3.16 Let w be s.t. $\Delta w = 0$ in Ω_e, $w = 0$ on $\partial\Omega$ and $|w| \leq M$. We may assume that $0 \in \Omega$. Let $0 < a < R$ s.t.

$$B_a(0) \subset \Omega \Subset B_R(0),$$

and let $u_R(x) := M \dfrac{\ln|x| - \ln a}{\ln R - \ln a}$. Define the bounded set $\Omega_{e,R} := \Omega_e \cap B_R(0)$. Then $\Omega_{e,R}$ is open, bounded and connected. Let $v_R(x) = u_R(x) - w(x)$. Then v_R is harmonic since both u_R and w are, and if $x \in \partial\Omega$ then

$$v_R(x) = \underbrace{M \frac{\ln|x| - \ln a}{\ln R - \ln a}}_{\geq 0} - \underbrace{w(x)}_{=0} \geq 0,$$

while if $x \in \partial B_R$ then

$$v_R(x) = \underbrace{M \frac{\ln|x| - \ln a}{\ln R - \ln a}}_{=1} - w(x) = M - w(x) \geq 0$$

as $|w| \leq M$. Hence, by the maximum principle,

$$0 \leq \min_{\partial\Omega_{e,R}} v_R \leq v_R(x) \qquad \forall\ x \in \Omega_{e,R}$$

$$\Rightarrow \quad v_R \geq 0 \text{ in } \Omega_{e,R} \quad \Rightarrow \quad u_R \geq w \text{ in } \Omega_{e,R}.$$

Since this holds for all $R > \operatorname{diam}(\Omega)$, and since $u_R \to 0$ as $R \to \infty$, it follows that $0 \geq w$. Conversely, the same argument with $\tilde{v}_R(x) := w(x) - u_R(x)$ shows that $w \geq 0$, whence $w = 0$.

$\blacksquare$

3.20 Let $u(x) = \frac{1-|x|^2}{2\pi} \int_{\partial B} \frac{g(\sigma)}{|x-\sigma|^2}\, d\sigma$, and let $\xi \in \partial B$. $(B = B_1(0))$.

Show: $\lim_{x \to \xi} u(x) = g(\xi)$.

Let $v(x) = \frac{1-|x|^2}{2\pi} \int_{\partial B} \frac{1}{|x-\sigma|^2}\, d\sigma$. Then v is harmonic in B and $v(0) = \frac{1}{2\pi} \int_{\partial B} \frac{1}{|\sigma|^2}\, d\sigma = \frac{1}{2\pi} \int_{\partial B} 1\, d\sigma = 1$. Moreover, v is rotationally invariant: $v(x) = w(|x|)$ for some $w : [0,1) \to \mathbb{R}$. Either $w(r) = 1$ for all $r \in [0,1)$, or there is some $r_0 > 0$ such that $w(r_0) > 1$ (or $w(r_0) < 1$) and $w(r) \geq 1$ (or $w(r) \leq 1$) for all $r \in [0, r_0)$. But then v is harmonic in $B_{r_0}(0)$, nonconstant and attains a minimum (maximum) in the interior of $B_{r_0}(0)$, which contradicts the maximum principle.

Therefore, $v(x) = 1$ for all $x \in B$.

Let $\delta > 0$, and let $E_\delta := \{\sigma \in \partial B : |\sigma - \xi| < \delta\}$ and $F_\delta = \partial B \setminus E_\delta$. We write

$$u(x) - g(\xi) = \frac{1-|x|^2}{2\pi} \int_{\partial B} \frac{g(\sigma) - g(\xi)}{|x-\sigma|^2}\, d\sigma$$

$$= \underbrace{\frac{1-|x|^2}{2\pi} \int_{E_\delta} \frac{g(\sigma)-g(\xi)}{|x-\sigma|^2}\, d\sigma}_{=: I} + \underbrace{\frac{1-|x|^2}{2\pi} \int_{F_\delta} \frac{g(\sigma)-g(\xi)}{|x-\sigma|^2}\, d\sigma}_{=: II}.$$

Let $\varepsilon > 0$, and let $\delta > 0$ be such that $|g(\sigma) - g(\xi)| < \varepsilon$ whenever $|\sigma - \xi| < \delta$. Then

$$|I| \leq \frac{1-|x|^2}{2\pi} \int_{E_\delta} \frac{\varepsilon}{|x-\sigma|^2}\, d\sigma \leq \varepsilon$$

for <u>all x</u>. Next, if $|x - \xi| < \delta/2$ then $|x - \sigma| > \delta/2$

for all $\sigma \in F_\delta$, so

$$|II| \leq \frac{1-|x|^2}{2\pi} \int_{F_\delta} \frac{|g(\sigma)-g(\xi)|}{\delta^2/4} d\sigma \leq C_\delta (1-|x|^2),$$

where $C_\delta = \frac{2}{\pi\delta^2} \int_{\partial B} |g(\sigma)-g(\xi)| d\sigma$. If x is so close to ξ that $1-|x|^2 < \frac{\varepsilon}{C_\delta}$ then also $|II| \leq \varepsilon$. Hence,

$$\lim_{x \to \xi} |u(x)-g(\xi)| = 0.$$

$\blacksquare$