What is a disk?

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Introduction

This paper should be considered as an attachment to F.W. Genring's survey lectures "Characterizations of quasidisks" given at this Summer School. Notation, definitions and results are given in that paper. Unless otherwise stated the reference numbers, for example Remark II. A.5, refer to Gehring's lectures.

For many of the characterizations of quasidisks Gehring has given the corresponding charcterizations for disks. In the cases where he has not given references for the proofs I will give the arguments below.

Reflection property characterization of a disk

Remark II.A.5.D is a half plane if and only if it has the reflection property with $L = 1$.

Proof of sufficiency. Fix $z_1 \in D$ and let $\zeta \in \partial D$, $z_1, \zeta \neq \infty$. Then

$$|f(z_1) - \zeta| = |f(z_1) - f(\zeta)| = |z_1 - \zeta|$$

Thus $\partial D \subset B$ where B is the perpendicular bisector of z_1 and $f(z_1)$. On the other hand let $b \in B - \{\infty\}$ and consider the broken line $\gamma = [z_1, b] \cup [b, f(z_1)]$. γ must meet $\partial D \subset B$ and so $\gamma \cap \partial D = B$. $\square$

Decomposition property charcterization of a disk

Remark II.D.1 D is a disk or half plane if and only if for each $z_1, z_2 \in D$ there exists a disk D' with

$$z_1, z_2 \in D' \subset D$$

Proof of sufficiency. We first prove that D is a Jordan domain. For this let $\zeta \in \partial D$ and consider $z_1, z_2 \in D \cap C$ where C is a circular neighborhood of ζ in $\overline{\mathbb{R}}^2$. By the hypothesis $z_1, z_2 \in D' \cap C$, a connected set, and D is locally connected at ζ. By Corollary 2 on p. 161 of [N] D is uniformally locally connected. Finally D is a Jordan domain by Theorem 16.2 on p. 167 of [ ].

Now we may assume that D is bounded. (If not we apply a suitable Möbius transformation.) We choose $z_j, z_j' \in D$ such that

$$|z_j - z_j'| \to \text{dia}(D) < \infty$$

By hypothesis we get w_j, r_j such that

$$z_j, z_j' \subset B(w_j, r_j) \subset D$$

and by passing to subsequenes we may assume

$$z_j \to z_0,\ z_j' \to z_0',\ w_j \to w_0,\ r_j \to r_0$$

Then $B(w_0, r_0) \subset \overline{D}$ and since D has no inner boundary $B(w_0, r_0) \subset D$. In particular $2r_0 \leq \text{dia}\, D$ but by construction $2r_0 \geq \text{dia}\, D$. We conclude that $B(w_0, r_0) = D$. □

Harmonic symmetry characterization of a disk

Remark III.C.8 D is a disk or a half plane if and only if it has the harmonic symmetry property with $c = 1$.

Proof of sufficiency. We consider conformal mappings

$$f : B \to D,\ g : B^* \to D^*$$

such that $f(0) = z_0,\ g(\infty) = z_0^*$.
Then f, g have homeomorphic extensions to $\overline{B}, \overline{B}^*$ and by a preliminary rotation we may assume that $f(1) = g(1)$. Then

$$h = g^{-1} \circ f : \partial B \to \partial B$$

is a sense preserving homeomorphism such that $h(1) = 1$. We want to show that $h(z) = z$.

For this let α, β be upper and lower halves of ∂B labeled so that $i \in \alpha$. By conformal invariance of harmonic measure

$$\omega(z_0, f(\alpha), D) = \omega(0, \alpha, B) = \frac{1}{2} = \omega(0, \beta, B) = \omega(z_0, f(\beta), D)$$

and by hypothesis

$$\begin{aligned}\omega(0,h(\alpha),B) &= \omega(\infty,h(\alpha),B^*) = \omega(z_0^*,g(h(\alpha)),D^*) \\ &= \omega(z_0^*,f(\alpha),D^*) = \omega(z_0^*,f(\beta),D^*) \\ &= \omega(z_0^*,g(h(\alpha)),D^*) = \omega(\infty,h(\beta),B^*) = \omega(0,h(\beta),B)\end{aligned}$$

In particular, this implies that $h(-1) = -1$, and, since h is sense preserving we have that $h(\alpha) = \alpha$.

Next let α, β be left and right halves of the upper half of ∂D labeled so that $e^{i\frac{\pi}{4}} \in \alpha$. Then as above

$$\omega(z_0, f(\alpha), D) = \omega(0, \alpha, B) = \frac{1}{4} = \omega(0, \beta, B) = \omega(z_0, f(\beta), D))$$

and

$$\omega(0, h(\alpha), B) = \omega(z_0^*, f(\alpha), D^*) = \omega(z_0^*, f(\beta), D^*) = \omega(0, h(\beta), B)$$

In particular we get that $h(i) = i$, and $h(\alpha) = \alpha$. Proceeding this way we see that

$$h(e^{2\pi it}) = e^{2\pi it}$$

for all $t \in [0, 2\pi)$ of the form $t = \frac{p}{2^n}$, $p \in \mathbb{Z}$, and by continuity $h(z) = z$.

This implies that the conformal mappings f and g together define a homeomorphism φ of $\overline{\mathbb{C}}$ which is conformal in $B \cup B^*$ and hence in $\overline{\mathbb{C}}$. In particular φ is a Möbius transformation and $D = \varphi(B)$ is a disk or half plane. $\square$

Other characterizations of a disk

For the characterizations in III.C.13, III.D.5 and V.C.3 names of people who have given the proofs are not written up but the arguments needed can be found in the proofs of the corresponding statements for quasidisks.

There ought to be characterizations for disks related to those for quasidisks in several other situations. However, no such formulation or conjecture has been stated!

Additional reference

[N]: M.H.A. Newman, *The topology of plane point sets.* Cambridge University Press 1951.

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