad 0 < t < T/2 \\ -1 & \text{if} \quad T/2 < t < T \end{cases}$$

$$u^*(t) = \begin{cases} -1 & \text{if} \quad 0 < t < T/2 \\ 1 & \text{if} \quad T/2 < t < T \end{cases}$$



The Value Function

$$\dot{x}(t) = f\big(x(t),\, \alpha(t)\big) \qquad\qquad s < t < T, \tag{4}$$

$$x(s) = y. \tag{5}$$

Here $x \in \mathbb{R}^n$, while the *control function* $\alpha : [s,T] \mapsto A$ is required to take values inside a compact set $A \subset \mathbb{R}^m$.

Optimization Problem: select a control $T \mapsto \alpha^*(t)$ which minimizes the cost

$$J(s,y,\alpha) \doteq \int_s^T h\big(x(t),\, \alpha(t)\big)\, dt + g\big(x(T)\big). \tag{6}$$

Assumption: The functions $f,\ g,\ h$ are Lipschitz continuous and uniformly bounded.

Dynamic Programming Approach: study the **value function**

$$u(s,y) \doteq \inf_{\alpha(\cdot)} J(s,y,\alpha). \tag{7}$$

Bellman's Dynamic Programming Principle

Theorem. For every $s < \tau < T$, $y \in \mathbb{R}^n$, one has

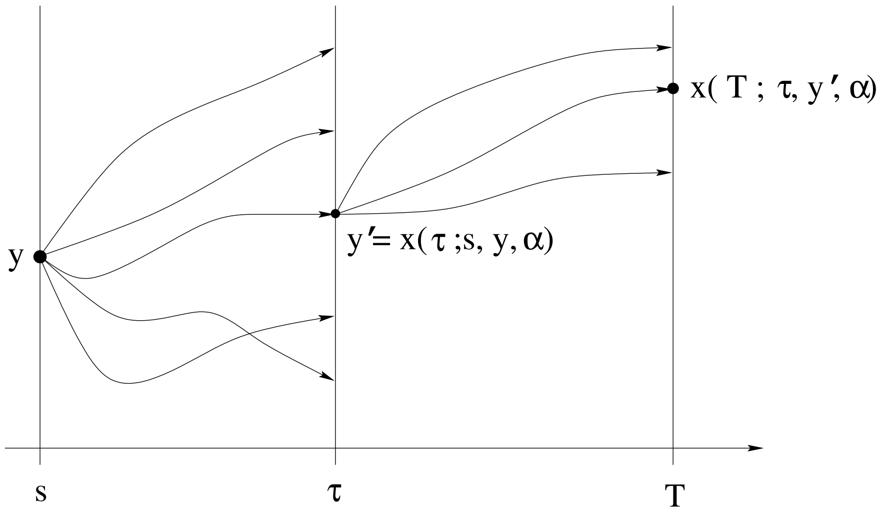
$$u(s,y) = \inf_{\alpha(\cdot)} \left\{ \int_s^\tau h\big(x(t;s,y,\alpha),\ \alpha(t)\big)\,dt \ + \ u\big(\tau,\ x(\tau;s,y,\alpha)\big) \right\}. \tag{8}$$

The optimization problem on $[s,T]$ can be split into two separate problems:

- As a first step, we solve the optimization problem on the sub-interval $[\tau,T]$, with running cost h and terminal cost g. In this way, we determine the value function $u(\tau,\cdot)$, at time τ.

- As a second step, we solve the optimization problem on the sub-interval $[s,\tau]$, with running cost h and terminal cost $u(\tau,\cdot)$, determined by the first step.

At the initial time s, by (8) the value function $u(s,\cdot)$ obtained in step 2 is the same as the value function corresponding to the global optimization problem over the whole interval $[s,T]$.



Mayer problem: $h \equiv 0$ (no running cost). In this case one has

- For every admissible trajectory $t \mapsto x(t)$, the value function $t \mapsto u\big(t,\,x(t)\big)$ is non-decreasing.

- A trajectory $t \mapsto x^*(t)$ is optimal if and only if $t \mapsto u\big(t,x^*(t)\big)$ is constant.

Proof. Call J_τ the right hand side of (8).

1. To prove that $J_\tau \leq u(s,y)$, fix $\varepsilon > 0$ and choose a control $\alpha : [s,T] \mapsto A$ such that

$$J(s,y,\alpha) \leq u(s,y) + \varepsilon.$$

Observing that

$$u\big(\tau, x(\tau;s,y,\alpha)\big) \leq \int_\tau^T h\big(t,\ x(t;s,y,\alpha)\big)\,dt + g\big(x(T;s,y,\alpha)\big),$$

we conclude

$$\begin{aligned} J_\tau &\leq \int_s^\tau h\big(t,\ x(t;s,y,\alpha)\big)\,dt + u\big(\tau, x(\tau;s,y,\alpha)\big) \\ &\leq J(s,y,\alpha) \\ &\leq u(s,y) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this first inequality is proved.

2. To prove that $u(s,y) \leq J_\tau$, fix $\varepsilon > 0$. Then there exists a control $\alpha' : [s,\tau] \mapsto A$ such that

$$\int_s^\tau h\big(t,\ x(t;s,y,\alpha')\big)\,dt + u\big(\tau,\ x(\tau;s,y,\alpha')\big) \leq J_\tau + \varepsilon$$

Moreover, there exists a control $\alpha'' : [\tau,T] \mapsto A$ such that

$$J\big(\tau,\ x(\tau;s,y,\alpha'),\ \alpha''\big) \leq u\big(\tau,\ x(\tau;s,y,\alpha')\big) + \varepsilon.$$

One can now define a new control $\alpha : [s,T] \mapsto A$ as the concatenation of α', α'':

$$\alpha(t) \doteq \begin{cases} \alpha'(t) & \text{if} \quad t \in [s,\tau], \\ \alpha''(t) & \text{if} \quad t \in\,]\tau,T]. \end{cases}$$

This yields

$$u(s,y) \leq J(s,y,\alpha) \leq J_\tau + 2\varepsilon.$$

with $\varepsilon > 0$ arbitrarily small.

The Hamilton-Jacobi-Bellman Equation

$$\dot{x}(t) = f\big(x(t),\, \alpha(t)\big) \qquad\qquad s < t < T, \tag{4}$$

$$x(s) = y, \qquad\qquad \alpha : [s,T] \mapsto A \tag{5}$$

$$J(s,y,\alpha) \doteq \int_s^T h\big(x(t),\, \alpha(t)\big)\, dt + g\big(x(T)\big). \tag{6}$$

$$u(s,y) \doteq \inf_{\alpha(\cdot)} J(s,y,\alpha). \tag{7}$$

Lemma (Regularity of the Value Function). *Let f, g, h, be uniformly bounded and Lipschitz continuous. Then the value function u is uniformly bounded and Lipschitz continuous on $[0,T] \times I\!R^n$.*

Theorem (Characterization of the Value Function). *The value function $u = u(s,y)$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-\Big[u_t + H(x, \nabla u)\Big] = 0 \qquad\qquad (t,x) \in\,]0,T[\, \times I\!R^n, \tag{9}$$

with terminal condition

$$u(T,x) = g(x) \qquad\qquad x \in I\!R^n, \tag{10}$$

and Hamiltonian function

$$H(x,p) \doteq \min_{a \in A} \big\{ f(x,a) \cdot p + h(x,a) \big\}. \tag{11}$$

Proof. To show that u is a **viscosity subsolution** of

$$-\Big[u_t + \min_{a\in A}\big\{f(x,a)\cdot\nabla u + h(x,a)\big\}\Big] = 0 \tag{9}$$

let $\varphi \in \mathcal{C}^1\big(]0,T[\times I\!\!R^n\big)$. We need to show:

(P1) If $u-\varphi$ attains a local maximum at a point $(t_0,x_0)\in]0,T[\times I\!\!R^n$, then

$$\varphi_t(t_0,x_0) + \min_{a\in A}\big\{f(x_0,a)\cdot\nabla\varphi(t_0,x_0) + h(x_0,a)\big\} \geq 0. \tag{12}$$

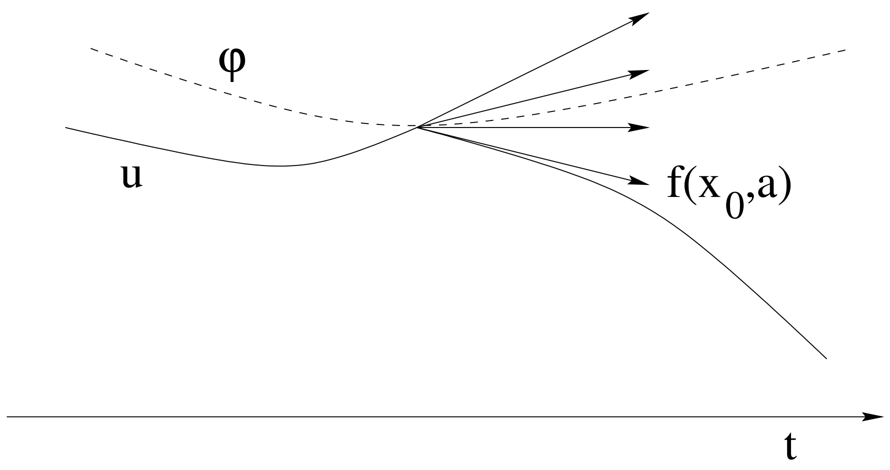
We can assume that

$$u(t_0,x_0) = \varphi(t_0,x_0), \qquad\qquad u(t,x)\leq\varphi(t,x) \quad \text{for all } t,x.$$

If (12) does not hold, then there exists $a\in A$ and $\theta>0$ such that

$$\varphi_t(t_0,x_0) + f(x_0,a)\cdot\nabla\varphi(t_0,x_0) + h(x_0,a) < -\theta. \tag{13}$$

We shall derive a contradiction by showing that this control value a is "too good to be true". Namely, by choosing a control function $\alpha(\cdot)$ with $\alpha(t)\equiv a$ for $t\in[t_0,\ t_0+\delta]$ and such that α is nearly optimal on the remaining interval $[t_0+\delta,\ T]$, we obtain a total cost $J(t_0,x_0,\alpha)$ strictly smaller than $u(t_0,x_0)$.



By continuity,

$$\varphi_t(t_0, x_0) + f(x_0, a) \cdot \nabla\varphi(t_0, x_0) + h(x_0, a) < -\theta \tag{13}$$

implies

$$\varphi_t(t, x) + f(x, a) \cdot \nabla\varphi(t, x) < -h(x, a) - \theta \tag{14}$$

whenever $t \approx t_0$ and $x \approx x_0$. Let $x(t) \doteq x(t; t_0, x_0, a)$ be the solution of

$$\dot{x}(t) = f\big(x(t), a\big) \qquad\qquad x(t_0) = x_0.$$

Then

$$\begin{aligned}
u\big(t_0+\delta,\ x(t_0+\delta)\big) - u(t_0, x_0) &\leq \varphi\big(t_0+\delta,\ x(t_0+\delta)\big) - \varphi(t_0, x_0) \\
&= \int_{t_0}^{t_0+\delta} \frac{d}{dt}\varphi\big(t,\, x(t)\big)\, dt \\
&= \int_{t_0}^{t_0+\delta} \Big\{\varphi_t\big(t,\, x(t)\big) + f\big(x(t), a\big) \cdot \nabla\varphi\big(t,\, x(t)\big)\Big\}\, dt \\
&\leq -\int_{t_0}^{t_0+\delta} h\big(x(t),\, a\big)\, dt - \delta\theta
\end{aligned} \tag{15}$$

On the other hand, the Dynamic Programming Principle

$$u(t_0, x_0) = \inf_{\alpha(\cdot)} \left\{ \int_{t_0}^{t_0+\delta} h\big(x(t; t_0, x_0, \alpha),\ \alpha(t)\big)\, dt \ + \ u\big(t_0+\delta,\ x(t_0+\delta;\ t_0,\ x_0, \alpha)\big) \right\}$$

yields

$$u\big(t_0+\delta,\ x(t_0+\delta)\big) - u(t_0, x_0) \geq -\int_{t_0}^{t_0+\delta} h\big(x(t),\, a\big)\, dt. \tag{16}$$

Together, (15) and (16) yield a contradiction, hence (P1) must hold.

To show that u is a **viscosity supersolution** of

$$-\Big[u_t + \min_{a\in A}\big\{f(x,a)\cdot\nabla u + h(x,a)\big\}\Big] = 0 \tag{9}$$

let $\varphi \in \mathcal{C}^1\big(]0,T[\times I\!R^n\big)$. We need to show:

(P2) If $u-\varphi$ attains a local minimum at a point $(t_0,x_0)\in]0,T[\times I\!R^n$, then

$$\varphi_t(t_0,x_0) + \min_{a\in A}\big\{f(x_0,a)\cdot\nabla\varphi(t_0,x_0) + h(x_0,a)\big\} \le 0. \tag{17}$$

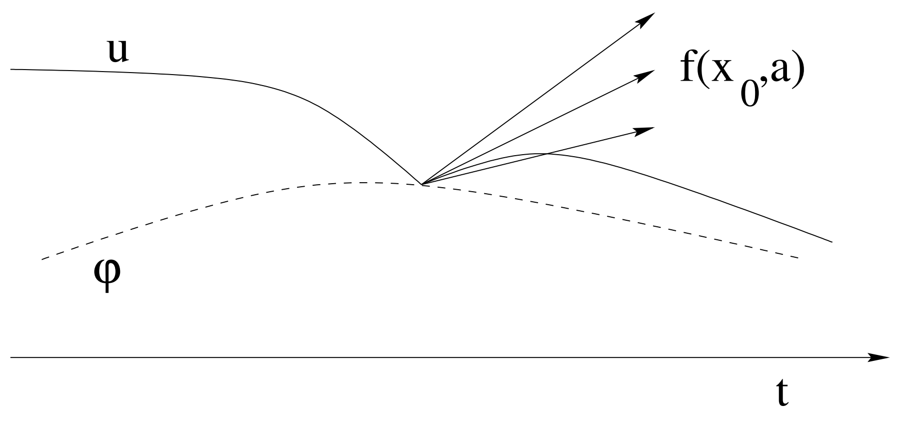
We can assume that

$$u(t_0,x_0) = \varphi(t_0,x_0), \qquad\qquad u(t,x)\ge\varphi(t,x) \quad \text{for all } t,x. \tag{18}$$

If (P2) fails, then there exists $\theta>0$ such that

$$\varphi_t(t_0,x_0) + f(x_0,a)\cdot\nabla\varphi(t_0,x_0) + h(x_0,a) > \theta \qquad \text{for all } a\in A. \tag{19}$$

In this case, we shall reach a contradiction by showing that no control function $\alpha(\cdot)$ is good enough. Namely, whatever control function $\alpha(\cdot)$ we choose on the initial interval $[t_0,\, t_0+\delta]$, even if during the remaining time $[t_0+\delta,\, T]$ our control is optimal, the total cost will still be considerably larger than $u(t_0,x_0)$.



By continuity,

$$\varphi_t(t_0, x_0) + f(x_0, a) \cdot \nabla\varphi(t_0, x_0) + h(x_0, a) > \theta \qquad \text{for all } a \in A \tag{17}$$

implies

$$\varphi_t(t, x) + f(x, a) \cdot \nabla\varphi(t, x) > \theta - h(x, a) \qquad \text{for all } a \in A \tag{20}$$

for all $t \approx t_0$ and $x \approx x_0$. Choose an arbitrary control function $\alpha : [t_0,\, t_0 + \delta] \mapsto A$, and call $t \mapsto x(t) = x(t; t_0, x_0, \alpha)$ the corresponding trajectory. We now have

$$\begin{aligned}
u\big(t_0 + \delta,\ x(t_0 + \delta)\big) - u(t_0, x_0) &\geq \varphi\big(t_0 + \delta,\ x(t_0 + \delta)\big) - \varphi(t_0, x_0) \\
&= \int_{t_0}^{t_0+\delta} \frac{d}{dt}\varphi\big(t,\, x(t)\big)\, dt \\
&= \int_{t_0}^{t_0+\delta} \varphi_t\big(t,\, x(t)\big) + f\big(x(t), \alpha(t)\big) \cdot \nabla\varphi\big(t,\, x(t)\big)\, dt \\
&\geq \int_{t_0}^{t_0+\delta} \theta - h\big(x(t), \alpha(t)\big)\, dt
\end{aligned} \tag{21}$$

Therefore, for every control function $\alpha(\cdot)$,

$$u\big(t_0 + \delta,\ x(t_0 + \delta)\big) + \int_{t_0}^{t_0+\delta} h\big(x(t), \alpha(t)\big)\, dt \geq u(t_0, x_0) + \delta\theta. \tag{22}$$

Taking the infimum of the left hand side of (22) over all control functions α, we see that this infimum is still $\geq u(t_0, x_0) + \delta\theta$.

On the other hand, the Dynamic Programming Principle states that

$$u(t_0, x_0) = \inf_{\alpha(\cdot)} \left\{ \int_{t_0}^{t_0+\delta} h\big(x(t; t_0, x_0, \alpha),\ \alpha(t)\big)\, dt \ + \ u\big(t_0 + \delta,\ x(t_0 + \delta;\ t_0,\ x_0, \alpha)\big) \right\}$$

in contradiction with (22). This establishes (P2).

Therefore, u is a **viscosity solution**.

Sufficient Conditions for Optimality

1. For each initial condition (s, y), construct a "candidate" optimal control $\alpha^{s,y} : [s, T] \mapsto A$, satisfying the PMP.

2. Consider the corresponding cost

$$u(s, y) \doteq J(s, y, \alpha^{s,y}) \tag{23}$$

3. Check whether v satisfies the Hamilton-Jacobi-Bellman equation

$$-\Big[u_t + \min_{a\in A} \big\{f(x, a) \cdot \nabla u + h(x, a)\big\}\Big] = 0 \qquad\qquad (t, x) \in]0, T[\, \times I\!R^n, \tag{9}$$

in the viscosity sense, with terminal condition

$$u(T, x) = g(x) \qquad\qquad x \in I\!R^n \tag{10}$$

In the positive case, by uniqueness, u coincides with the minimum value function. Hence all controls $\alpha^{s,y}$ are optimal.

The optimal control is found in feedback form:

$$\alpha^*(t, x) = \operatorname*{argmin}_{a\in A} \big\{f(x, a) \cdot \nabla u + h(x, a)\big\}$$

at every point (t, x) where u is differentiable.

Regular Synthesis

For the optimization problem (4)–(6), let $u = u(t,x)$ be the minimum value function, defined at (7).

Theorem. *Let $v : [0,T] \times I\!\!R^n \mapsto I\!\!R$ be a continuous function such that*

$$v \geq u \tag{23}$$

$$v(T,x) = g(x) \qquad\qquad x \in I\!\!R^n\,. \tag{24}$$

Assume that there exist $\mathcal{C}^1$ manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_N \subset [0,T] \times I\!\!R^n$ of dimension $\leq n$, such that v is continuously differentiable and satisfies

$$v_t + \min_{a\in A} \big\{ f(x,a) \cdot \nabla v + h(x,a) \big\} = 0 \tag{9}$$

on the open set $]0,T[\times I\!\!R^n \setminus \bigcup \mathcal{M}_j$.

Then $v = u$ on $[0,T] \times I\!\!R^n$.

Remark. Here we only require (9) to be satisfied on an open set where v is $\mathcal{C}^1$. Nothing has to be checked at points $(t,x) \in \mathcal{M}_j$.

On the other hand, we assume that v is sufficiently regular, i.e. $v \in \mathcal{C}^1$ outside finitely many manifolds.

In this case, we say that the optimization problem admits a **regular synthesis**.

Proof. We only need to show that $u \geq v$.

1. Assume $u(t_0,x_0) < v(t_0,x_0)$. By continuity,

$$u(t,x) + 3\varepsilon \leq v(t,x) \tag{24}$$

for some $\varepsilon > 0$ and all (t,x) in a neighborhood of (t_0,x_0).

2. Since u is the infimum of the cost, there exists a control $\alpha^* : [t_0, T] \mapsto A$ such that the trajectory $t \mapsto x^*(t)$ satisfies

$$x^*(t_0) = x_0, \qquad \dot{x}^*(t) = f\big(x^*(t), \alpha^*(t)\big) \qquad \text{for a.e. } t \in [t_0, T]$$

$$\int_{t_0}^{T} h\big(x^*(t), \alpha^*(t)\big)\, dt + g\big(x^*(T)\big) \le u(t_0, x_0) + \varepsilon \tag{25}$$

3. The set of piecewise constant controls is $\mathbf{L}^1$ dense on the set of all measurable controls $\alpha : [t_0, T] \mapsto A$. Hence we can find a piecewise constant control $\alpha^\sharp$ such that, calling $t \mapsto x^\sharp(t)$ the solution to

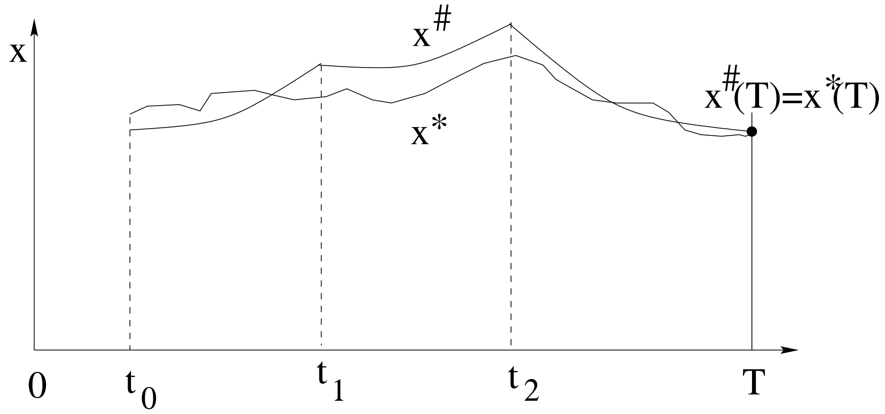
$$\dot{x}^\sharp(t) = f\big(x^\sharp(t),\, \alpha^\sharp(t)\big), \qquad x^\sharp(T) = x^*(T),$$

also $\alpha^\sharp$ is nearly optimal. Its cost is

$$J(\alpha^\sharp) = \int_{t_0}^{T} h\big(x^\sharp(t), \alpha^\sharp(t)\big)\, dt + g\big(x^\sharp(T)\big) \le u\big(t_0, x^\sharp(t_0)\big) + 2\varepsilon \tag{26}$$

We shall obtain a contradiction by showing that the total cost for $\alpha^\sharp$ is

$$J(\alpha^\sharp) \ge v\big(t_0, x^\sharp(t_0)\big) \ge u\big(t_0, x^\sharp(t_0)\big) + 3\varepsilon$$



4. Let $t_0 < t_1 < \cdots < t_m = T$ be the times where $u^\sharp$ has a jump.

If we show that, for every $j = 1, \ldots, m$

$$\int_{t_{j-1}}^{t_j} h\big(x^\sharp(t),\, u^\sharp(t)\big)\, dt \geq v\big(t_{j-1},\, x^\sharp(t_{j-1})\big) - v\big(t_j,\, x^\sharp(t_j)\big) \tag{27}$$

we reach a contradiction. Indeed, summing over j we obtain

$$\int_{t_0}^{T} h\big(x^\sharp(t),\, u^\sharp(t)\big)\, dt \geq v\big(t_0,\, x^\sharp(t_0)\big) - v\big(T, x^\sharp(T)\big).$$

Then the total cost using the control $\alpha^\sharp$ is

$$\int_{t_0}^{T} h\big(x^\sharp(t),\, u^\sharp(t)\big)\, dt + g\big(x^*(T)\big) \geq v\big(t_0, x^\sharp(t_0)\big). \tag{28}$$

5. It remains to prove (27). More generally we show that, for every constant control $\alpha(t) \equiv \tilde{a}$ and every trajectory

$$\dot{x}(t) = f\big(x(t),\, \tilde{a}\big) \qquad\qquad t \in [\tau, \tau']$$

we have

$$\int_{\tau}^{\tau'} h\big(x(t),\, \tilde{a}\big)\, dt \geq v\big(\tau,\, x(\tau)\big) - v\big(\tau',\, x(\tau')\big) \tag{29}$$

By a transversality argument, we can find a sequence of trajectories

$$\dot{x}_m = f\big(x_m(t),\, \tilde{a}\big)$$

uniformly converging to x, such that each x_m crosses each manifold $\mathcal{M}_j$ only at finitely many times. Since $x_m(t) \notin \cup \mathcal{M}_j$ outside these crossing times, the Hamilton-Jacobi-Bellman equation

$$v_t + \min_{a \in A} \big\{ f(x,a) \cdot \nabla v + h(x,a) \big\} = 0 \tag{9}$$

yields

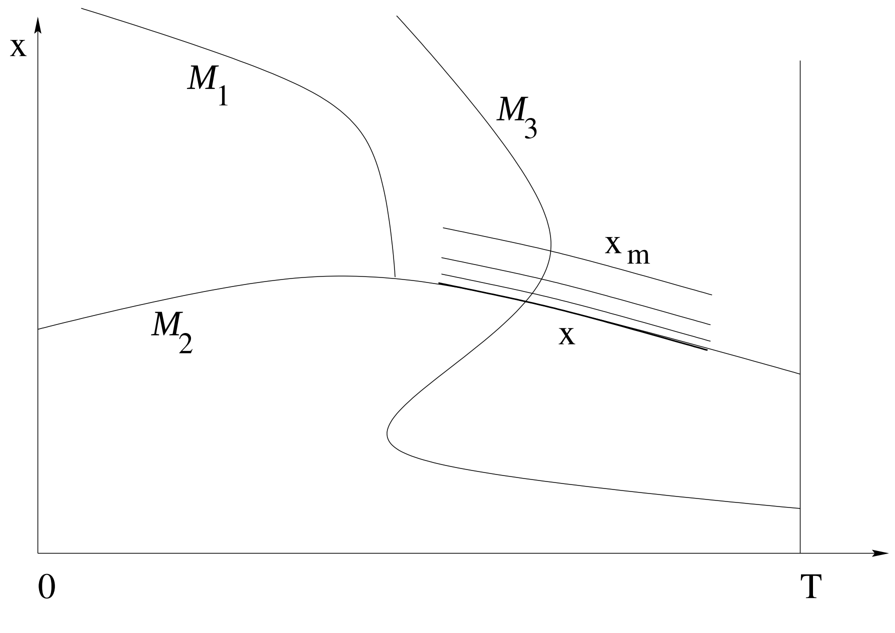
$$\frac{d}{dt} v\big(t,\, x_m(t)\big) + h(x_m, \tilde{a}) = v_t + \nabla v \cdot f(x_m,\, \tilde{a}) + h(x_m, \tilde{a}) \geq 0$$

Therefore, integrating over $[\tau, \tau']$

$$\int_\tau^{\tau'} h\big(x_m(t),\, \tilde{a}\big)\, dt + v\big(\tau',\, x_m(\tau')\big) - v\big(\tau,\, x_m(\tau)\big) \geq 0$$

Letting $m \to \infty$ we conclude

$$\int_\tau^{\tau'} h\big(x(t),\, \tilde{a}\big)\, dt \geq v\big(\tau,\, x(\tau)\big) - v\big(\tau',\, x(\tau')\big). \tag{29}$$



Nonlinear First Order P.D.E.

The Method of Characteristics

$$v_t + H(x, \nabla v) = 0 \tag{30}$$

$$v(\tau, x) = \bar{v}(x), \tag{31}$$

Assume H, v smooth. Call $p \doteq \nabla v$, so that $p = (p_1, \ldots, p_n) = (v_{x_1}, \ \ldots, \ v_{x_n})$

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}\,.$$

Differentiating (30) w.r.t. x_i one obtains

$$\frac{\partial p_i}{\partial t} = \frac{\partial^2 v}{\partial x_i \partial t} = -\frac{\partial H}{\partial x_i} - \sum_j \frac{\partial H}{\partial p_j}\,\frac{\partial p_i}{\partial x_j}$$

The total derivative of p_i along a curve $t \mapsto x(t)$ is

$$\begin{aligned} \frac{d}{dt} p_i\big(t, x(t)\big) &= \frac{\partial p_i}{\partial t} + \sum_j \dot{x}_j\,\frac{\partial p_i}{\partial x_j} \\ &= -\frac{\partial H}{\partial x_i} + \sum_j \left(\dot{x}_j - \frac{\partial H}{\partial p_j}\right)\frac{\partial p_i}{\partial x_j} \end{aligned} \tag{32}$$

By choosing $\dot{x} = \partial H/\partial p$, the last term in (32) disappears.

Method of characteristics:

For each $\bar{x}$, solve the Hamiltonian system of O.D.E's

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial p_i}(x,p)\,, \\ \dot{p}_i = -\dfrac{\partial H}{\partial x_i}(x,p)\,, \end{cases} \qquad\qquad \begin{cases} x_i(\tau) = \bar{x}_i\,, \\ p_i(\tau) = \dfrac{\partial \bar{v}}{\partial x_i}(\bar{x})\,. \end{cases} \tag{33}$$

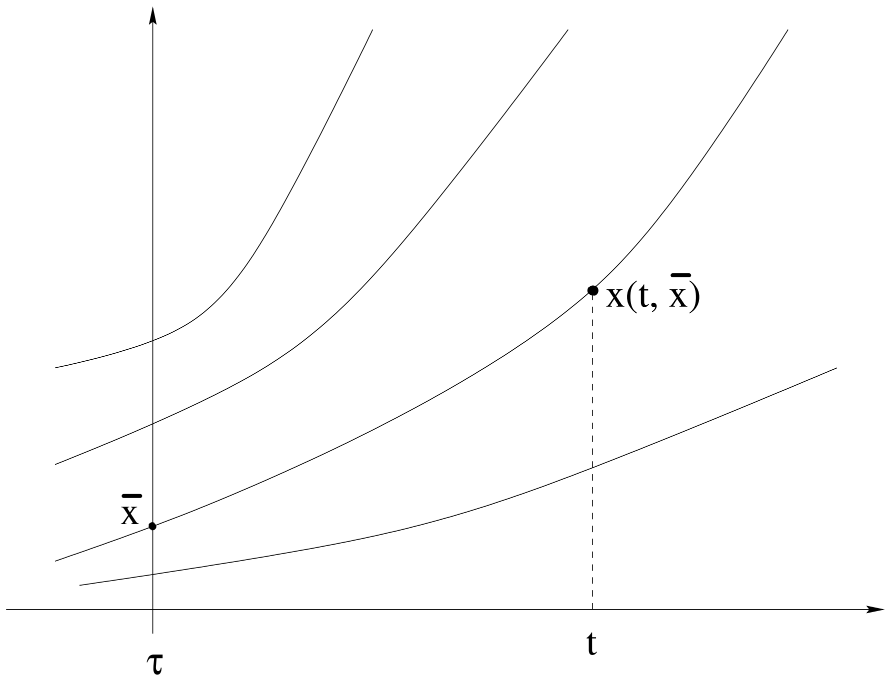
Write the solution as $\quad t \mapsto x(t,\bar{x}), \qquad t \mapsto p(t,\bar{x})$.

Observing that $\nabla v\big(t,\, x(t,\bar{x})\big) = p(t,\bar{x})$ and

$$\frac{d}{dt} v\big(t,\, x(t,\bar{x})\big) = v_t + \nabla v \cdot \dot{x} = -H(x,p) + p \cdot \frac{\partial H}{\partial p}$$

we obtain

$$v\big(t,\, x(t,\bar{x})\big) = v(\tau,\bar{x}) + \int_\tau^t \left(-H(x,p) + p \cdot \frac{\partial H}{\partial p} \right) ds$$



Pontryagin's Maximum Principle and the P.D.E. of Dynamic Programming

$$\dot{x} = f(x, \alpha) \qquad\qquad \alpha(t) \in A \tag{34}$$

Minimize a cost functional involving a running cost h and a terminal cost g:

$$J^{s,y}(\alpha) \;\doteq\; \int_s^T h\big(x(t),\, \alpha(t)\big)\, dt \;+\; g\big(x(T)\big)\,. \tag{35}$$

Here $t \mapsto x(t)$ is the trajectory of (34) corresponding to the control $\alpha : [s, T] \mapsto A$ and with initial data $x(s) = y$.

$$\textbf{Value function:} \qquad\qquad v(t, x) \doteq \inf_{\alpha(\cdot)} J^{t,x}(\alpha) \tag{36}$$

is a **viscosity solution** of the Hamilton-Jacobi-Bellman equation

$$-\Big[v_t + H(x, \nabla v)\Big] = 0 \tag{37}$$

with

$$H(x, p) \doteq \min_{a \in A} \big\{p \cdot f(x, a) + h(x, a)\big\}. \tag{38}$$

On a region where v is smooth, the H-J equation (37) can be solved by the method of characteristics. Assume

$$H(x, p) = p \cdot f\big(x,\, a^*(x, p)\big) + h\big(x,\, a^*(x, p)\big) = \min_{a \in A} \big\{p \cdot f(x, a) + h(x, a)\big\}\,. \tag{38}$$

At the point α^* where the minimum is attained, one has

$$p \cdot \frac{\partial f}{\partial a}(x, \alpha^*) + \frac{\partial h}{\partial a}(x, \alpha^*) = 0\,.$$

Hence the Hamiltonian system (33) takes the form

$$\begin{cases} \dot{x} = f\big(x,\, \alpha^*(x, p)\big) \\ \dot{p} = -p \cdot \dfrac{\partial f}{\partial x}\big(x,\, \alpha^*(x, p)\big) - \dfrac{\partial h}{\partial x}\big(x,\, \alpha^*(x, p)\big) \end{cases} \tag{39}$$

The existence of an adjoint vector $t \mapsto p(t)$ satisfying the linear evolution equation in (39) and the minimality condition (38) is a well known necessary condition for optimality, stated in the **Pontryagin Maximum Principle**.

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