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★**Soliton equations and their algebro-geometric solutions. Vol. II.**
$(1+1)$-dimensional discrete models.
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This is the second volume of a planned set of three. Its topic is differential-difference equations, completely integrable via spectral data of an associated Lax pair. The choice of specific integrability (algebro-geometric, in the sense that the spectrum is a compact Riemann surface, or a complex curve with some singular points in limiting cases) grounds to a satisfactory extent the elusive concept of "complete integrability". As in the first volume [F. Gesztesy and H. Holden, *Soliton equations and their algebro-geometric solutions. Vol. I*, Cambridge Univ. Press, Cambridge, 2003; MR1992536 (2005b:37173)], the equations in the Lax pair are second-order (the Riemann surface is hyperelliptic), as the $(1+1)$-case in the subtitle says; Volume III will take up Riemann surfaces that cover the sphere to higher degree. The two original authors enlisted two collaborators in the project, and a set of errata to Volume I is provided at the end. Despite these small differences, there is a constant: like Volume I, this work stands out for focus and thoroughness.

The focus of the work is classical complex analysis and spectral theory. In their research, the authors unraveled many of the mysteries of algebro-geometric integrability. Once the spectral problem is found, or posed, and the isospectral deformations are written algebraically, the criterion of algebraic integrability is encoded in the common eigenfunction, which has special analytic properties. This is one reason why the authors' approach is so valuable (self-help Appendices on Riemann surfaces, especially hyperelliptic; on spectral theory; and some algebraic combinatorics (interpolation) are provided for completeness—with some inevitable overlap from Volume I).

The format is the same as in Volume I, with a thorough Introduction that provides a road map through the development of each chapter (spectral version of the stationary problem, solution in terms of theta functions, time deformation and solution, conservation laws and Hamiltonian formalism); the three chapters take up progressively more general cases of the differential-difference problem (the Toda lattice, the Kac–van Moerbeke hierarchy, the Ablowitz-Ladik hierarchy). Another unique feature is the thorough set of notes at the end of each chapter which provides history and highlights on special features (special classes of solutions, additional conditions, open questions).

The bibliography is again thorough and up-to-date; some papers, cited for their connection with the Toda system and its generalizations, of course are not contextualized—I'll give an example I like. The beautiful (cited) work by P. Iliev [Selecta Math. (N.S.) **13** (2007), no. 3, 497–530; MR2383604 (2009a:37133)] uses the previous [J. Phys. A **34** (2001), no. 11, 2445–2457; MR1831308 (2002g:39008)] by L. Haine and Iliev and [Math. Phys. Anal. Geom. **5** (2002), no. 2, 183–200; MR1918052 (2003i:37080)] by F. A. Grünbaum and Iliev, where the differential-difference operators in question are defined: Iliev's stunning result is that their ring is bispectral if and only if the expansion of their fundamental solution in terms of (generalized) Hadamard coefficients satisfies a finiteness condition. In the process he provides closed-form expressions, and deformations, in terms of the τ-function. However, this inevitable bibliographical incompleteness, accompanied, as was the case with Volume I, by the exclusion of many of the other beautiful mysteries of soliton theory (links with representation theory, algebraic topology, applied mathematics), is acknowledged in the Introduction, if not intentional. Focus is what makes the book readable, usable, inspiring for both experienced researchers and beginners, and beautiful. But let me try to "show, not tell".

The Toda system, historically the first completely integrable nonlinear-interaction (1-dimensional) lattice, is the following set of equations on the displacements $x(n,t)$ of the n-th particle from its equilibrium position at time t:

$$x_{tt}(n,t) = e^{(x(n-1,t)-x(n,t))} - e^{(x(n,t)-x(n+1,t))}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}.$$

The Flaschka transformation:

$$a(n,t) = \frac{1}{2} \exp\left(\frac{1}{2}((x(n,t) - x(n+1,t))\right),$$
$$b(n,t) = -\frac{1}{2} x_t(n,t)$$

allows a Lax representation of the system by a tridiagonal (infinite) matrix $L(t)$ with $b(n,t)$ along the diagonal and $a(n,t)$ on the two adjacent diagonals; the typical row reads: $[\dots\ 0\ a(n,t)\ b(n+1,t)\ a(n+1,t)\ 0\ \dots]$. The Lax pair, for t and more generally the r-th time flow t_r of the hierarchy, reads:

$$\frac{\partial}{\partial t_r} L - [P_{2r+2}, L] = 0,$$

where P_{2r+2} is a shift-differential operator for whose coefficients recursion equations are given, so that they can be calculated. The algebro-geometric ansatz:

$$[P_{2p+2}, L] = 0$$

provides a hyperelliptic Riemann surface $\mathcal{K}$ of genus p (essentially, the joint spectrum of P_{2p+2} and L) and explicit integration of the Toda system by the mixed algebraic theory (divisors of degree p on a plane curve $\mathcal{K}'$, birational to $\mathcal{K}$, essentially given by imposing Dirichlet boundary conditions on the eigenfunctions) and analytic theory (the divisors correspond to points on the Jacobi variety of $\mathcal{K}$ and move "linearly" on it, which in time means along lines on the universal cover $\mathbb{C}^p$, and in the discrete variable means by iterating the addition of a fixed point, essentially $P_{\infty+} - P_{\infty-}$, where the two points at infinity are the hyperelliptic fibre over the point at infinity of the Riemann sphere). The Toda displacements are expressed both in terms of the algebraic coordinates of the points of the divisor, meromorphic functions on $\mathcal{K}$, and of Riemann's theta function. The motion of the points is found by integrating the Dubrovin equations in the continuous case, whereas for the discrete parameter the authors set up a recursion themselves (in the Notes section). This is a good time to say that the authors originally contributed much explicitness and detail to the tremendous amount of existing literature on the Toda lattice. Lastly, in treating the conserved quantities ("trace formulas" of Volume I), an exquisitely analytic (spectral) property, the scope is considerably extended. As in Volume I, the authors consider the most general, not necessarily algebro-geometric, situation, and present much original work on the Green's function of the Lax-pair operators; they give a summary of functional Poisson geometry and recursion formulas, including some discrete variational calculus, and also work out the bi-Hamiltonian model (in each of Chapters 1 and 3, the model of Chapter 2 being subsumed). As they modestly point out in the Notes section, despite the enormous amount of work on the Hamiltonian model of the Toda lattice, their spectral-invariant approach and recursion formulas are novel. To demonstrate the potential growth of the field, I will just issue a call for an R-matrix theory (developed in the context of representation theory, which is outside the scope of this book) to include this work.

In conclusion, the authors do a superb job at clarity and detail; very subtle features of existence, almost periodicity and reality conditions are also developed (there are no complete answers at present).

Paraphrasing journalist John Gunther, "tell your audience what you are going to say, say it, then tell them what you said". I called the theory beautiful because it combines algebra and analysis, to provide satisfying, exact solutions to important nonlinear problems that arise in the real world. It does so by giving a geometric explanation of the persistence of symmetry, and allows the modeling of (multi-)soliton solutions. This book, in my opinion, does the theory full justice.

Reviewed by *Emma Previato*