Compact and discrete subgroups of algebraic quantum groups I

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Abstract

Let G be a locally compact group. Consider the C*-algebra $C_0(G)$ of continuous complex functions on G, tending to 0 at infinity. The product in G gives rise to a coproduct Δ_G on the C*-algebra $C_0(G)$. A locally compact *quantum* group is a pair (A, Δ) of a C*-algebra A with a coproduct Δ on A, satisfying certain conditions. The definition guarantees that the pair $(C_0(G), \Delta_G)$ is a locally compact quantum group and that conversely, every locally compact quantum group (A, Δ) is of this form when the underlying C*-algebra A is abelian.

Assume now that G is a locally compact group with a compact open subgroup K. A continuous complex function f with compact support on G is called of *polynomial type* if there exist finitely many continuous functions (f_i) on G and (g_i) on K such that $f(pk) = \sum f_i(p)g_i(k)$ for all $p \in G$ and $k \in K$. The set $P(G)$ of such functions is a dense *-subalgebra of $C_0(G)$, independent of the choice of K. The comultiplication Δ_G leaves $P(G)$ invariant and the pair $(P(G), \Delta_G)$ is a so-called *algebraic quantum group*. Algebraic quantum groups are a special class of locally compact quantum groups. This class contains the discrete quantum groups, the compact quantum groups and it is self-dual. As mentioned above, also a locally compact group with a compact open subgroup falls into this class. In fact, if a locally compact group G gives rise to an algebraic quantum group, it must contain a compact open subgroup.

When K is a compact open subgroup of G, the characteristic function χ_K of K is a self-adjoint idempotent h satisfying $\Delta_G(h)(1 \otimes h) = h \otimes h$. In this paper, we consider such *group-like projections* in a general algebraic quantum group. In other words, we study compact quantum subgroups of algebraic quantum groups. For a general algebraic quantum group, we look at the analogues of the *-algebra of continuous complex functions on the subgroup K and the *-algebra of complex functions with finite support on the quotient space G/K in the group case, as well as their dual objects. In a forthcoming paper on this subject, we plan to include more examples and special cases to illustrate the different notions and results of this paper.

Results in algebraic quantum groups usually are also true (in an adapted form) in general locally compact quantum groups and we discuss further research in this direction. In particular, we will propose a definition of a *totally disconnected (locally compact) quantum group.*

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0. Introduction

This paper studies some material related to the theory of locally compact quantum groups. Let us therefore first recall this concept.

Locally compact quantum groups

Let G be a locally compact group. Consider the C*-algebra $C_0(G)$ of continuous complex functions on G tending to 0 at infinity. It is well-known that the topological structure of G is encoded in the C*-algebraic structure of $C_0(G)$. The product in G gives rise to a coproduct on $C_0(G)$ in the following way. First we identify the minimal C*-tensor product $C_0(G) \otimes C_0(G)$ of $C_0(G)$ with itself with the C*-algebra $C_0(G \times G)$ of continuous complex functions, tending to 0 at infinity on the cartesian product $G \times G$ of G with itself. Then we identify the C*-algebra $C_b(G \times G)$ of continuous bounded complex functions on $G \times G$ with the multiplier algebra $M(C_0(G) \otimes C_0(G))$ of $C_0(G) \otimes C_0(G)$. Now we define a *-homomorphism $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G))$ by $(\Delta(f))(p,q) = f(pq)$ whenever $f \in C_0(G)$ and $p, q \in G$. This *-homomorphism is a comultiplication on the C*-algebra $C_0(G)$ and it carries all the information about the product in G. Indeed, the product in G can be recovered from this coproduct on $C_0(G)$.

The theory of locally compact quantum groups is motivated by the above observation. Now a pair (A, Δ) of a C*-algebra A (not necessarily abelian) and a comultiplication Δ on A is considered. The comultiplication Δ on A is a *-homomorphism from A to the multiplier algebra $M(A \otimes A)$ of the minimal C*-tensor product $A \otimes A$ satifsying certain properties (such as coassociativity).

The pair (A, Δ) will be a locally compact quantum group provided there exist left and right Haar weights. These are the non-commutative analogues of the left and the right Haar measures in the group case. In the general quantum case, the existence of such invariant weights is a part of the definition.

For more information about the general theory of locally compact quantum groups, we refer to [K-V1], [K-V2] and [K-V3]. See also [VD6] and [VD7] for a new, von Neumann algebraic approach to this theory.

About this paper

Very little knowledge about general locally compact quantum groups however is assumed for understanding this paper. Some notions of this general theory will help to have a better idea of the main results of this work. This paper should be readable (and of interest) for algebraists, familiar with some Hopf algebra theory, as well as operator algebraists, with some knowlegde about the operator algebra approach to quantum groups.

Indeed, this paper mainly deals with multiplier Hopf *-algebras with positive integrals; the so-called *algebraic quantum groups* (cf. [VD3]). It is well-known that the purely algebraic aspects of locally compact quantum groups can be studied - up to a certain level - within the special case of such algebraic quantum groups (see [VD5]). In fact, historically, the development of the general theory was greatly motivated by the results on algebraic quantum groups. In the same spirit, the material studied in this paper is also expected to contribute to the further development of the general theory of locally compact quantum groups.

Motivation

The interest for what we are doing here, finds its origin in a question related with the above discussed role of the algebraic quantum groups in the general theory. Let G be a locally compact group. When is the associated pair $(C_0(G), \Delta)$ an algebraic quantum group? More precisely, when is there a dense *-subalgebra A of $C_0(G)$ such that (A, Δ) is an algebraic quantum group? This is the case when G is discrete. Then A is the *-algebra of complex functions on G with finite support. It is also the case when G is a compact group. Now A is the *-algebra of polynomial functions (i.e. matrix elements for finite-dimensional representations, considered as complex functions on the group G). And of course, trivial cases build from these two special cases, like the direct product of a discrete and a compact group, will also give rise to an algebraic quantum group.

In [L-VD1] an answer to this question is given. It turns out that $C_0(G)$ is an algebraic quantum group (in the above sense), if and only if the group G contains a compact open subgroup.

Now, if K is a compact open subgroup of a locally compact group G, and if we let h be the characteristic function χ_K of K, then h is a projection (i.e. a self-adjoint idempotent) in $C_0(G)$ and it satisfies the equation $\Delta(h)(1 \otimes h) = h \otimes h$. One of the main objects that we study in this paper are precisely such projections in multiplier Hopf *-algebras.

Throughout the paper, we will have this motivating example in mind. We will use it to illustrate the various objects that we introduce and the properties that we prove. In two forthcoming papers, we treat some other, more complicated examples together with some special cases (see [L-VD2] and [L-VD3]).

Later in this introduction, we will recall some notions and conventions, used in this paper, together with the basic references. We also refer to the appendix for a review of the basic properties of multiplier Hopf algebras.

Content of the paper

In *Section 1* of this paper, we will initiate the study of what we will call *group-like* projections in a multiplier Hopf *-algebra. These are (non-zero) elements h in a multiplier Hopf *-algebra A satisfying $h^2 = h = h^*$ (self-adjoint idempotents or projections) and moreover $\Delta(h)(1 \otimes h) = h \otimes h$. We call such a projection group-like because, in the abelian case, it really comes from a subgroup (see Proposition 1.4 in Section 1). We found it quite remarkable how many properties of such a group-like projection can be proven from these elementary assumptions. This is done in Section 1.

In *Section 2*, we begin with showing that, if h is a *central* group-like projection in A (that is if $ha = ah$ for all $a \in A$), then, as expected, the *-algebra Ah can be made into a compact quantum group by cutting down the multiplication, i.e. by defining $\Delta_0 : Ah \to Ah \otimes Ah$

as $\Delta_0(a) = \Delta(a)(h \otimes h)$ when $a \in Ah$. The integral on Ah is obtained by restricting the integrals of A. This result is not very difficult. The rest of the section is devoted to the general case, where h is no longer assumed to be central. Now we take hAh as the underlying algebra and we have to cut down Δ on both sides with $h \otimes h$. We are left with a positive comultiplication which is no longer a homomorphism. We get a so-called *compact quantum hypergroup* (or an algebraic quantum hypergroup of compact type in the sense of [De-VD1]).

In *Section 3* we study the equivalent of the algebra of functions that are constant on right cosets. More precisely, we study the 'right leg' of $\Delta(h)$ for a group-like projection h. It can be characterized as the smallest subspace C of A such that $\Delta(h)(A \otimes 1) \subseteq A \otimes C$. It turns out that C is a *-subalgebra of A. It is also left invariant in the sense that the right leg of $\Delta(C)$ is contained in C. In the event that C is invariant under the antipode (this is the case when the left and the right leg of $\Delta(h)$ are the same), then (C, Δ) is a discrete quantum group. In the abelian case, this happens when the compact open subgroup K is normal. Then C is the discrete quantum group of functions of finite support on the quotient group G/K, now a discrete group. Again, in the general case, we have to cut down the comultiplication, still restrict to a smaller *-subalgebra C_1 and then we get what can be considered as a *discrete quantum hypergroup* (or an algebraic quantum hypergroup of discrete type in the sense of [De-VD1]).

Throughout these three preceding sections, we also consider the case of a pair of group-like projections, one smaller than the other. This is important for later work on what will be defined as a totally disconnected locally compact quantum group.

In the last section, *Section 4*, we formulate some conclusions and we discuss future research.

In an *Appendix* we first collect the basic definitions and properties about multiplier Hopf *-algebras (A, Δ), algebraic quantum groups (multiplier Hopf *-algebras with positive integrals) and duality. We also introduce and study the notion of the legs of $\Delta(a)$ for elements $a \in A$. We define left invariant subalgebras and prove some properties. These objects and results are used in the paper, but as they are of some general interest and not dependent upon the notion of a group-like projection, we have put this material in the appendix.

Notions and conventions, basic references

As we have mentioned already, this paper will mainly deal with purely algebraic objects. We will work with algebras A over $\mathbb{C}$, possibly without identity, but always with a non-degenerate product. We use A' for the space of linear functionals on A. We will use $M(A)$ to denote the multiplier algebra of A. The identity in an algebra, e.g. in $M(A)$, will be denoted by 1 (while e will be used to denote the identity in a group). The identity map, say from A to itself, will always be denoted by ι. The (algebraic) tensor product $A \otimes A$ is again an algebra over $\mathbb{C}$ with a non-degenerate product. We identify e.g. $A \otimes \mathbb{C}$ with A and therefore, the slice map $\iota \otimes \omega$ for an element $\omega \in A'$ is considered to be a map from $A \otimes A$ to A.

A *comultiplication* on A is a homomorphism $\Delta : A \to M(A \otimes A)$ satisfying certain properties (such as coassociativity). A *multiplier Hopf (*-)algebra* is a (*-)algebra A with a non-degenerate product and a comultiplication Δ satisfying certain conditions. A multiplier Hopf algebra is called *regular* if the opposite comultiplication $\Delta^{\rm op}$, obtained by composing Δ with the flip, also satisfies these properties. In the case of a *-algebra it is assumed that Δ is a *-homomorphism and regularity is automatic. For a multiplier Hopf algebra (A, Δ), we have the existence of a unique *counit* and a unique *antipode*.

Any Hopf (*-)algebra is a multiplier Hopf (*-)algebra. Conversely, if (A, Δ) is a multiplier Hopf (*-)algebra and if A has an identity, it is a Hopf (*-)algebra. So we see that the theory of multiplier Hopf algebras extends in a natural way the theory of Hopf algebras to the case where the underlying algebras are not required to have an identity. The use of Sweedler's notation in the case of Hopf algebras is a common practice. It is also justified in the case of multiplier Hopf algebras (see e.g. [Dr-VD] and [Dr-VD-Z]) and whenever convenient, we will also do so in this paper.

Let (A, Δ) be a regular multiplier Hopf algebra. A linear functional φ on A is called left invariant if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. A *left integral* is a non-zero left invariant linear functional on A. Similarly, a non-zero right invariant linear functional on A is called a *right integral*.

If (A, Δ) is a multiplier Hopf *-algebra with a positive left integral, then there is also a positive right integral. We will use the term *algebraic quantum group* for such a multiplier Hopf *-algebra. In this paper, we mainly deal with algebraic quantum groups, defined in this sense. However, although for many of the results we need an integral, it is not so clear if the positivity of this integral is important. Still, we make the convention that integrals are assumed to be positive. This implies that our *-algebras are in fact operator algebras, i.e. that they can be represented as *-algebras of bounded operators on a Hilbert space.

We have collected the most important definitions and properties about multiplier Hopf *-algebras and algebraic quantum groups in an appendix. For more details, we refer to [VD1], [VD3] and [VD-Z]. For results about ordinary Hopf algebras, we refer to the basic works [A] and [S].

We will freely use results from these basic references in this paper. In particular, we will use φ to denote a left integral and we use ψ for a right integral. We use σ for the *modular automorphism* of φ, satisfying $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. Similarly σ' is used for the modular automorphism of ψ. The *modular element* δ is a multiplier in $M(A)$, defined and characterized by $(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta$ for all $a \in A$. It is invertible and the inverse satisfies $(\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$. Finally, there is the *scaling constant* ν defined by $\varphi(S^2(a)) = \nu\varphi(a)$ where S is the antipode. We also have collected some of the main formulas relating these various objects, associated with the original pair (A, Δ) as well as with the dual pair $(\hat{A}, \hat{\Delta})$, in the appendix.

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1. Group-like projections in multiplier Hopf *-algebras

Let (A, Δ) be a multiplier Hopf *-algebra. We begin with the definition of a group-like projection. This is essentially the basic object for the rest of the paper. We illustrate the notion by some basic examples and a simple, but important result (Proposition 1.4 below). We prove elementary properties of such group-like projections. We also show that, in the case of algebraic quantum groups, the 'Fourier transform' of a group-like projection is again a group-like projection in the dual $(\hat{A}, \hat{\Delta})$.

Definition and examples of group-like projections

1.1 Definition Let h be a self-adjoint projection in A. We call h *group-like* if

$$\Delta(h)(1 \otimes h) = h \otimes h.$$

That h is a self-adjoint projection means that $h^2 = h = h^*$. Recall also that in a multiplier Hopf algebra, we have $\Delta(a)(1 \otimes b) \in A \otimes A$ for all $a, b \in A \otimes A$ so that the equation in the definition in principle can occur.

We will always assume that $h \neq 0$ when we consider a group-like projection h in this paper.

Recall that an element a in a Hopf algebra is called group-like if $\Delta(a) = a \otimes a$. In general, this notion does not make much sense in a multiplier Hopf algebra. One rather has to look at elements in the multiplier algebra $M(A)$ of A with this property. Here however, we use the term in a different sense. We call the element h group-like because, in the classical case, it behaves like the characteristic function of a subgroup. This was mentioned already in the introduction. It will also become clear from the following examples. They will motivate and justify the use of this terminology.

The first two examples are very simple and easy to understand. We include them mainly for didactical purposes.

1.2 Examples i) Let G be a (discrete) group and let A be the algebra of complex functions on G with finite support with pointwise operations. Consider the obvious comultiplication Δ on A given by $(\Delta(f))(p, q) = f(pq)$ when $f \in A$ and $p, q \in G$. Let K be any finite subgroup and let h be the characteristic function on K, i.e. $h(p) = 1$ if $p \in K$ and $h(p) = 0$ if $p \notin K$. Then $(\Delta(h)(1 \otimes h))(p, q) = h(pq)h(q)$. This will be non-zero if both pq and q are in K. This is true if and only if both p and q are in K. So $\Delta(h)(1 \otimes h) = h \otimes h$.

ii) Again let G be a discrete group, but now consider the group algebra B over $\mathbb{C}$. The comultiplication Δ is given by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ where $p \mapsto \lambda_p$ is the canonical imbedding of G in B. Again let K be a finite subgroup of G. Now put

$$h = \frac{1}{n} \sum_{p \in K} \lambda_p$$

where n is the number of elements in K. It is easy to verify that h is a self-adjoint projection (observe that $\lambda_p^* = \lambda_{p^{-1}}$) and that it satisfies $\Delta(h)(1 \otimes h) = h \otimes h$.

Here are the (more interesting) topological equivalents of the examples in 1.2.

1.3 Examples i) Let G be a locally compact group. Consider the C*-algebra $C_0(G)$ of continuous complex functions on G vanishing at infiniy. The comultiplication Δ is now defined by $(\Delta(f))(p, q) = f(p, q)$ when $f \in C_0(G)$ and $p, q \in G$. This will not be a multiplier Hopf *-algebra in general. However, if there is a compact open subgroup K of G, then it is shown in [L-VD1] that there is a dense *-subalgebra A of $C_0(G)$ such that (A, Δ) is a multiplier Hopf *-algebra with integrals (i.e. an algebraic quantum group).

The characteristic function h of K will be a self-adjoint projection in A satisfying the equation $\Delta(h)(1 \otimes h) = h \otimes h$.

ii) Let G be as in i), that is a locally compact group with a compact open subgroup K. Now consider the reduced group C*-algebra $C_r^*(G)$. This is the C*-algebra generated by operators of the form

$$\int_{p \in G} f(p) \lambda_p dp$$

where integration is over the left Haar measure on G, where λ_p is the translation operator on $L^2(G)$, defined by $(\lambda_p \xi)(q) = \xi(p^{-1}q)$ when $\xi \in L^2(G)$ and $p, q \in G$ and where $f \in C_c(G)$, the space of continuous complex functions with compact support on G. The comultiplication on $C_r^*(G)$ is given by

$$\Delta\left(\int f(p) \lambda_p dp\right) = \int f(p)(\lambda_p \otimes \lambda_p) dp$$

for $f \in C_c(G)$. Also in this case, there exists a dense *-subalgebra that is a multiplier Hopf *-algebra with integrals (see [L-VD1]).

Again h, defined by

$$h = \frac{1}{n} \int_{p \in K} \lambda_p dp,$$

where n is the Haar measure of K, will be a group-like projection in the sense of Definition 1.1.

In both cases, the dense *-subalgebra is constructed by taking *polynomial functions* on the group G w.r.t. the open compact subgroup K. These are continuous complex functions f on G with compact support such that there exist finitely many continuous functions (f_i) on G and (g_i) on K satisfying

$$f(pk) = \sum_i f_i(p)g_i(k)$$

for all $p \in G$ and $k \in K$. Again see [L-VD1].

There is also the following result, related with the first example above. Because it is rather fundamental for the motivation of the basic definition (Definition 1.1 of a group-like projection) and hence for the rest of this paper, let us formulate it as a separate result.

1.4 Proposition Let G be a locally compact group and let K be a compact and open subset of G. Let h be the characteristic function of K. Then K is a subgroup if and only if h is a group-like projection.

Proof: First assume that $\Delta(h)(1 \otimes h) = h \otimes h$. Take points p, q in G. If $pq \in K$ and $q \in K$, then

$$h(p) = h(p)h(q) = (h \otimes h)(p, q) = (\Delta(h)(1 \otimes h))(p, q) = h(pq)h(q) = 1.$$

Hence $p \in K$. So $KK^{-1} \subseteq K$. As it is assumed that $h \neq 0$, we have K non-empty and so it follows that the identity e of the group must be in K. Then it follows that $K^{-1} \subseteq K$. Taking inverses, we also get the other inclusion and so $K^{-1} = K$. Combined with the first result, namely that $KK^{-1} \subseteq K$, we finally get $KK \subseteq K$ and we see that K is a group.

Conversely, if K is a group, then for any pair of elements $p, q \in G$ we have $p, q \in K$ if and only if $pq, q \in K$ and then $\Delta(h)(1 \otimes h) = h \otimes h$.

Apart from these motivating examples, we will also keep in mind some special cases:

1.5 Examples i) Assume that (A, Δ) is of compact type (see Definition 5.2 in [VD3]), that is when A has an identity and hence when (A, Δ) is actually a Hopf *-algebra. Then obviously 1 is a group-like projection in the sense of the above definition.

ii) Assume that (A, Δ) is of discrete type (see again Definition 5.2 in [VD3]), that is when there exists a non-zero element $h \in A$ satisfying $ah = \varepsilon(a)h$ for all $a \in A$. Then

$$\Delta(h)(1 \otimes h) = (\iota \otimes \varepsilon)\Delta(h) \otimes h = h \otimes h$$

so that h is group-like.

These are two extreme cases. Taking the tensor product of a compact type with a discrete type, we get an example 'in between' by considering $1 \otimes h$. In the group case (Example

1.3.i) this would correspond to taking a direct product of a compact group with a discrete group and looking at the compact group as a subgroup of this direct product.

Most of these examples are rather trivial. Even the two examples coming from a locally compact group with a compact open subgroup are not very difficult. We will use these examples throughout the paper for illustrating various results. However, all these examples are in a way too simple for a complete understanding. In [L-VD2] we give more and less trivial examples. One of them is the quantum double. In a sense, this is a twisted version of the example mentioned above of the tensor product of a compact type with a discrete type.

Properties of group-like projections in algebraic quantum groups

Now, we give the first properties of such group-like projections. We will mainly deal with algebraic quantum groups.

1.6 Proposition Let (A, Δ) be an algebraic quantum group and h a group-like projection in A. Then also $\Delta(h)(h \otimes 1) = h \otimes h$. Moreover $\varepsilon(h) = 1$ and $S(h) = h$ (where ε is the counit and S the antipode of A).

Proof: Apply $\varepsilon \otimes \iota$ to the defining equation $\Delta(h)(1 \otimes h) = h \otimes h$. Then we get $h^2 = \varepsilon(h)h$. As $h^2 = h$ and h is assumed to be non-zero, we obtain $\varepsilon(h) = 1$.

Now, apply $\iota \otimes \Delta$ to the defining equation and use the Sweedler notation (see the appendix) to obtain

$$(h_{(1)} \otimes h_{(2)} \otimes h_{(3)})(1 \otimes \Delta(h)) = h \otimes \Delta(h).$$

Then apply S on the first leg, and multiply the first leg with the second one. We get

$$\begin{aligned}(S(h) \otimes 1)\Delta(h) &= (S(h_{(1)})h_{(2)} \otimes h_{(3)})\Delta(h) \\ &= (1 \otimes h)\Delta(h) \\ &= h \otimes h.\end{aligned}$$

The last equality follows from the defining equation by taking adjoints.

Now apply $\iota \otimes \varphi$, where φ is a positive left integral, to get $S(h)\varphi(h) = h\varphi(h)$. Because φ is faitful and $h \neq 0$, we have $\varphi(h) = \varphi(h^*h) \neq 0$. Therefore $S(h) = h$.

Combining with the above equation, we now get

$$(h \otimes 1)\Delta(h) = h \otimes h$$

and taking adjoints, we get the desired equation.

It is not clear if these results will still be true for any multiplier Hopf *-algebra (not an algebraic quantum group). Clearly still $\varepsilon(h) = 1$ will hold. When we apply $\iota \otimes \varepsilon$ to the equation $(S(h) \otimes 1)\Delta(h) = h \otimes h$, we get $S(h)h = h$. If we take adjoints, we get $h = hS(h)^* = hS^{-1}(h)$. If we apply again S, we get $S(h) = hS(h)$. If we also assume that $\Delta(h)(h \otimes 1) = h \otimes h$, we get using a similar argument, that $h = hS(h)$. Then also $S(h) = h$.

In some sense, the result proven earlier in Proposition 1.4, can be seen as an illustration of Proposition 1.6. Indeed, using the notations of 1.4 and 1.6, we see that $\varepsilon(h) = 1$ means $e \in K$ while $S(h) = h$ means $K^{-1} = K$.

The following results relate the various data of an algebraic quantum group with such a group-like projection.

1.7 Proposition Let (A, Δ) be an algebraic quantum group and let h be a group-like projection in A. Then the scaling constant ν is 1. We have $h\delta = h$ where δ is the modular element relating the left and the right integrals. Also $\sigma(h) = \sigma'(h) = h$ where σ and σ' are the modular automorphisms for respectively the left and the right integrals.

Proof: Because $S(h) = h$ we get $S^2(h) = h$ and so $\varphi(h) = \varphi(S^2(h)) = \nu\varphi(h)$ where φ is the left integral. We have seen already that $\varphi(h) \neq 0$. So $\nu = 1$.

If we apply $\varphi \otimes \iota$ to $\Delta(h)(1 \otimes h) = h \otimes h$, we get $\delta h = h$ as $(\varphi \otimes \iota)\Delta(h) = \varphi(h)\delta$. Taking adjoints gives $h\delta = h$.

We have $\Delta(h) = \Delta(S^2(h)) = (\sigma \otimes {\sigma'}^{-1})\Delta(h)$ (see Lemma 3.10 in [K-VD] and Proposition A.6 of the appendix). If we multiply on the left with $h \otimes 1$ and on the right with $1 \otimes {\sigma'}^{-1}(h)$ we get

$$h \otimes h{\sigma'}^{-1}(h) = h\sigma(h) \otimes {\sigma'}^{-1}(h).$$

So $h\sigma(h)$ is a scalar multiple of h. However, as $\varphi(h\sigma(h)) = \varphi(hh) = \varphi(h)$, we must have $h\sigma(h) = h$. Taking adjoints, and using that $\sigma(h)^* = \sigma^{-1}(h^*)$ (cf. Appendix), we also get $h = \sigma^{-1}(h)h$ and if we apply σ, we find $\sigma(h) = h\sigma(h)$. Hence, as also $h = h\sigma(h)$, we get $\sigma(h) = h$. Similarly $\sigma'(h) = h$.

We see that the existence of a group-like projection implies that the scaling constant is trivial. For general locally compact quantum groups, we know examples with a non-trivial scaling constant (cf. [VD4]). For algebraic quantum groups however, the question was open for a long time. There were no known examples of algebraic quantum groups with non-trivial scaling constants. The above result is an indication that such examples can not be found. Indeed, very recently, it is shown in [DC-VD] that for algebraic quantum groups (with positive integrals), the scaling constant is always trivial. Observe however that there are multiplier Hopf algebras with integrals and non-trivial scaling constant (see some of the examples in [VD3]).

It is not possible to illustrate the previous proposition with the given examples. These are too simple for this purpose. Indeed, $\nu = 1$ in these examples anyway and the modular automorphisms σ and σ' are also trivial. In the more general case of a (non-discrete) locally compact group, the modular function can be non-trivial. Then the property $h\delta = h$

is related with the fact that compact groups are unimodular. We refer to [L-VD2] with non-trivial examples for an illustration of the above results.

In most of the paper, we will deal with a single group-like projection h. Then we can *normalize* the left integral φ and the right integral so that $\varphi(h) = 1$ and $\psi(h) = 1$. This is possible because we have that $\varphi(h) > 0$ and $\psi(h) > 0$ in general. Because the scaling constant ν equals 1, this joint normalization of the left and the right integral will give that $\psi = \varphi \circ S$. So, in most of the paper, when we only deal with one specific group-like projection h, we will assume that $\varphi(h) = \psi(h) = 1$.

On a few occasions, we will treat several group-like projections together. This will be done e.g. furhter in this section. In that case, we will not assume the normalizations above. We will clearly mention this fact when it appears.

The Fourier transform of a group-like projection

Now we want to study the Fourier transform of a group-like projection. The remarkable property is that it is again a group-like projection.

First we recall the framework, as it is reviewed in the appendix, needed to consider the Fourier transform.

So, as before, (A, Δ) is a multiplier Hopf *-algebra with positive integrals, i.e. an algebraic quantum group. We use B for the dual $\hat{A}$ and we take for the comultiplication Δ on B the opposite comultiplication $\hat{\Delta}^{\mathrm{op}}$ of the dual comultiplication $\hat{\Delta}$ on $\hat{A}$. We also use $\langle \,\cdot\,, \,\cdot\,\rangle$ for the pairing between A and B. Observe that one of the consequences of this convention is that $\langle S(a), b\rangle = \langle a, S^{-1}(b)\rangle$ when $a \in A$ and $b \in B$. As we explain in the appendix, this convention is very common in the theory of locally compact quantum groups.

For the Fourier transform, we make the choice $F(a) = \varphi(\,\cdot\, a)$ where φ is a left integral on A. We know that then the inverse transform is given by $a = \varphi(S^{-1}(\,\cdot\,)b)$ when $b = \varphi(\,\cdot\, a)$, provided we have the correct relative normalization of the left integrals on A and on B (Proposition 3.4 in [VD5]). Again, see the appendix for more details about this Fourier transform.

We have the following result.

1.8 Proposition Let h be a group-like projection in the algebraic quantum group (A, Δ). Put $k = F(h) = \varphi(\,\cdot\, h)$. Then k is a group-like projection in the dual B.

Proof: Recall that we assume $\varphi(h) = 1$.

We first show that $k^2 = k$. We will again make use of the Sweedler notation for convenience. For any $a \in A$ we have

$$\begin{aligned}
\langle a, k^2\rangle &= \langle a_{(1)}, k\rangle\langle a_{(2)}, k\rangle = \varphi(a_{(1)}h)\varphi(a_{(2)}h) \\
&= (\varphi \otimes \varphi)(\Delta(a)(h \otimes h)) = (\varphi \otimes \varphi)(\Delta(a)\Delta(h)(h \otimes 1)) \\
&= \varphi(ah)\varphi(h) = \varphi(ah) \\
&= \langle a, k\rangle.
\end{aligned}$$

Next we show that $k^* = k$. By the definition of the adjoint on the dual, we have $\langle a, k^*\rangle = \langle S(a)^*, k\rangle^- = \varphi(S(a)^*h)^-$ for any $a \in A$. As φ is positive, hence self-adjoint, we get $\varphi(S(a)^*h)^- = \varphi(hS(a))$. Because $S(h) = h$ we have $\varphi(hS(a)) = \varphi(S(ah))$. Using the property $\varphi(S(x)) = \varphi(x\delta)$ for all $x \in A$, we get $\varphi(S(ah)) = \varphi(ah\delta)$. In Proposition 1.7 we found $h\delta = h$. Combining all these equations, we get $\langle a, k^*\rangle = \varphi(ah) = \langle a, k\rangle$ for all a so that $k^* = k$.

Finally, we show that $\Delta(k)(1 \otimes k) = k \otimes k$. In the next series of equalities, we use that the product in B is dual to the coproduct in A, as well as many of the results about h from Propositions 1.6 and 1.7. We will need $\Delta(h)(h \otimes 1) = h \otimes h$, $S(h) = h$, $h\delta = h$ and $\sigma(h) = h$. For any pair $a, a' \in A$ we have

$$\begin{aligned}
\langle a \otimes a', \Delta(k)(1 \otimes k)\rangle &= \langle a \otimes a'_{(1)}, \Delta(k)\rangle\langle a'_{(2)}, k\rangle = \langle a'_{(1)}a, k\rangle\langle a'_{(2)}, k\rangle \\
&= (\varphi \otimes \varphi)(\Delta(a')(a \otimes 1)(h \otimes h)) \\
&= (\varphi \otimes \varphi)((h \otimes h)\Delta(a')(a \otimes 1)) \\
&= (\varphi \otimes \varphi)((h \otimes 1)\Delta(ha')(a \otimes 1)) \\
&= \varphi(ha')\varphi(ha) = \varphi(a'h)\varphi(ah) \\
&= \langle a \otimes a', k \otimes k\rangle.
\end{aligned}$$

Of course, the formulas that we have proven for h in the Propositions 1.6 and 1.7 will also be valid for k. And in fact, it is interesting to see how these various properties of k relate with the various properties of h. In [L-VD2], where we consider duality, we will see some of these results. We will e.g. show that h and k commute in the Heisenberg algebra.

At this point, it is interesting to have a look at *the case of two group like projections*, one smaller than the other.

As before, let (A, Δ) be an algebraic quantum group and let h and h' be two group-like projections in A. Now, we will not assume any specific normalization of the integrals. We consider the case $h \leq h'$. In this algebraic setting, the inequality $h \leq h'$ means that $hh' = h'h = h$. Observe that one equation $hh' = h$ implies the other $h'h = h$ by taking adjoints.

Using that h' is group-like and $h'h = h$, we get immediately $\Delta(h')(1 \otimes h) = h' \otimes h$. Similarly $\Delta(h')(h \otimes 1) = h \otimes h'$. Then, we also get the following property.

1.9 Proposition As before, let h and h' be group-like projections in an algebraic quantum group. Consider the (properly scaled) Fourier transforms k and k' of h and h' respectively. If $h \leq h'$ then $k' \leq k$.

Proof: Denote $c = \varphi(h)$ and $c' = \varphi(h')$. Then $k = c^{-1}\varphi(\,\cdot\,h)$ and $k' = c'^{-1}\varphi(\,\cdot\,h')$. For any $a \in A$ we get

$$\begin{aligned}
cc'\langle a, kk'\rangle &= \varphi(a_{(1)}h)\varphi(a_{(2)}h') \\
&= \varphi(a_{(1)}h'_{(2)}S^{-1}(h'_{(1)})h)\varphi(a_{(2)}h'_{(3)}).
\end{aligned}$$

Because $S(h) = h$ we get

$$\begin{aligned} S^{-1}(h'_{(1)})h \otimes h'_{(2)} &= (S^{-1} \otimes \iota)((h \otimes 1)\Delta(h')) \\ &= (S^{-1} \otimes \iota)(h \otimes h') = h \otimes h'. \end{aligned}$$

Hence

$$\begin{aligned} cc'\langle a, kk'\rangle &= \varphi(a_{(1)}h'_{(1)}h)\varphi(a_{(2)}h'_{(2)}) \\ &= \varphi(h)\varphi(ah') = cc'\langle a, k'\rangle. \end{aligned}$$

So $kk' = k'$ and therefore $k' \leq k$.

We see that increasing projections $h \leq h'$ yield decreasing Fourier transforms $k' \leq k$.

Examples and some special cases

It is not hard to illustrate the results of Proposition 1.8 in the case of our motivating examples. Take the example of a discrete group (Example 1.2.i and 1.2.ii). These two cases are dual to each other. Using the pairing $\langle f, \lambda_p\rangle = f(p)$, we see that k, defined as $\frac{1}{n}\sum_{p\in k}\lambda_p$ (with $n = \#K$), and h, defined as χ_K, satisfy

$$\langle f, k\rangle = \frac{1}{n}\sum_{p\in K} f(p) = \varphi(fh).$$

We use $\varphi(f) = \frac{1}{n}\sum_{p\in G} f(p)$ (taking care of the correct normalization of φ).

The last thing we will do in this section is to consider two special cases: i) h is central in A and ii) k is central in B. In the case of the example i) in 1.2, of course A is abelian and h is automatically central. When the group is not abelian, B is not abelian. It turns out that k being central is related with the fact that K is a normal subgroup. The reader should have this example in mind for a better understanding of some aspects of the rest of the paper.

We first formulate (and prove) the main result in this connection. Then we give some comments.

1.10 Proposition With the notations as before, the following are equivalent:

i) h is central in A,
ii) $hA = Ah$,
iii) $\Delta^{\mathrm{op}}(k) = (\iota \otimes S^2)(\Delta(k)(1 \otimes \delta^{-1}))$,
iv) $\{\langle a, k_{(1)}\rangle k_{(2)} \mid a \in A\} = \{\langle a, k_{(2)}\rangle k_{(1)} \mid a \in A\}$.

Proof: Let us first show that i) and ii) are equivalent. It is clear that i) implies ii) and we only have to show the converse. So, assume that $hA = Ah$. Take any $a \in A$. Then there exists an $a' \in A$ so that $ha = a'h$. Then

$$ha = a'h = a'hh = hah.$$

So $ha = hah$ for all $a \in A$. Taking adjoints, we get that also $ah = hah$ for all a. Hence $ha = ah$ for all $a \in A$. Therefore also ii) implies i).

Now we will prove that i) and iii) are equivalent. Take a pair of elements $a, a' \in A$. Using that φ is σ-invariant, we get

$$\varphi(a'\sigma(a)h) = \varphi(\sigma^{-1}(a')ah) = \langle \sigma^{-1}(a')a, k\rangle = \langle a \otimes \sigma^{-1}(a'), \Delta(k)\rangle.$$

Now we know from the appendix that

$$\langle \sigma^{-1}(a), b\rangle = \langle a, S^2(b)\delta^{-1}\rangle$$

for all $a \in A$ and $b \in B$. If we use this formula in the previous one, we get

$$\varphi(a'\sigma(a)h) = \langle a \otimes a', (\iota \otimes S^2)\Delta(k)(1 \otimes \delta^{-1})\rangle.$$

On the other hand, we also have

$$\langle a \otimes a', \Delta^{\mathrm{op}}(k)\rangle = \langle aa', k\rangle = \varphi(aa'h) = \varphi(a'h\sigma(a)).$$

So we see that condition iii) is equivalent with the equality

$$\varphi(a'h\sigma(a)) = \varphi(a'\sigma(a)h)$$

for all $a, a' \in A$. This is of course true when h is central. So i) implies iii). Conversely, if iii) holds, then this equaltiy is true for all $a, a' \in A$. By the faithfulness of φ, it then follows that $h\sigma(a) = \sigma(a)h$ for all $a \in A$. Then h is central. This proves the equivalence of i) and iii)

Finally, let us show that ii) and iv) are equivalent. From the definition of k (see also the previous calculations), we get

$$\{\langle a, k_{(1)}\rangle k_{(2)} \mid a \in A\} = \{\varphi(\,\cdot\, ah) \mid a \in A\}.$$

Similarly

$$\begin{aligned}\{\langle a, k_{(2)}\rangle k_{(1)} \mid a \in A\} &= \{\varphi(a\,\cdot\, h) \mid a \in A\}.\\ &= \{\varphi(\,\cdot\, h\sigma(a)) \mid a \in A\}\\ &= \{\varphi(\,\cdot\, ha) \mid a \in A\}.\end{aligned}$$

So, condition iv) will hold if and only if $Ah = hA$, that is if ii) holds.

Condition iv) simply means that 'the right leg' of $\Delta(k)$ is the same as its 'left leg' (see the appendix for some more details about how to define these 'legs'). Condition iii) looks strange, but, as we saw in the proof, it is the dual version of condition i). So, the equivalence of iii) and iv) is as the equivalence between i) and ii).

By duality, we also have this result when we replace h by k. So, we get e.g. that k is central if and only if the two legs of $\Delta(h)$ coincide.

In the case of a group, we have $S^2 = \iota$ and also $\delta = 1$ in B. Then h is central and $\Delta(k)$ is symmetric. In this case, k will be central if the left leg of $\Delta(h)$ coincides with its right leg. Now, recall that h is the characteristic function of the subgroup K. These two legs of $\Delta(h)$ are nothing else but the functions that are constant on right, respectively left cosets. These are the same if and only if K is a normal subgroup.

We will come back to these various cases later, especially, in the beginning of Section 2 and of Section 3.

2. Compact quantum hypergroups

As before, let (A, Δ) be a multiplier Hopf *-algebra and assume that h is a group-like projection (cf. Definition 1.1). So h is a non-zero element in A satisfying $h^2 = h = h^*$ and $\Delta(h)(1 \otimes h) = h \otimes h$. We have seen that, in the case of an algebraic quantum group, automatically also $\Delta(h)(h \otimes 1) = h \otimes h$ and that $\varepsilon(h) = 1$ and $S(h) = h$ (cf. Proposition 1.6). In any case, let us assume that also $\Delta(h)(h \otimes 1) = h \otimes h$. We know that then the two other formulas, namely $\varepsilon(h) = 1$ and $S(h) = h$, are also true.

In this section, we will first consider the case where h is a central projection. Then, we will see that Ah is a compact quantum group. In this paper, by a compact quantum group, we mean a Hopf *-algebra with a positive integral (i.e. an algebraic quantum group of compact type in the sense of [VD3]). This is essentially the algebraic ingredient of a compact quantum group as defined in the C*-algebraic sense by Woronowicz (see [W]). In the general case, when h is not a central projection, we consider hAh and we encounter a compact quantum hypergroup (in the sense of [De-VD1]). We use the basic examples to illustrate the results. They turn out to give the classical Hecke algebras.

The case of a central group-like projection

The following result is fairly easy to obtain.

2.1 Proposition Assume that h is central in A and denote $A_0 = Ah$. Then A_0 is a Hopf *-algebra when we define the coproduct Δ_0 on A_0 by $\Delta_0(a) = \Delta(a)(h \otimes h)$.

Proof: First observe that $\Delta_0(A_0) \subseteq A_0 \otimes A_0$. Indeed,

$$\Delta(a)(h \otimes h) = (\Delta(a)(1 \otimes h))(h \otimes h)$$

and $\Delta(a)(1 \otimes h) \in A \otimes A$. Next remark that Δ_0 is a *-homomorphism from A_0 to $A_0 \otimes A_0$, precisely because h is central in A. Of course, h is the identity in A_0 and $\Delta_0(h) = \Delta(h)(h \otimes h) = h \otimes h$ so that Δ_0 is unital.

To prove coassociativity of Δ_0, let $a \in A_0$ and write

$$\begin{aligned}(\Delta_0 \otimes \iota)\Delta_0(a) &= ((\Delta \otimes \iota)\Delta_0(a))(h \otimes h \otimes 1)\\ &= (\Delta \otimes \iota)(\Delta(a)(h \otimes h))(h \otimes h \otimes 1)\\ &= ((\Delta \otimes \iota)\Delta(a))(\Delta(h)(h \otimes h) \otimes h)\\ &= ((\Delta \otimes \iota)\Delta(a))(h \otimes h \otimes h).\end{aligned}$$

Similarly

$$(\iota \otimes \Delta_0)\Delta_0(a) = ((\iota \otimes \Delta)\Delta(a))(h \otimes h \otimes h)$$

and the coassociativity of Δ_0 follows from the coassociativity of Δ.

We also see that

$$\begin{aligned}\Delta_0(A_0)(1 \otimes A_0) &= (\Delta(A)(1 \otimes A))(h \otimes h)\\ &= (A \otimes A)(h \otimes h)\\ &= A_0 \otimes A_0\end{aligned}$$

and similarly

$$(A_0 \otimes 1)\Delta_0(A_0) = A_0 \otimes A_0.$$

Finally, it is easily verified that the linear maps $a \otimes b \mapsto \Delta_0(a)(1 \otimes b)$ and $a \otimes b \mapsto (a \otimes 1)\Delta_0(b)$ are injective on $A_0 \otimes A_0$. It follows that A_0 is a Hopf *-algebra (see e.g. [VD1]).

It is not so hard to verify that the counit ε_0 for Δ_0 is simply the restriction of ε to A_0 because $\varepsilon(h) = 1$. Similarly, the antipode S_0 is given by the restriction of S to A_0 as $S(h) = h$.

If (A, Δ) is a multiplier Hopf *-algebra with integrals, we get the following.

2.2 Theorem Let (A, Δ) be an algebraic quantum group. Assume that h is a central group-like projection. Then (A_0, Δ_0) is a compact quantum group when, as in the previous proposition, $A_0 = Ah$ and Δ_0 is defined by $\Delta_0(a) = \Delta(a)(h \otimes h)$. The Haar functional on A_0 is given by the restriction of the left integral on A to A_0.

Observe that the property $h\delta = h$, proven in Proposition 1.7, gives that the left and the right integrals on A have the same restriction to A_0 (as expected).

The typical example to illustrate the above result is coming from a compact open subgroup K of a locally compact group G. In this case, as we saw in Example 1.3.i, A is the *-algebra of 'polynomial functions' on G and h is the characteristic function of K. As A is abelian, h will be central and (A_0, Δ_0) is nothing else but the Hopf *-algebra of polynomial functions on the compact group K.

The general case

In the general case, when h is no longer assumed to be central, we have a far more complicated situation.

To begin with, Ah is no longer a *-algebra. If it is a *-algebra, then $Ah = hA$ and we have seen in Proposition 1.10 that then h is central.

There are two ways to get a *-algebra. On can either take hAh or one can consider AhA. But only on the first one, it seems to be possible to define some kind of a comultiplication. Now, we must restrict on both sides by $h \otimes h$ as we show in the following proposition.

2.3 Proposition Let $A_0 = hAh$. Define Δ_0 on A_0 by

$$\Delta_0(a) = (h \otimes h)\Delta(a)(h \otimes h).$$

Then Δ_0 is a unital, positive map from A_0 to $A_0 \otimes A_0$ and it is coassociative.

Proof: This is all more or less obvious. The proof of the coassociativity is essentially the same as in Proposition 2.1. The positivity can be seen as follows:

$$\begin{aligned}\Delta_0(a^*a) &= (h \otimes h)\Delta(a)^*\Delta(a)(h \otimes h) \\ &= (\Delta(a)(h \otimes h))^*(\Delta(a)(h \otimes h))\end{aligned}$$

for any $a \in A_0$.

One can show that Δ_0 is not only positive, but also completely positive. Recall that *complete positivity* for Δ_0 means that $\Delta_0 \otimes \iota$ defined on $A_0 \otimes M_n(\mathbb{C})$ is still positive for all n where $M_n(\mathbb{C})$ is the *-algebra of n by n complex matrices with the usual *-operation. Because it is not essential here, we will not give details.

The counit ε can be restricted to A_0 and will give a unital *-homomorphism ε_0 from A_0 to $\mathbb{C}$ satisfying the usual conditions:

$$(\varepsilon_0 \otimes \iota)\Delta_0(a) = a = (\iota \otimes \varepsilon_0)\Delta_0(a)$$

for all $a \in A_0$.

The antipode S can also be restricted to A_0 because $S(h) = h$. This restriction S_0 is a anti-homomorphism of A_0 and it satisfies $S_0(S_0(a)^*)^* = a$ and

$$\Delta_0(S_0(a)) = (S_0 \otimes S_0)\Delta_0^{\mathrm{op}}(a)$$

for all $a \in A_0$. Recall that Δ_0^{op} is the opposite comultiplication, obtained by composing Δ_0 with the flip. Unfortunately, these conditions are too weak to fully characterize the antipode. If e.g. A_0 is abelian and coabelian, the identity map will also satisfy these formulas.

Let us now proceed in the case of an *algebraic quantum group*. We get the following results about the integrals. We denote by φ_0 the restriction of the left integral φ to the algebra A_0.

2.4 Proposition Assume that (A, Δ) is an algebraic quantum group and that h is a group-like projection in A. Let A_0 and Δ_0 be as in Proposition 2.3. The left integral φ and the right integral ψ coincide on A_0. The restriction φ_0 of φ to A_0 is a positive linear functional on A_0 satisfying

$$(\iota \otimes \varphi_0)\Delta_0(a) = \varphi_0(a)h$$
$$(\varphi_0 \otimes \iota)\Delta_0(a) = \varphi_0(a)h$$

for all $a \in A_0$.

Proof: This is essentially straightforward. For the second equatity, we use that the left integral φ and the right integral ψ coincide on A_0. This is so because $h\delta = h$.

Remark that h is the identity in A_0 and so these are the usual formulas to express left and right invariance of the functional φ_0. Observe also that $\varphi_0(h) = \varphi(h)$ and that this is non-zero. It follows that left and right invariant functionals are unique (and equal).

In the case of an algebraic quantum group, the antipode on the pair (A_0, Δ_0) can be characterized using the relation with the invariant integral in the following way.

2.5 Proposition We assume again that h is a group-like projection in the algebraic quantum group (A, Δ). We will use the notations of before. So again, S_0 is the restriction of the antipode to the *-subalgebra A_0. Also for this restriction we have

$$S_0((\iota \otimes \varphi_0)(\Delta_0(a)(1 \otimes b))) = (\iota \otimes \varphi_0)((1 \otimes a)\Delta_0(b))$$

for all $a, b \in A_0$.

Proof: For $a, b \in A_0$ we have

$$(\iota \otimes \varphi_0)(\Delta_0(a)(1 \otimes b)) = (\iota \otimes \varphi)((h \otimes h)\Delta(a)(h \otimes h)(1 \otimes b)).$$

This is hxh where $x = (\iota \otimes \varphi)((1 \otimes h)\Delta(a)(1 \otimes hb))$. We get

$$x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes hbh)) = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b))$$

because $\sigma(h) = h$ and $hbh = b$. We know that

$$S(x) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b))$$

(see e.g. Proposition A.4 in the appendix). A similar calculation will give that

$$hS(x)h = (\iota \otimes \varphi_0)((1 \otimes a)\Delta_0(b)).$$

So, because $S(h) = h$, we get the desired formula.

We know that the formula in the previous proposition characterizes the antipode in the case of an algebraic quantum group. However, we need to know that all elements in A_0 can be written as (a linear combination of) elements of the form

$$(\iota \otimes \varphi_0)(\Delta_0(a)(1 \otimes b))$$

where $a, b \in A_0$. This is true for (A, Δ). Indeed, the linear span of

$$\{(\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) \mid a, b \in A\}$$

is A simply because $\Delta(A)(1 \otimes A) = A \otimes A$. This stronger condition will not remain valid for the pair (A_0, Δ_0) however. In general, it will not be true that $\Delta_0(A_0)(1 \otimes A_0) = A_0 \otimes A_0$. See the examples below.

On the other hand, we do have the weaker condition. This is the content of the next proposition.

2.6 Proposition We have

$$A_0 = \operatorname{span}\{(\iota \otimes \varphi_0)((\Delta_0(a)(1 \otimes b)) \mid a, b \in A_0\}.$$

Proof: If the result is not true, then there is a non-zero linear functional ω on A_0 that kills all the elements of the right hand side. By the fact that φ_0 remains faithful on the subalgebra A_0, it follows that $(\omega \otimes \iota)\Delta_0(a) = 0$ for all $a \in A_0$. Now apply the counit ε_0 to get $\omega(a) = 0$ for all a. This is a contradiction.

If we combine all these results, we get the following.

2.7 Theorem The pair (A_0, Δ_0) is an algebraic quantum hypergroup of compact type (in the sense of [De-VD1]).

Recall that it is called of compact type because the algebra A_0 has an identity. We get an algebraic version of a compact quantum hypergroup as it was introduced by Chapovsky and Vainerman, see e.g. [C-V] (and some earlier references therein).

In the next section, we will encounter an algebraic quantum hypergroup of discrete type (see Theorem 3.17).

In the previous section, we have looked at the situation of a pair of group-like projections $h, h' \in A$ such that $h \leq h'$. For the results in this section, this situation is not really very interesting because the corresponding algebra hAh is simply a subalgebra of $h'Ah'$ (because $h = h'h = hh'$. Remark however that the coproduct on the smaller one is not just the restriction of the coproduct on the bigger one. One has to cut down further with $h \otimes h$ on each side.

Examples

Let us now look at our examples and see what we get for the pair (A_0, Δ_0) in these simple special cases.

2.8 Examples i) First take the case of a (discrete) group G with a finite subgroup K. Let A be the group algebra $\mathbb{C}G$ and put $h = \frac{1}{n}\sum_{p \in K} \lambda_p$ as in Example 1.2.ii.

When the group K is a normal subgroup, we have, for any $q \in G$,

$$\lambda_q h = \frac{1}{n} \sum_{p \in K} \lambda_{qp} = \frac{1}{n} \sum_{p \in K} \lambda_{qpq^{-1}} \lambda_q = h \lambda_q$$

and h is central. Here Ah is the group algebra of the quotient group G/K.

ii) Again, let K be a finite subgroup of G, not necessarily normal. Let A and h be as in i). For any $q \in G$ we have

$$h \lambda_q h = \frac{1}{n^2} \sum_{p,p' \in K} \lambda_{pqp'}.$$

Let us denote this element by $\pi(q)$ and observe that it only depends on the double coset KqK. For the product of such elements, we have

$$\begin{aligned}
\pi(q)\pi(q') &= \frac{1}{n^4} \sum_{p,p',r,r'} \lambda_{pqp'rq'r'} \\
&= \frac{1}{n^3} \sum_{p,p',r} \lambda_{pqrq'p'} \\
&= \frac{1}{n} \sum_{r \in K} \pi(qrq').
\end{aligned}$$

For the coproduct Δ_0, we get

$$\begin{aligned}
\Delta_0(\pi(q)) &= (1 \otimes h)\Delta(\pi(q))(1 \otimes h) \\
&= \frac{1}{n^4} \sum_{p,p',r,r'} \lambda_{pqp'} \otimes \lambda_{rpqp'r'} \\
&= \frac{1}{n^4} \sum_{p,p',r,r'} \lambda_{pqp'} \otimes \lambda_{rqr'} \\
&= \pi(q) \otimes \pi(q).
\end{aligned}$$

The counit ε_0 is given by the restriction of ε and so $\varepsilon(\pi(q)) = 1$ for all q. The antipode S_0 is given by the restriction of S and so $S_0(\pi(q)) = \pi(q^{-1})$. Observe that in general,

$$\pi(q^{-1})\pi(q) \neq 1$$

so that the formula $m(S_0 \otimes \iota)\Delta_0(a) = \varepsilon_0(a)1$, true in a Hopf algebra, will not hold here. Similarly it will not be true that $\Delta_0(A_0)(1 \otimes A_0) = A_0 \otimes A_0$.

For the integral, we have $\varphi(\lambda_p) = 0$ except for $p = e$, the identity in the group. So, when we restrict, we will get $\varphi_0(\pi(q)) = 0$ when $q \notin K$. Remark that $\pi(q) = h$ if and only if $q \in K$. In that case of course, $\varphi_0(\pi(q)) \neq 0$.

Finally, let us look at the formula in Proposition 2.5. And first, let us calculate

$$(\iota \otimes \varphi_0)(\Delta_0(\pi(q))(1 \otimes \pi(q'))$$

for any two elements $q, q' \in K$. We get

$$\pi(q)\varphi_0(\pi(q)\pi(q')) = \pi(q)\frac{1}{n}\sum_{r\in K}\varphi_0(\pi(qrq')).$$

This is 0 except when $\pi(q') = \pi(q^{-1})$ and in that case, it is $\pi(q)$, provided we use the normalization $\varphi(h) = 1$.

Similarly,

$$(\iota \otimes \varphi_0)(1 \otimes \pi(q))\Delta_0(\pi(q')) = 0$$

except again when $\pi(q') = \pi(q^{-1})$, in which case the expression is $\pi(q')$. We see that these elements indeed span A_0 and that the antipode S_0 satisfies the equality of Proposition 2.5.

The case of Example 1.2.i. is too trivial to mention.

Let us now also consider the example with a compact open subgroup of a locally compact group. Again Example 1.3.i. is not very interesting and also here, we only look at the second case in Example 1.3.

2.9 Example Let G be a locally compact group with a compact open subgroup K. Consider the reduced C*-algebra $C_r^*(G)$ and its dense multiplier Hopf *-algebra A of polynomial functions (as in Example 1.3.ii). Take for h the characteristic function of K, as sitting in A. We have seen that h is a group-like projection.

If K is a normal subgroup, the element h is central and we end up by considering the group algebra of the (disrete) quotient group G/K.

In general, when K is not assumed to be a normal subgroup, we must consider the algebra A_0, defined as hAh. This consists of elements of the form $\int_G f(p)\lambda_p dp$ where f is a continuous function with compact support on G, constant on double cosets $K\backslash G/K$. The coproduct Δ_0 on hAh is given by the formula

$$\Delta_0(\int_G f(p)\lambda_p dp) = \frac{1}{n^2}\int_G\int_K\int_K f(p)\,\lambda_p \otimes \lambda_{kpk'}\,dk\,dk'\,dp$$

when $f \in hAh$ where as before, n is the measure of K. The counit and antipode on A_0 are the restrictions of the counit and antipode on A and the same is true for the integral φ_0 on A_0.

In fact, because the space of double cosets will be a discrete space, we get a compact quantum hypergroup of the same type as in the previous example. Indeed, it is still possible to consider the elements $\pi(q)$ for any $q \in G$. Now they are defined, similarly as in Example 2.8.ii by

$$\pi(q) = h\lambda_q h = \frac{1}{n^2}\int_K\int_K \lambda_{kqk'}\,dk\,dk'$$

and A_0 is again the linear span of these element $\pi(q)$ with $q \in G$. As before, we will get $\Delta_0(\pi(q)) = \pi(q) \otimes \pi(q)$ for all $q \in G$. Also $\varepsilon_0(\pi(q)) = 1$ and $S_0(\pi(q)) = \pi(q^{-1})$ for all q. Finally, also $\varphi_0(\pi(q)) = 0$ except for $q \in K$.

We finish this section with some remarks.

2.10 Remarks i) As we saw in Proposition 2.3, the situation is quite nice, and in fact not very surprising, when h is a *central* group-like projection. When the projection is not central, we have considered the *-algebra hAh as this algebra still carries a reasonable structure (it is a compact quantum hypergroup).

ii) In the Example 2.8.ii, where we have a (discrete) group G with a finite subgroup K, the algebra we get is nothing else but the classical Hecke algebra $H(G, K)$ (see e.g. Section 2.10 in [G-H-J]). Also the Example 2.9 gives an algebra of this type.

iii) In general however, this algebra may be rather small and contain too little information. Consider e.g. the case of $G = GL(2, \mathbb{Q})$ with the subgroup $K = SL(2, \mathbb{Z})$. Consider the group algebra (as in Example 2.9) so that A_0 is the convolution algebra with functions constant on double cosets. In this example, this algebra turns out to be abelian and it will not contain much information about the pair (G, K). See e.g. [Kr].

iv) Finally, let us make a short remark about terminology. In the Examples 1.2.i and 1.3.i, the group-like projection h is central as the algebras are abelian. We have that A is an algebra of functions on the group G and Ah is an algebra of functions on the subgroup K. From this point of view, in the general case, we can think of Ah (if h is central) and hAh (in general) as a compact quantum (hyper) *sub*group. However, if we consider Example 2.8 (cf. Example 1.2.ii) and Example 2.9 (cf. Example 1.3.ii), we see that A is a group algebra and, in the case of a normal subgroup, that Ah is a group algebra of the quotient group. From this point of view, in the general case, we can think of Ah (if h is central) and hAh (in general) as a compact quantum (hyper) *quotient* group.

In [L-VD2] we plan to give some other, perhaps more interesting examples of compact quantum hypergroups obtained in this way.

3. Discrete quantum hypergroups

In this section, we will study another *-algebra associated with a group-like projection. The case is dual to the case considered in the previous section as we show in the second paper on this subject ([L-VD2]. We will consider the dual objects to Ah, hA and hAh.

In the previous section, we got a compact quantum group if h is central. In this section, we first look at the case where the two legs of $\Delta(h)$ coincide. This will yield a discrete quantum group. Recall that in this paper, a discrete quantum group is an algebraic quantum group with co-integrals (i.e. an algebraic quantum group of discrete type as in [VD3], see also [VD2]). In the general case, we will encounter discrete quantum hypergroups (in the sense of [De-VD1]). Again, we also consider the basic examples. In this case, the underlying algebras consist of functions that are constant on cosets (whereas in the previous section, we rather looked at the functions living on the subgroup).

We begin with considering the algebra C. The reader should have in mind that this is dual to Ah and in fact the (inverse) Fourier transform of the space Bk (see again [L-VD2] for more details about this point of view).

The left invariant *-subalgebra C

Again, we start with the general situation and see what is possible. So, to begin with, let (A, Δ) be any multiplier Hopf *-algebra and h a group-like projection in A. We will assume that (A, Δ) is regular.

In what follows, we will use the definitions and results of the appendix concerning the legs of the coproduct (cf. Definition A.11 and following).

3.1 Notation Let C denote the right leg of $\Delta(h)$. If we work with several group-like projections (see further in this section), we will write C_h. But we can drop the index because most of the time, we only work with a single group-like projection.

By the self-adjointness of $\Delta(h)$, we certainly have that C is a *-subspace of A. But we can prove more. We have the following properties of elements in C.

3.2 Proposition For any $a \in C$ we have

i) $ah = \varepsilon(a)h$,
ii) $\Delta(a)(1 \otimes h) = a \otimes h$,
iii) $(S(a) \otimes 1)\Delta(h) = (1 \otimes a)\Delta(h)$.

Proof: Start from the equation $\Delta(h)(1 \otimes h) = h \otimes h$. If we apply a linear functional ω on the first leg of this equation, we get $ah = \omega(h)h$ where $a = (\omega \otimes \iota)\Delta(h)$. Clearly $\omega(h) = \varepsilon(a)$ and because every element in C is of this form, we have proven i).

To prove ii), start again with $\Delta(h)(1 \otimes h) = h \otimes h$, apply $\Delta \otimes \iota$, use coassociativity and multiply with $a \otimes 1 \otimes 1$ to get

$$((\iota \otimes \Delta)(\Delta(h)(a \otimes 1))(1 \otimes 1 \otimes h) = (\Delta(h)(a \otimes 1)) \otimes h.$$

Now take a linear functional ω on A and apply $\omega \otimes \iota \otimes \iota$ to get $\Delta(x)(1 \otimes h) = x \otimes h$ where $x = (\omega \otimes \iota)(\Delta(h)(a \otimes 1))$. Again, we get all elements x in C and so we proved ii).

To prove iii), we start with $\Delta(a)(1 \otimes h) = a \otimes h$, where $a \in C$, and we apply $\iota \otimes \Delta$. We will also use the Sweedler notation and write

$$(\Delta \otimes \iota)\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

Then

$$a_{(1)} \otimes (a_{(2)} \otimes a_{(3)})\Delta(h) = a \otimes \Delta(h).$$

Now apply S on the first leg of this equation and multiply to get

$$(S(a_{(1)})a_{(2)} \otimes a_{(3)})\Delta(h) = (S(a) \otimes 1)\Delta(h).$$

The left hand side is $(1 \otimes a)\Delta(h)$ and so we get the result.

Observe that we have proven condition iii) from condition ii) without the need for any other property of h. This fact will be used later, in the proof of Proposition 3.4.

3.3 Proposition The space C is a *-algebra and $\Delta(C)(A \otimes 1) \subseteq A \otimes C$. We have $h \in C$.

Proof: We have shown already that C is *-invariant. To show that it is a subalgebra, take any $a \in C$ and $a' \in A$. Then, from iii) in the previous proposition, we get

$$(1 \otimes a)\Delta(h)(a' \otimes 1) = (S(a) \otimes 1)\Delta(h)(a' \otimes 1) \subseteq A \otimes C.$$

Take an element $\omega \in A'$ and apply $\omega \otimes \iota$ to this equation. We get $ab \in C$ where $b = (\omega \otimes \iota)(\Delta(h)(a' \otimes 1))$. This proves that C is an algebra.

We have that $\Delta(h)(A \otimes 1)$ is a subspace of $A \otimes C$. If we apply $\Delta \otimes \iota$ and multiply with $A \otimes 1 \otimes 1$ we get

$$((\iota \otimes \Delta)\Delta(h))(A \otimes A \otimes 1) \subseteq A \otimes A \otimes C.$$

When we apply a linear functional ω on the first leg of this equation, we get

$$\Delta(a)(A \otimes 1) \subseteq A \otimes C$$

for all $a \in C$.

Finally, it is not difficult to see that $h \in C$. Indeed, for any $a \in A$ with $\varepsilon(a) = 1$ we get $h \in C$ from applying $\varepsilon \otimes \iota$ to the expression $\Delta(h)(a \otimes 1)$.

We see that the right leg of $\Delta(C)$ lies again in C. Therefore, C is a *left invariant* *-*subalgebra* of A (see Definition A.13 in the appendix).

Observe that condition i) in Proposition 3.2, together with the left invariance of C also implies ii) in that proposition.

The algebra C in the case of an algebraic quantum group

Now, and further in this section, we will assume that the pair (A, Δ) is an algebraic quantum group. As before, we will use φ for a left integral and ψ for a right integral. Given the group-like projection h, we assume that these two integrals are normalized so that $\varphi(h) = 1$ and $\psi(h) = 1$.

First of all, in this case, we can use the formulas ii) and iii) in Proposition 3.2 to characterize elements in C. This is done in the following proposition.

3.4 Proposition Let (A, Δ) be an algebraic quantum group and let C be the right leg of a group-like projection h in A. Then

$$\begin{aligned} C &= \{a \in A \mid \Delta(a)(1 \otimes h) = a \otimes h\} \\ &= \{a \in A \mid (1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)\}. \end{aligned}$$

Proof: Suppose that $a \in A$ and that $\Delta(a)(1 \otimes h) = a \otimes h$. We saw in the proof of Proposition 3.2 that it then follows that also $(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$. Apply the right invariant functional ψ to get $\psi(h)a \in C$. As we know that $\psi(h) \neq 0$, we will have $a \in C$. Then this proposition follows from Proposition 3.2.

It is not clear if this result is true in general (that is if we do not assume the existence of integrals).

As an application of the above result, let us look again at the case of two group-like projections h and h' with $h \leq h'$.

We have seen already in Section 1 that

$$\Delta(h')(1 \otimes h) = h' \otimes h \qquad \text{and} \qquad \Delta(h')(h \otimes 1) = h' \otimes h.$$

This is a simple consequence of the definitions and the formula $h = h'h$.

Now denote by C_h and by $C_{h'}$ the right legs of $\Delta(h)$ and $\Delta(h')$ respectively.

3.5 Proposition With the notation and conditions as above, we have $C_{h'} \subseteq C_h$.

Proof: Because $\Delta(h')(1 \otimes h) = h' \otimes h$, it follows from Proposition 3.4 that $h' \in C_h$. By the left invariance of C_h, it follows that also the right leg of $\Delta(h')$ sits in C_h. This proves the result.

We see that, increasing projections give rise to decreasing algebras. Compare with the result we found in Proposition 1.9 where we saw that a similar sitation happened with the Fourier transform of group-like projections.

Now again, consider the case of a single group-like projection. We prove some more properties about C that are easy consequences of the properties of h and the definition of C.

3.6 Proposition Let (A, Δ) be an algebraic quantum group and as before, let C be the right leg of $\Delta(h)$ for a group-like projection h in A. Then $S^2(C) = C$ and also $\sigma(C) = C$. In fact, $\sigma(a) = S^2(a)$ for all $a \in C$. We also have $C\delta = \delta C = C$.

Proof: We have seen (in Proposition 1.6 and Proposition 1.7) that $\sigma(h) = h$ and $S(h) = h$, so also $S^2(h) = h$. Then

$$(S^2 \otimes S^2)\Delta(h) = \Delta(h)$$
$$(S^2 \otimes \sigma)\Delta(h) = \Delta(h).$$

We use that $\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a)$ for all $a \in A$ to get the last formula (see Proposition A.5 in the appendix). It follows that indeed $S^2(C) = C$ and $\sigma(C) = C$. Moreover, combining the two formulas we get

$$(\iota \otimes S^2)\Delta(h) = (\iota \otimes \sigma)\Delta(h)$$

and this will imply that σ and S^2 coincide on C.

To prove the second statement, start from the equality $h\delta = h$, proven in Proposition 1.7 and apply Δ to get $\Delta(h)(\delta \otimes \delta) = \Delta(h)$. It follows that $C\delta = C$ and by taking adjoints also that $\delta C = C$.

From one of the formulas in Proposition A.9 in the appendix, it follows that for any algebraic quantum group (A, Δ) with a unimodular dual (i.e. when left and right integrals on the dual coincide), we have that $\sigma = S^2$ on A. In particular, because a compact quantum group is unimodular, we have this property on a discrete quantum group. In the previous proposition, we see that also $\sigma = S^2$ on this subalgebra C. This is not unexpected as we will see later that C shares also other properties with discrete quantum groups.

The last formula in the previous proposition says that δ and δ^{-1} are also multipliers of C.

In general, we will not have that C is left invariant by the antipode S. We have that $S(C)$ is the left leg of $\Delta(h)$. It will also be a *-subalgebra of A, now a right invariant one. We will only have $S(C) = C$ if the two legs of $\Delta(h)$ are the same. This is one of the special cases we have considered in Section 1 (cf. Proposition 1.10), but in its dual form. We will show that the pair (C, Δ) is a discrete quantum group if $S(C) = C$ (see Theorem 3.8 below). Then we know that we must have that the *-algebra C is a direct sum of matrix algebras. Now, it turns out that this is already the case, also when we do not assume that $S(C) = C$. We obtain this in the next proposition.

3.7 Proposition Let (A, Δ) be an algebraic quantum group and h a group-like projection. Let C be the right leg of $\Delta(h)$. Then C is a direct sum of matrix algebras.

Proof: Take $a \in A$ and write $\Delta(h)(1 \otimes a) = \sum a_i \otimes b_i$. Let V be the vector space spanned by the elements b_i. We see that $Ca \subseteq V$. In particular, Ca is finite-dimensional for all $a \in A$. Similarly, aC is finite-dimensional for all a. Then also CaC will be finite-dimensional for all $a \in A$. So, CaC will be a finite-dimensional two-sided *-ideal of C for all self-adjoint elements $a \in C$. Because we are working with operator algebras (i.e. *-algebras with enough positive linear functionals), such a two-sided *-ideal must be in itself a finite-dimensional operator algebra, so a direct sum of matrix algebras.

Now, take any $a \in C$. We know that $\psi(h)a = (\psi \otimes \iota)(\Delta(h)(1 \otimes a))$ when ψ is a right integral. We also know that there is an element $e \in A$ such that $(e \otimes 1)\Delta(h)(1 \otimes a) = \Delta(h)(1 \otimes a)$ (see Proposition A.7 in the appendix). If we apply $\psi \otimes \iota$, we see that $a \in Ca$. Taking adjoints, also $a \in aC$. Combining, we get $a \in CaC$. So any element of C belongs to such a sum of matrix algebras in C.

Hence C will be a direct sum of matrix algebras.

We have seen already that $ah = \varepsilon(a)h$ for all $a \in C$ and so $\mathbb{C}h$ is a one-dimensional component of C. The complement of this component in this direct sum decomposition is precisely the kernel of ε on C.

Observe that not very much is needed about h, except for the fact that C is a *-algebra. Indeed, in [L-VD2], we will obtain a similar result as above for any left invariant *-subalgebra with a non-trivial relative commutant in the dual algebra B.

Let us now combine the result of Proposition A.15 in the appendix (exitence of local units in C) with the one of Proposition 3.7. Let elements $a_1, a_2, \dots, a_n$ in A be given and let $e \in C$ so that $a_i e = a_i$ for all i. Because of the structure of C (being a direct sum of matrix algebras), we can find a projection f in C such that $ef = e$. This element will still satifsfy $a_i f = a_i$ for all i. It is not difficult to show that in fact, we have two-sided local units for A that are projections (in C). This strengthens the result in Proposition A.15.

The comultiplication Δ on C in the case when $S(C) = C$

As mentioned before, the algebra C should be considered as the dual object to Ah, or more precisely, as the inverse Fourier transform of Bk where k is the Fourier transform of h. This means that the condition $S(C) = C$ should be viewed as dual to the condition $S(Ah) = Ah$, or again more precisely, as $S(Bk) = Bk$. This is the case when $Bk = kB$ since $S(k) = k$. This means that k is central. Therefore it is quite natural to start with this case, just as in Section 1 where we first considered the case where h is central.

In the special case where $S(C) = C$, we have that C is left invariant by Δ and that (C, Δ) is actually a discrete quantum group. This is the content of what follows.

3.8 Theorem Let (A, Δ) be an algebraic quantum group and assume that h is a group-like projection such that the left and the right legs of $\Delta(h)$ coincide. Then (C, Δ) is a discrete quantum group. The element h is the co-integral.

Proof: We have $\Delta(C)(A \otimes 1) \subseteq A \otimes C$ and by the symmetry also $\Delta(C)(1 \otimes A) \subseteq C \otimes A$. This will give $\Delta(C)(C \otimes 1) \subseteq C \otimes C$ and $\Delta(C)(1 \otimes C) \subseteq C \otimes C$. So, the maps T_1 and T_2, defined by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ will go from $C \otimes C$ to $C \otimes C$. Of course they are injective on $C \otimes C$ as they are already injective on $A \otimes A$. The inverse of the map T_1 on $A \otimes A$ is given by

$$T_1^{-1}(a \otimes b) = (\iota \otimes S)((1 \otimes S^{-1}(b))\Delta(a))$$

and it follows from the fact that C is invariant under S, that also T_1^{-1} will map $C \otimes C$ into itself. Similarly for T_2. Therefore (C, Δ) is a multiplier Hopf *-algebra.

Clearly h is a co-integral as $ah = \varepsilon(a)h$ for all $a \in C$ (and also $ha = \varepsilon(a)h$ by taking adjoints). It follows that C is a discrete quantum group (Definition 5.2 in [VD3], see also Definition A.8 in the appendix).

So, this theorem is a dual version of Theorem 2.2 in the previous section. In [L-VD2], where we study all these objects in duality, we will come back to these two special cases and see how they are dual one to the other.

Let us now illustrate this case with two examples.

3.9 Examples i) First let K be a finite subgroup of a (discrete) group G. Let A be the group algebra and let $h = \frac{1}{n}\sum_{p\in K}\lambda_p$ (see Example 1.2.ii). Of course, as A is co-abelian, $\Delta(h)$ is symmetric. Here, the algebra C is nothing else but the group algebra of K.

ii) Let K be an open compact subgroup of a locally compact group G and let A be the algebra of polynomial functions on G as in Example 1.3.i. Assume that K is normal and let $h = \chi_K$, the characteristic function of K. The right leg of $\Delta(h)$ consists of continuous functions with compact support and constant on right cosets. Because K is assumed to be a normal subgroup, the right cosets are the same as the left cosets and so we get that the two legs of $\Delta(h)$ are the same. The discrete quantum group we get here is precisely the one associated to the discrete group G/K. The algebra C is now the algebra of complex functions with finite support on G/K.

The *-algebra $C_1 = C \cap S(C)$

Now we will see what is still possible in the general case, that is when the algebra C is not invariant under the antipode S. It is more or less obvious that we have to consider the intersection $C \cap S(C)$. This is the step dual to taking hAh in the previous section. Indeed, $hAh = Ah \cap hA$ and $hA = S(Ah)$.

3.10 Proposition Let $C_1 = C \cap S(C)$. Then C_1 is a *-subalgebra of A, invariant under the antipode. We still have $AC_1 = A$.

Proof: We know that C and $S(C)$ are *-subalgebras. Therefore, the intersection is also a *-subalgebra. Because $S^2(C) = C$, we have $S(C_1) = C_1$.

We claim that elements x of the form $(\psi \otimes \iota)(\Delta(a)(b \otimes 1))$ with $a, b \in C$ belong to C_1. Indeed, by the left invariance of C we get $x \in C$ because $a \in C$. We also know that $x = S(y)$ where $y = (\psi \otimes \iota)((a \otimes 1)\Delta(b))$ (see Proposition A.4 in the appendix) and so $y \in C$ because $b \in C$. Hence $x \in C \cap S(C)$ if both $a, b \in C$.

Next, take any finite number of elements $a_1, a_2, \ldots, a_n$ in A. By Proposition A.15 we have an element $p \in C$ such that

$$(1 \otimes a_i)\Delta(h)(p \otimes 1) = (1 \otimes a_i)\Delta(h)$$

for all i. If we apply ψ we get again $a_i e = a_i$ for all i with $e = (\psi \otimes \iota)\Delta(h)(p \otimes 1)$. By the result we just proved, $e \in C_1$. So we find local units for A in C_1. Then $AC_1 = A$.

We see from this proposition that C_1 is still imbedded non-degenerately in A (as $AC_1 = A$ and by taking adjoints, also $C_1 A = A$). Again, we can consider the multiplier algebra $M(C_1)$ as sitting inside $M(A)$ (see a remark in the appendix, following Proposition A.15).
We have $h \in C_1$ because $h \in C$ and $S(h) = h$. Clearly $S^2(C_1) = C_1$ because this holds for C. And as $\sigma = S^2$ on C, the same is true on the smaller algebra C_1. Finally, we still have $C_1\delta = \delta C_1 = C_1$ because this holds for C and because $S(\delta) = \delta^{-1}$. So, δ and δ^{-1} are still multipliers of C_1.
Of course, also C_1 is a direct sum of matrix algebras.

Now, we would like to consider the comultiplication on C_1. We saw in Theorem 3.8 that Δ leaves C invariant if $S(C) = C$. We can not expect that Δ leaves C_1 invariant. Just as in the compact case (Section 2), we have to use a projection map to get down to C_1. As in that case, also here we will no longer have a comultiplication in the usual sense as it will not be a homomorphism anymore. It will be a positive linear map, still satisfying coassociativity (as was the case in Proposition 2.3).
We will use the projection maps $E : A \to C$ and $E' : A \to S(C)$ that we will introduce in the following proposition. Recall that we use left and right integrals φ and ψ resp. that are normalized w.r.t. our group-like projection h. So $\varphi(h) = \psi(h) = 1$. Remember that $\psi(\,\cdot\, h) = \varphi(\,\cdot\, h)$ because $h\delta = h$ and that we use φ_0 for this functional.

3.11 Proposition Define linear maps E, E' from A to itself by

$$E(a) = (\iota \otimes \varphi)(\Delta(a)(1 \otimes h)) = (\iota \otimes \varphi_0)\Delta(a),$$
$$E'(a) = (\psi \otimes \iota)(\Delta(a)(h \otimes 1)) = (\varphi_0 \otimes \iota)\Delta(a).$$

Then E and E' are positive, faithful conditional expectations from A onto C and $S(C)$ respectively. The antipode converts one to the other and they commute with each other.

Proof: First observe that the linear map E is positive because

$$E(a^*a) = (\iota \otimes \varphi)((1 \otimes h)\Delta(a^*a)(1 \otimes h))$$

where we have used that $h^2 = h = \sigma(h)$. To show that E is faithful, take $a \in A$ and assume that $E(a^*a) = 0$. It follows from the fact that φ is faithful, that then $\Delta(a)(1 \otimes h) = 0$. As $h \neq 0$, we must have $a = 0$ (by the very definition of a multiplier Hopf algebra). So, E is a faithful positive map.
If $a \in C$, then $\Delta(a)(1 \otimes h) = a \otimes h$ and we see that $E(a) = a$ because $\varphi(h)$ is assumed to be 1. If $a \in C$ and $b \in A$ we have

$$\begin{aligned} E(ba) &= (\iota \otimes \varphi)(\Delta(b)\Delta(a)(1 \otimes h)) \\ &= (\iota \otimes \varphi)(\Delta(b)(a \otimes h)) \\ &= E(b)a. \end{aligned}$$

Similarly, or by taking adjoints, we get also $E(ab) = aE(b)$ if $a \in C$ and $b \in A$.

Finally, we need to show that $E(A) \subseteq C$. We know from Proposition A.4 in the appendix that $E(a) = S(x)$ where $x = (\iota \otimes \varphi)(\Delta(h)(1 \otimes a))$ (using that $\sigma(h) = h$). So x belongs to the left leg of $\Delta(h)$ and consequently $S(x)$ sits in the right leg of $\Delta(h)$. We find $E(a) \in C$ for all $a \in A$.

A similar argument will work for E'.

Now, for any $a \in A$ we have

$$\begin{aligned}
E(S(a)) &= (\iota \otimes \varphi)(\Delta(S(a))(1 \otimes h)) \\
&= (\varphi \otimes \iota)(((S \otimes S)\Delta(a))(h \otimes 1)) \\
&= (\varphi \otimes \iota)((S \otimes S)((h \otimes 1)\Delta(a))) \\
&= S((\psi \otimes \iota)((h \otimes 1)\Delta(a))) \\
&= S(E'(a)).
\end{aligned}$$

Similarly we can get $E'(S(a)) = S(E(a))$. It is also possible to obtain this from the previous formula, using that S^2 commutes with both E and E'.

Finally, for any $a \in A$, we have

$$\begin{aligned}
E'(E(a)) &= E'((\iota \otimes \varphi)(\Delta(a)(1 \otimes h))) \\
&= (\psi \otimes \iota \otimes \varphi)((\Delta \otimes \iota)\Delta(a)(h \otimes 1 \otimes h))
\end{aligned}$$

and by the coassociativity of Δ this turns out to be the same as $E(E'(a))$. This proves the proposition.

It follows that the composition $E'E$ will be a positive, faithful conditional expectation from A onto C_1. We can write $E'E(a) = (\varphi_0 \otimes \iota \otimes \varphi_0)\Delta^{(2)}(a)$ for all $a \in A$.

Before we continue, let us make the following observation. One can verify that the maps E and E' are adjoint to the maps from B to B given by $b \mapsto bk$ and $b \mapsto kb$ respectively. This explains part of the results of the above proposition. The composition EE' is the adjoint of the map $b \mapsto kbk$ on B.

Now, we are almost ready to define a positive comultiplication Δ_1 on C_1. The following lemma will be provide the last formula to make this possible.

3.12 Lemma For any $a \in A$, we get

$$(E \otimes \iota)\Delta(a) = (\iota \otimes E')\Delta(a).$$

Proof: We have by the definitions of E and E'

$$\begin{aligned}
(E \otimes \iota)\Delta(a) &= (\iota \otimes \varphi_0 \otimes \iota)((\Delta \otimes \iota)\Delta(a)), \\
(\iota \otimes E')\Delta(a) &= (\iota \otimes \varphi_0 \otimes \iota)((\iota \otimes \Delta)\Delta(a))
\end{aligned}$$

and the result follows from the coassociativity of Δ.

Before we continue, we must make a remark about the formula in the lemma. Indeed, we are applying the maps $E \otimes \iota$ and $\iota \otimes E'$ to an element in $M(A \otimes A)$ and this is not completely obvious. One solution to this problem is to give a meaning to the equation in the lemma by multiplying at the right places with elements of A (or better, with elements in C_1). Another, more elegant solution is to extend these maps to $M(A \otimes A)$. We know how to do this for non-degenerate homomorphisms. In a similar way, this can be done for these conditional expectations. Take e.g. an element $m \in M(A)$ and define a multiplier $E(m)$ in $M(C)$ by

$$E(m)c = E(mc)$$
$$cE(m) = E(cm)$$

when $c \in C$. It is easy to verify that $E(m)$ is a well-defined element in $M(C)$. Also E, as a map from $M(A)$ to $M(C)$, satisfies the expected properties (like $E(1) = 1$). In a similar way, one can extend the maps E', $E \otimes \iota$ and $\iota \otimes E'$.

The comultiplication Δ_1 on C_1

The following is an easy consequence of Lemma 3.12.

3.13 Proposition Define Δ_1 on A by

$$\Delta_1(a) = (E \otimes \iota)\Delta(a) = (\iota \otimes E')\Delta(a).$$

Then $\Delta_1(C_1)(1 \otimes C_1) \subseteq C_1 \otimes C_1$ and $\Delta_1(C_1)(C_1 \otimes 1) \subseteq C_1 \otimes C_1$. We also have that Δ_1 is coassociative on C_1.

Proof: We know that $\Delta(a)(A \otimes 1) \subseteq A \otimes C$ when $a \in C$. Because $\Delta_1 = (\iota \otimes E')\Delta$ and $E'(C) \subseteq C_1$ we also get $\Delta_1(a)(A \otimes 1) \subseteq A \otimes C_1$ for all $a \in C$. Similarly, we know that $\Delta(a)(1 \otimes A) \subseteq S(C) \otimes A$ when $a \in S(C)$. And now, because also $\Delta_1 = (E \otimes \iota)\Delta$ and $E(S(C)) \subseteq C_1$ we get $\Delta_1(a)(1 \otimes A) \subseteq C_1 \otimes A$ for all $a \in S(C)$. So

$$\Delta_1(C_1)(1 \otimes A) \subseteq C_1 \otimes A \qquad \text{and} \qquad \Delta_1(C_1)(A \otimes 1) \subseteq A \otimes C_1.$$

Then also

$$\Delta_1(C_1)(1 \otimes C_1) \subseteq C_1 \otimes C_1 \qquad \text{and} \qquad \Delta_1(C_1)(C_1 \otimes 1) \subseteq C_1 \otimes C_1.$$

To prove coassociativity we use that $\Delta_1(a) = (\iota \otimes \varphi_0 \otimes \iota)\Delta^{(2)}(a)$ so that

$$(\iota \otimes \Delta_1)\Delta_1(a) = (\iota \otimes \varphi_0 \otimes \iota \otimes \varphi_0 \otimes \iota)\Delta^{(4)}(a).$$

We get the same right hand side if we calculate $(\Delta_1 \otimes \iota)\Delta_1(a)$.

This map Δ_1 will not be a homomorphism, but it will again be positive. The reason for this is that Δ is a *-homomorphism and that the maps E, E' are positive. It also follows from the expression

$$\Delta_1(a) = (\iota \otimes \varphi_0 \otimes \iota)\Delta^{(2)}(a)$$

and the positivity of φ_0. Also here, as in the case of Δ_0, defined in Proposition 2.3, it can be shown that Δ_1 is completely positive, but again this fact is of no interest for this paper.

We have a *counit* for this new comultiplication on C_1. It is simply the restriction of the original comultiplication ε to C_1. We obtain a *-homomorphism from C_1 to $\mathbb{C}$ still satisfying the axioms for a counit on the new comultiplication. Indeed, when $a \in C_1$ we have

$$\begin{aligned}(\varepsilon \otimes \iota)\Delta_1(a) &= (\varepsilon \otimes \iota)(\iota \otimes E)\Delta(a) \\ &= E((\varepsilon \otimes \iota)\Delta(a)) = E(a) = a\end{aligned}$$

and similarly $(\iota \otimes \varepsilon)\Delta(a) = a$.

The integrals on the pair (C_1, Δ_1) and the antipode.

First, we look for the left and right integrals on the algebra C_1, considered with the positive comultiplication Δ_1. These present no problem because of the following result.

3.14 Lemma The left and right integrals φ and ψ on A are both invariant under E and E'.

Proof: We have

$$\begin{aligned}\psi(E(a)) &= (\psi \otimes \varphi_0)\Delta(a) = \psi(a)\varphi_0(1) = \psi(a) \\ \varphi(E'(a)) &= (\varphi_0 \otimes \varphi)\Delta(a) = \varphi_0(1)\varphi(a) = \varphi(a).\end{aligned}$$

On the other hand, because $\varphi_0(\delta) = \varphi(\delta h) = \varphi(h) = 1$ and also $\varphi_0(\delta^{-1}) = 1$, we have

$$\begin{aligned}\varphi(E(a)) &= (\varphi \otimes \varphi_0)\Delta(a) = \varphi(a)\varphi_0(\delta) = \varphi(a) \\ \psi(E'(a)) &= (\varphi_0 \otimes \psi)\Delta(a) = \varphi_0(\delta^{-1})\psi(a) = \psi(a).\end{aligned}$$

It follows easily from this that the restrictions of φ and ψ to C_1 are still left, resp. right invariant for the new comultiplication. Indeed, let e.g. $a \in C_1$. Then

$$(\varphi \otimes \iota)\Delta_1(a) = (\varphi \circ E \otimes \iota)\Delta(a) = (\varphi \otimes \iota)\Delta(a) = \varphi(a)1.$$

Observe also that the automorphisms σ and σ' leave C_1 globally invariant and so they will still be automorphisms for the restrictions.

Now we look at the *antipode* and its relation with Δ_1 and with the integrals.

We have seen already that S leaves C_1 invariant. Also $E'(S(a)) = S(E(a))$ for all $a \in A$. Then we get, for all $a \in A$,

$$\begin{aligned}\Delta_1(S(a)) &= (\iota \otimes E')\Delta(S(a)) \\ &= (\iota \otimes E')(S \otimes S)\Delta^{\mathrm{op}}(a) \\ &= (S \otimes S)(\iota \otimes E)\Delta^{\mathrm{op}}(a) \\ &= (S \otimes S)\Delta_1^{\mathrm{op}}(a)\end{aligned}$$

as expected. However, we also know that this property does not characterize the antipode completely. We must, as we did also in Section 2, look at the relation with the integrals.

We have the following result.

3.15 Proposition For all $a, b \in C_1$ we have

$$S((\iota \otimes \varphi)(\Delta_1(a)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a)\Delta_1(b)).$$

Proof: Let $a, b \in C_1$. Then

$$\begin{aligned}\Delta_1(a)(1 \otimes b) &= (\iota \otimes E')(\Delta(a))(1 \otimes b) \\ &= (\iota \otimes E')(\Delta(a)(1 \otimes b))\end{aligned}$$

where we use that $b \in S(C)$ and therefore $E'(xb) = E'(x)b$ for all $x \in A$.

Now, apply the left integral φ and use that φ is invariant for E' to get

$$(\iota \otimes \varphi)(\Delta_1(a)(1 \otimes b)) = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)).$$

Similarly we have

$$(\iota \otimes \varphi)((1 \otimes a)\Delta_1(b)) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)).$$

Then the result follows because we already have

$$S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b))$$

for all $a, b \in A$.

Also here it is possible to prove the following result. It is the dual version of Proposition 2.6 of the previous section.

3.16 Proposition We have

$$C_1 = \mathrm{span}\{(\iota \otimes \varphi)(\Delta_1(a)(1 \otimes b)) \mid a, b \in C_1\}.$$

Again, the result follows essentially from the faithfulness of φ and the existence of the counit (cf. the proof of Proposition 2.6).

If we combine the previous results, we arrive at the following theorem, which is a dual version of Theorem 2.7 in the previous section.

3.17 Theorem Let (A, Δ) be an algebraic quantum group and h a group-like projection in A. Let $C_1 = C \cap S(C))$ where C is the right leg of $\Delta(h)$ and let Δ_1 be the coproduct defined on C_1 as in Proposition 3.13. Then (C_1, Δ_1) is an algebraic quantum hypergroup of discrete type (as introduced in [De-VD1]).

It is of discete type because there is a co-integral, namely h itself. In our second paper on this subject [L-VD2], we will see (C_1, Δ_1) is the dual version of the compact quantum hypergroup (A_0, Δ_0) from Theorem 2.7. More precisely, it is in duality with (B_0, Δ_0), defined in a similar way as A_0.

Let us now finish this section with the basic examples.

3.18 Examples i) Let K be a finite subgroup of a (discrete) group G. Consider the group algebra of G and let $h = \frac{1}{n}\sum_{p\in K} \lambda_p$ as in Example 1.2.ii (where n is the number of elements in K). Because Δ is coabelian, the left and right legs of $\Delta(h)$ coincide so that $C_1 = C$. Then also $\Delta_1 = \Delta$ and we are in the situation of Example 3.9.i. The pair (C, Δ) is nothing else but the group algebra of K. A similar situation occurs if we take a compact open subgroup of a locally compact group G and h as in Example 1.3.ii. Now, we will get the *-subalgebra of $C_r^*(K)$ consisting of polynomial functions on K. This is a discrete quantum group.

ii) Again, let K be a finite subgroup of a group G, but consider the algebra A of functions with finite support on G and pointwise operations. Take for h the characteristic function of K as in Example 1.2.i. The right leg C of $\Delta(h)$ is the *-algebra of functions f with finite support on G and such that $f(pk) = f(p)$ for all $p \in G$ and $k \in K$. Similarly, C_1 will be the *-algebra of functions f with finite support on G and such that $f(kpk') = f(p)$ for all $p \in G$ and $k, k' \in K$. The conditional expectations E and E' are given by

$$E(f)(p) = \frac{1}{n}\sum_{k\in K} f(pk)$$
$$E'(f)(p) = \frac{1}{n}\sum_{k\in K} f(kp)$$

where again, n is the number of elements in K, and where $f \in A$ and $p \in G$. The coproduct Δ_1 on C_1 satisfies

$$\Delta_1(f)(p, q) = \frac{1}{n}\sum_{k\in K} f(pkq).$$

It is easily verified that the pair (C_1, Δ_1) is indeed an algebraic quantum hypergroup of discrete type.

iii) In a completely similar way, we can look at the case of a locally compact group G with a compact open subgroup. Now, the algebra A is the algebra of polynomial functions and also h is the characteristic function of K (as in Example 1.3.i). For C we get the *-algebra of continuous complex functions with compact support on G, constant on right K-cosets. For C_1 we get the *-algebra of continous functions with compact support on G, constant on double K-cosets. In the first case, this is the algebra of complex functions with finite support on the qoutient space G/K, while in the second case, this is the algebra of complex functions with finite support on the double quotient space $K\backslash G/K$. The sums in the previous example become (normalized) integrals over K. So, the coproduct Δ_1 on C_1 is given by the formula

$$\Delta_1(f)(p,q) = \int_K f(pkq)\,dk$$

where the normalized Haar measure on K is used.

We can make a remark, similar as Remark 2.10.iv in the previous section. Depending on the point of view, we can think of the algebras C and C_1 either as discrete quantum (hyper) qoutient groups or discrete quantum (hyper) subgroups.

4. Conclusions and further research

In a *forthcoming paper* [L-VD2], we study the objects, obtained in this paper, in duality. This will only be possible for algebraic quantum groups. In Section 1 of this paper, we have already shown that the Fourier transform k of a group-like projection h is again a group-like projection in the dual $(\hat{A}, \hat{\Delta})$ of (A, Δ). Therefore, we can also consider the algebras, associated with the projection k in the dual: $\hat{A}k$, $k\hat{A}k$, D and D_1. This will be done in [L-VD2].

We will look at the relative position of the algebras C and D in the Heisenberg algebra. Here, the Heisenberg algebra is the algebra generated by A and its dual $\hat{A}$ with the Heisenberg commutation relations between elements of A and $\hat{A}$. It turns out that C and D are exactly each others relative commutants:

$$C = \{a \in A \mid ab = ba \text{ for all } b \in D\}$$
$$D = \{b \in \hat{A} \mid ab = ba \text{ for all } a \in C\}.$$

This result is of the same type as a well-known result in the theory of crossed products, cf. [L]. We also prove some result in the other direction. We show that any left invariant *-subalgebra C of A with a non-trivial relative commutant D, must be the right leg of $\Delta(h)$

for a group-like projection $h \in A$. It follows that then D is the relative commutant of C. So, we obtain a one-to-one correspondence between certain left invariant *-subalgebras and group-like projections.

In [L-VD2], we also investigate two types of relations between the objects obtained in Section 2 (the compact quantum (hyper) subgroups) and those obtained in Section 3 (the discrete quantum (hyper) subgroups). One aspect is the duality between those two. The other one is the Fourier transform that maps one object to its dual object.

We will also include more examples in [L-VD2] and illustrate our results using these examples. Related is also the work on algebraic quantum hypergroups [De-VD2].

Further research

In this paper, we have essentially only treated the case of an algebraic quantum group. A few results were also shown to be true for all multiplier Hopf *-algebras and probably more results are correct in this more general situation. It is one possible direction of future research: What can still be done for general multiplier Hopf algebras? In [L-VD2] and [L-VD3], when studying special cases and examples, we also go beyond the case of algebraic quantum groups. It is an indication that some of the objects we studied and results we obtained, can also be studied in the more general setting of locally compact quantum groups. Remember that already in this introduction, we mentioned that algebraic quantum groups serve as a laboratory for work on general locally compact quantum groups.

One of the related problems is that of characterizing the algebraic quantum groups among the general locally compact quantum groups. There is now a solution to this problem in the abelian case (i.e. for $C_0(G)$) and in the dual, coabelian case (i.e. for the dual of $C_0(G)$, the reduced group C*-algebra $C_r^*(G)$ of G), see [L-VD1]. The solution in these two special cases may inspire a possible solution in the general case. A related question that remains open is the following. Is there always a group-like projection in an algebraic quantum group? Observe that the answer is positive in the abelian and the coabelian case.

Another related problem is to find some structure theorem for algebraic quantum groups. What we have in mind is the special case of a locally compact group G with a normal compact open subgroup K. Then G is a cocycle crossed product of K with G/K. Similar constructions exist in the theory of locally compact quantum groups (see [V-V]). One can hope that this work could be used to understand the structure of an algebraic quantum group (more likely even when there exists a group-like projection).

Totally disconnected locally compact groups have a local basis of the identity consisting of compact open subgroups. Taking into account the results in this paper, we propose the following definition of a *totally disconnected locally compact quantum group*. It should be an algebraic quantum group (A, Δ) with the property that there exists a decreasing net $(h_\lambda)_\lambda$ of group-like projections in A such that the associated *-algebras C_λ, the right legs of $\Delta(h_\lambda)$, increase to A. Remark that we have shown in Section 3 that indeed, these algebras will give a increasing net. So, we just require that any element of A belongs to such an algeba C_λ for some λ. It is not so hard to show that in this case, also the associated smaller *-algebras $C_\lambda \cap S(C_\lambda)$ will increase to A. In any case, it is expected that the development of a theory of totally disconnected locally compact quantum groups will depend heavily on the results in this paper.

Finally, not only compact open quantum subgroups of a locally compact quantum groups can be studied, but also compact quantum subgroups, closed quantum subgroups, ... and similar questions can be asked about such subgroups.

Appendix

In this appendix, we will first recall the notion of a multiplier Hopf *-algebra and invariant integrals. Then, we recall some of the main properties and in particular, we look at the concept of the Fourier transform. All these things are available in the literature (cf. the references given in the introduction). We also introduce the notion of the legs of a comultiplication as this concept is used in this paper. Finally we discuss some properties of left invariant algebras as they also appear in this work.

Multiplier Hopf *-algebras and algebraic quantum groups

Let A be an algebra over $\mathbb{C}$, possibly without identity, but always with a non-degenerate product. This means that $b = 0$ whenever either $ab = 0$ for all a or $ba = 0$ for all a. This property is automatic when the algebra has an identity. The *multiplier algebra* $M(A)$ of an algebra A can be characterized as the largest algebra with identity, containing A as a two-sided ideal with the property that, if $x \in M(A)$ then $x = 0$ if either $xa = 0$ for all $a \in A$ or $ax = 0$ for all $a \in A$. If A is a *-algebra, then $M(A)$ is also a *-algebra. The typical example to have in mind is the algebra of complex functions with finite support on a set (with pointwise operations). In that case, the multiplier algebra is the algebra of all complex functions on this set. For a precise treatment of the multiplier algebra of an algebra, we refer to [VD3].

If A is an algebra with a non-degenerate product, the tensor product $A \otimes A$ of A with itself is again an algebra with a non-degenerate product. We have natural inclusions

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

In most cases, these two inclusions are strict (when the algebra has no identity). A *comultiplication* on A is a homomorphism $\Delta : A \to M(A \otimes A)$ satisfying certain properties. If A has an identity, it is natural to assume that $\Delta(1) = 1 \otimes 1$. If the algebra has no identity, this condition is replaced by non-degeneracy of the comultiplication. This means that

$$\Delta(A)(A \otimes A) = (A \otimes A)\Delta(A) = A \otimes A.$$

Then, Δ can be uniquely extended to a homomorphism from $M(A)$ to $M(A \otimes A)$ and this extension (still denoted by Δ) will be unital. We can consider the homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ from $A \otimes A$ to $M(A \otimes A \otimes A)$ where ι is used to denote the identity map from

A to itself. These two homomorphisms are again non-degenerate and can be extended to $M(A \otimes A)$. If again we use the same symbols for these extensions, it makes sense to require coassociativity for Δ in the form $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. If A is a *-algebra, also $A \otimes A$ is a *-algebra and in this case, we require Δ to be a *-homomorphism.

Here, the motivating example is the algebra A of complex functions with finite support on a group G. In this case, Δ is defined on A by $(\Delta(f))(p, q) = f(pq)$ whenever $f \in A$ and $p, q \in G$. One has to consider elements in $M(A \otimes A)$ as complex functions on $G \times G$. Coassociativity is an easy consequence of the associativity of the product in G.

We are now ready to recall the definition of a multiplier Hopf algebra (cf. [VD3]).

A.1 Definition A *multiplier Hopf (*-)algebra* is a (*-)algebra A with a non-degenerate product and a comultiplication Δ satisfying certain conditions: It is assumed that the two linear maps, defined from $A \otimes A$ to $M(A \otimes A)$ by

$$a \otimes b \mapsto \Delta(a)(1 \otimes b) \qquad\qquad a \otimes b \mapsto (a \otimes 1)\Delta(b),$$

are injective, map into $A \otimes A$ and actually have all of $A \otimes A$ as their range. A multiplier Hopf algebra is called *regular* if the opposite comultiplication Δ^{op}, obtained by composing Δ with the flip, also satisfies these properties.

In the case of a *-algebra, regularity is automatic.

Again the motivating example is the algebra of complex functions with finite support on a group G with the comultiplication Δ defined as before. The necessary conditions on Δ are an easy consequence of the fact that the maps

$$(p, q) \mapsto (pq, q) \qquad\qquad (p, q) \mapsto (p, pq)$$

are bijective from $G \times G$ to itself in the case of a group.

For a multiplier Hopf algebra (A, Δ), we have the existence of a unique *counit* and a unique *antipode* as in the following proposition.

A.2 Proposition There exists a unique homomorphism ε from A to $\mathbb{C}$, called the counit, satisfying

$$(\varepsilon \otimes \iota)\Delta(a) = a \qquad\qquad (\iota \otimes \varepsilon)\Delta(a) = a$$

for all $a \in A$. We give a meaning to these equations by multiplying, left or right, with elements in A. There also exists a unique anti-homomorphism S from A to $M(A)$, called the antipode, satisfying

$$m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \qquad\qquad m(\iota \otimes S)\Delta(a) = \varepsilon(a)1$$

for all $a \in A$, where m stands for the multiplication on A, viewed as a linear map from $M(A) \otimes A$ or $A \otimes M(A)$ to A. Again one multiplies with an element in A to give a meaning to these formulas. If A is a regular multiplier Hopf algebra, then S maps into

A and it is bijective. If A is a multiplier Hopf *-algebra, then ε is a *-homomorphism while S satisfies $S(a^*) = (S^{-1}(a))^*$ for all a.

Any Hopf (*-)algebra is a multiplier Hopf (*-)algebra. Conversely, if (A, Δ) is a multiplier Hopf (*-)algebra and if A has an identity, it is a Hopf (*-)algebra. So we see that the theory of multiplier Hopf algebras extends in a natural way the theory of Hopf algebras to the case where the underlying algebras are not required to have an identity.

For the theory of multiplier Hopf (*-)algebras, we refer to [VD3] and [VD-Z3].

Now, let (A, Δ) be a regular multiplier Hopf algebra.

A.3 Definition A linear functional φ on A is called left invariant if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. Again ι stands for the identity map and the equation is given a meaning by multiplying with any $b \in A$ from the left. A *left integral* is a non-zero left invariant linear functional on A. Similarly, a right invariant linear functional on A is defined and it is called a *right integral* if it is non-zero.

Such integrals on regular multiplier Hopf algebras are unique (up to a scalar) and if a left integral φ exists, also a right integral exists (namely $\varphi \circ S$). Integrals are automatically faithful. This means (for the left integral φ) that $\varphi(ab) = 0$ for all a will imply $b = 0$ and similarly when $\varphi(ba) = 0$ for all a.

If (A, Δ) is a multiplier Hopf *-algebra with a positive left integral, then there is also a positive right integral. This result is non-trivial. It was first shown in [K-VD] but recently, a simpler proof has been obtained in [DC-VD]. We will use the term *algebraic quantum group* for such a multiplier Hopf *-algebra. In this paper, we mainly deal with algebraic quantum groups, defined in this sense (although probably, some of the results will still be true without the assumption of positivity of the integrals).

We refer to [VD7] and [VD-Z3] for details and we will freely use results from these basic references in this paper. In particular, we use φ to denote a left integral and we use ψ for a right integral. We use σ for the *modular automorphism* of φ, satisfying $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. Similarly σ' is used for the modular automorphism of ψ. The *modular element* δ is a multiplier in $M(A)$, defined and characterized by $(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta$ for all $a \in A$. It is invertible and the inverse satisfies $(\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$. It is also characterized by the formula $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$. Finally, there is the *scaling constant* ν defined by $\varphi(S^2(a)) = \nu\varphi(a)$ where S is the antipode. Recently, it is shown in [DC-VD] that this scaling constant is trivial ($\nu = 1$) for any multiplier Hopf *-algebra with positive integrals. There are however examples with non-trivial scaling constant if positivity of the integrals is not assumed.

There are many formulas relating these various objects and they can be found in the basic references. Nevertheless, let us recall the most important onces, used in this paper.

In the first proposition below, we formulate a well-known relation of the antipode with the left and the right integrals, important for this paper (see e.g. the proof of Proposition 3.11 in [VD7]).

A.4 Proposition For any $a, b \in A$, when

$$x = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) \qquad \text{then} \qquad S(x) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)).$$

Similarly, when

$$y = (\psi \otimes \iota)((b \otimes \iota)\Delta(a)) \qquad \text{then} \qquad S(y) = (\psi \otimes \iota)(\Delta(b)(a \otimes 1)).$$

Furthermore, we have the following various formulas.

A.5 Proposition We have $\varepsilon(\delta) = 1$, $S(\delta) = \delta^{-1}$ and $\sigma(\delta) = \sigma'(\delta) = \nu^{-1}\delta$. Furthermore $\sigma'(a) = \delta\sigma(a)\delta^{-1}$ for all a and $S\sigma' = \sigma^{-1}S$. All the automorphisms S^2, σ and σ' mutually commute and we have

$$\Delta(\sigma(a)) = (S^2 \otimes \sigma)\Delta(a)$$
$$\Delta(\sigma'(a)) = (\sigma' \otimes S^{-2})\Delta(a)$$

for all a in A. In the case of a multiplier Hopf *-algebra, we have $\sigma(a)^* = \sigma^{-1}(a^*)$ for all a.

All these formulas can be found already in the original paper [VD7]. There is also the following result. It was first proven in [K-VD], Lemma 3.10, but we will give here another (new and simpler) argument.

A.6 Proposition For all $a \in A$ we have

$$\Delta(S^2(a)) = (\sigma \otimes \sigma'^{-1})\Delta(a).$$

Proof: Apply ε on the second leg in the formula for $\Delta \circ \sigma$ to get

$$S^{-2}\sigma(a) = (\iota \otimes \varepsilon \circ \sigma)\Delta(a).$$

Next apply ε on the first leg in the formula for $\Delta \circ \sigma'$ to obtain

$$S^2\sigma'(a) = (\varepsilon \circ \sigma' \otimes \iota)\Delta(a).$$

Because $\varepsilon(\delta) = 1$, it follows from $\sigma'(a) = \delta\sigma(a)\delta^{-1}$ that $\varepsilon \circ \sigma = \varepsilon \circ \sigma'$. Therefore

$$(\iota \otimes \varepsilon \circ \sigma \otimes \iota)\Delta^{(2)}(a) = (\iota \otimes \varepsilon \circ \sigma' \otimes \iota)\Delta^{(2)}(a)$$

and if we use the two previous formulas, we get from this that

$$(S^{-2}\sigma \otimes \iota)\Delta(a) = (\iota \otimes S^2\sigma')\Delta(a).$$

Because $\Delta(S^2(a)) = (S^2 \otimes S^2)\Delta(a)$, we get the desired formula.

For an algebraic quantum group, we have the existence of *local units* as in the following proposition.

A.7 Proposition Let (A, Δ) be an algebraic quantum group. For any finite number of elements $\{a_1, a_2, \dots, a_n\}$ in A there exists an element $e \in A$ such that $a_j e = a_j$ and $e a_j = a_j$ for all j.

The proof of this result is found in [Dr-VD-Z] and the result is used a few times in this paper (in the proof of Proposition 3.7 and further in this appendix).

In this paper, also the following special cases of algebraic quantum groups play a role.

A.8 Definition Let (A, Δ) be an algebraic quantum group. It is called of *compact type* if the algebra A has an identity. It is called of *discrete type* if there is a left co-integral.

Recall that a left co-integral is a non-zero element h in A such that $ah = \varepsilon(a)h$ for all a in A. If a left co-integral exists, there is also a right co-integral k satisfying $ka = \varepsilon(a)k$, namely $k = S(h)$. If co-integrals exist, they are also unique up to a scalar.

The dual of an algebraic quantum group and the Fourier transform

For an algebraic quantum group (A, Δ), one can define a dual object $(\widehat{A}, \widehat{\Delta})$. As a vector space, $\widehat{A} = \{\varphi(\,\cdot\, a) \mid a \in A\}$. Because of the existence of σ, it is also true that $\widehat{A} = \{\varphi(a\,\cdot\,) \mid a \in A\}$. Moreover, in these two cases, the left integral φ can be replaced by the right integral ψ (by applying the antipode). The product on $\widehat{A}$ is dual to the coproduct on A while the coproduct $\widehat{\Delta}$ is dual to the product in A. It is again an algebraic quantum group. The dual right integral $\widehat{\psi}$ is given by the formula $\widehat{\psi}(\omega) = \varepsilon(a)$ when $\omega = \varphi(\,\cdot\, a)$. Observe that the dual of a compact type is of discrete type and vice versa.

In this paper, we use B for $\widehat{A}$ and we equip B with the opposite coproduct $\widehat{\Delta}^{\mathrm{op}}$. We will now denote this coproduct on B by Δ. We get a dual pair of multiplier Hopf algebras (A, B) which is somewhat different than in the paper [Dr-VD]. The pairing is denoted by $\langle a, b\rangle$ for $a \in A$ and $b \in B$. It is one of the consequences of this choice that $\langle S(a), b\rangle = \langle a, S^{-1}(b)\rangle$ for all $a \in A$ and $b \in B$. The dual right integral $\widehat{\psi}$ becomes the left integral on B. The reason for making this choice is to make notations in accordance with the ones that are common in the general theory of locally compact quantum groups as developed in [K-V1] and [K-V2].

It is important to notice that the Sweedler notation can safely be used, also in the theory of multiplier Hopf algebras. One can write $a_{(1)} \otimes a_{(2)}$ for $\Delta(a)$ provided we have a *covering* of the first or the second factor, either through a product, from the left or the right, by an element in the algebra, or through a pairing with an element in the dual algebra B. We have used the Sweedler notation e.g. in the proofs of Proposition 1.6 and 1.8 of Section 1 (and also further in the paper). We refer to [Dr-VD] and [Dr-VD-Z] for more information about the use of the Sweedler notation and this technique of covering (see also [VD-Z3]).

All the objects that we have introduced for A, like δ, σ and σ', also exist for B. We will use the same symbols in most cases as there will be no confusion. Only for the modular elements, we have to be a little more attentive. Remark that in general, the scaling constant for B is ν^{-1} when ν is the scaling constant for A. This is not so important for this paper because the algebraic quantum groups that are studied here have a trivial scaling constant (see Proposition 1.7 and following comments).

We will not only have all the formulas for these objects in B as we have for A, we also get some new ones, relating the various objects for A with those of B. Many of these results can be found e.g. in [Ku]. We collect some of them in the following proposition. The results are related with the proof we have given before of Proposition A.6.

A.9 Proposition For the modular element δ in B we have $\langle a, \delta\rangle = \varepsilon(\sigma(a)) = \varepsilon(\sigma'(a))$ and $\langle a, \delta^{-1}\rangle = \varepsilon(\sigma^{-1}(a)) = \varepsilon({\sigma'}^{-1}(a))$ for all $a \in A$. Furthermore, for the modular automorphism on B, we get $\langle a, \sigma(b)\rangle = \langle S^2(a)\delta, b\rangle$ and $\langle a, \sigma^{-1}(b)\rangle = \langle S^{-2}(a)\delta^{-1}, b\rangle$ for $a \in A$ and $b \in B$.

Proof: Let $a, a' \in A$ and $b = \varphi(\,\cdot\, a)$ so that $\varphi(b) = \varepsilon(a)$. If we let $b' = \langle a', b_{(2)}\rangle b_{(1)}$, then one can calculate that $b' = \varphi(\,\cdot\, a\sigma(a'))$ and so $\varphi(b') = \varepsilon(a)\varepsilon(\sigma(a'))$. Then, from the formula $(\varphi \otimes \iota)\Delta(b) = \varphi(b)\delta$, it will follow that $\langle a', \delta\rangle = \varepsilon(\sigma(a'))$. It follows from the relation $\sigma'(a) = \delta\sigma(a)\delta^{-1}$ (see Proposition A.5) and $\varepsilon(\delta) = 1$ that $\varepsilon(\sigma(a)) = \varepsilon(\sigma'(a))$ for all a and this proves the first part of the proposition.

We will prove the second part of the proposition first in its dual form. We start with the formula $S^{-2}\sigma(a) = (\iota\otimes\varepsilon\circ\sigma)\Delta(a)$ (cf. the proof of A.6). Pairing with an element $b \in B$ and combining with the first result in the proposition, we get precisely $\langle S^{-2}\sigma(a), b\rangle = \langle a, b\delta\rangle$ for all b. Moving S to the other side, we get $\langle \sigma(a), b\rangle = \langle a, S^{-2}(b)\delta\rangle$ for all b. If we dualize this formula, we get $\langle a, \sigma(b)\rangle = \langle S^2(a)\delta, b\rangle$, proving the first part of the desired result. The second part is obtained in a similar way, but it also follows easily from the first formula.

We must make a few comments before we continue. First, the formula $\langle a, \delta\rangle = \varepsilon(\sigma(a))$ involves a pairing between an element in A and an element in $M(B)$. This can be given a meaning, but the formula can also be interpreted by considering the second form $\langle a', b\rangle = \langle a, b\delta\rangle$ with $a' = (\iota \otimes \varepsilon \circ \sigma)\Delta(a)$.

Similarly, we find formulas involving σ' (using e.g. that $S\sigma' = \sigma^{-1}S$).

Let us now formulate some results about the so-called Fourier transform in this context. These results are not so essential for this paper, but they are of great importance in the second paper [L-VD2].

The *Fourier transform* F is considered here as a map from A to B and it is defined by $F(a) = \varphi(\,\cdot\, a)$. Let us fix the left integral φ on B with the formula given before and so $\varphi(b) = \varepsilon(a)$ when $b = F(a)$. If $b = F(a)$, then it is not hard to show that $a = \varphi(S^{-1}(\,\cdot\,)b)$ giving the formula for the inverse transform. Indeed, let $a \in A$ and $b = \varphi(\,\cdot\, a)$. If $x \in A$ and $y \in B$ we find

$$\begin{aligned}\langle x, S^{-1}(y)b\rangle &= \langle x_{(1)}, S^{-1}(y)\rangle\varphi(x_{(2)}a)\\ &= \langle a_{(1)}, y\rangle\varphi(xa_{(2)})\end{aligned}$$

using Proposition A.4. Then, using the definition of φ on B, we find

$$\varphi(S^{-1}(\,\cdot\,)b) = \varepsilon(a_{(2)})a_{(1)} = a.$$

In the case of a *-algebra, when we assume a positive left integral φ on A, we get $\varphi(b^*b) = \varphi(a^*a)$ when $b = \varphi(\,\cdot\, a)$. Indeed, using the result above, we get

$$\begin{aligned}\varphi(b^*b) &= \varphi(S^{-1}(S(b^*))b) = \langle a, S(b^*)\rangle \\ &= \langle S^{-1}(a), b^*\rangle = \langle a^*, b\rangle^- = \varphi(a^*a).\end{aligned}$$

This is 'Plancherel's formula'. It shows that the dual left integral on B is positive when the left integral on A is positive.

Finally, observe that, if A has an identity (i.e. when it is of compact type), then we have that the left integral φ on A is in B. It is clearly a left co-integral in B (and so B is of discrete type). On the other hand, if A is of discrete type and if h is a left co-integral in A, we see easily that $\varphi(\,\cdot\, h)$ will be the identity in B (provided $\varphi(h) = 1$). Remark that $\varphi(h)$ can not be 0 as this would imply that $\varphi(ah) = 0$ for all a and this contradicts the fact that φ is faithful. Hence, B is of compact type.

The left and right legs of $\Delta(a)$ for $a \in A$

In our paper, we use the notion of the legs of $\Delta(a)$ for a in a regular multiplier Hopf algebra. We will discuss this concept here.

Again, let (A, Δ) be a regular multiplier Hopf algebra.

We will first say what we precisely mean by 'the right leg' (respectively 'the left leg') of $\Delta(a)$ for a single element $a \in A$ or for a belonging to a subspace $V \subseteq A$. We begin with a lemma that prepares for this definition.

A.10 Lemma Let $a \in A$ and assume that C is a subspace of A. Then, the following are equivalent:

i) $\Delta(a)(a' \otimes 1) \in A \otimes C$ for all $a' \in A$,

ii) $(a' \otimes 1)\Delta(a) \in A \otimes C$ for all $a' \in A$.

Proof: Suppose i) holds. Take $a' \in A$ and write $(a' \otimes 1)\Delta(a) = \sum a_i \otimes b_i$. Choose $e \in A$ such that $a_i e = a_i$ for all i (cf. Proposition A.7). Then

$$\begin{aligned}(a' \otimes 1)\Delta(a) &= (a' \otimes 1)\Delta(a)(e \otimes 1) \\ &\subseteq (a' \otimes 1)(A \otimes C) \subseteq A \otimes C.\end{aligned}$$

Therefore ii) holds. Similarly, ii) implies i) and this proves the lemma.

A.11 Definition The smallest subspace C with the properties in the lemma will be called *the right leg* of $\Delta(a)$. In a similar way, we can define the left leg of $\Delta(a)$ for a single element $a \in A$, as well as the left and right legs of $\Delta(V)$ for a subspace V of A.

It is immediatley clear that the legs of a self-adjoint element and of a self-adjoint subspace (in the case of a multiplier Hopf *-algebra) are again self-adjoint subspaces.

In the case of an algebraic quantum group, we can prove a bit more.

A.12 Lemma Let (A, Δ) be an algebraic quantum group and let B be the dual. Let $a \in A$ and let C be a subspace of A. Then condition i) and ii) in Lemma A.10 are also equivalent with the following condition:

iii) $\langle a_{(1)}, b\rangle a_{(2)} \in C$ for all $b \in B$.

Proof: Assume that conditon i) of Lemma A.10 holds. Take any $a' \in A$ and apply the right integral ψ on the first leg of $\Delta(a)(a' \otimes 1)$. We see that $(\psi(\,\cdot\, a') \otimes \iota)\Delta(a) \in C$ and so $\langle a_{(1)}, b\rangle a_{(2)} \in C$ for all $b \in B$. This proves iii). Conversely, assume iii) and take $a' \in A$. Write $\Delta(a)(a' \otimes 1) = \sum p_i \otimes q_i$ with the (p_i) linearly independent. By assumption, $\sum \psi(p_i x)q_i \in C$ for all $x \in A$. Let ω be a linear functional on A such that $\omega|_C = 0$. Then $\sum \psi(p_i x)\omega(q_i) = 0$ for all $x \in A$ and by the faithfulness of ψ, we have $\sum \omega(q_i)p_i = 0$. As the (p_i) are assumed to be linearly independent, we must have $\omega(q_i) = 0$ for all i. Hence $q_i \in C$ for all i. So $\Delta(a)(a' \otimes 1) \in A \otimes C$. Therefore also iii) implies condition i) of Lemma A.10.

We see that condition i) of Lemma A.10 is equivalent with iii) and so the lemma is proved.

So in the case of an algebraic quantum group, the right leg of $\Delta(a)$ can also be characterized as the smallest subspace C of A such that $\langle a_{(1)}, b\rangle a_{(2)} \in C$ for all $b \in B$.

Left invariant subalgebras

As before let (A, Δ) be a regular multiplier Hopf algebra. Using the notion of the right leg, we can define the following:

A.13 Definition Let C be a subalgebra of A. Then it is called *left invariant* if the right leg of $\Delta(C)$ belongs to C.

We call it *left* invariant because $(\omega \otimes \iota)\Delta(C) \subseteq C$ for appropriate linear functionals ω on A.

We now prove some useful properties of such left invariant subalgebras.

A.14 Proposition Let C be a (non-trivial) left invariant subalgebra of a regular multiplier Hopf algebra. Then the product in C is (still) non-degenerate.

Proof: Suppose that $b \in C$ and that $ba = 0$ for all $a \in C$. Then $(1 \otimes b)\Delta(a)(a' \otimes 1) = 0$ for all $a \in C$ and $a' \in A$. So, $(1 \otimes b)\Delta(a) = 0$ for all $a \in C$. If C is non-trivial, this already implies that $b = 0$ (cf. Definition A.1). Similarly on the other side.

There is even a stronger result. Also for elements $a \in A$ we have that $a = 0$ if $ac = 0$ for all $c \in C$ and similarly on the other side (the proof is as above). This means that A is a *non-degenerate* C-bimodule in the sense of [Dr-VD-Z].

Because the product in C is non-degenerate, it makes sense to consider the multiplier algebras $M(C)$ and $M(A \otimes C)$. It also follows that $\Delta(C) \subseteq M(A \otimes C)$ for a left invariant subalgebra. It is not clear however if, for a general subalgebra C (with a non-degenerate product) of A, this condition will be sufficient to guarantee that we also have the stronger property, namely that the right leg of $\Delta(C)$ is contained in C. Indeed, assume that C is a subalgebra of A with a non-degenerate product and such that $\Delta(C) \subseteq M(A \otimes C)$. Then it is not hard to see that the second leg of $\Delta(C)$ belongs to the set C_1, defined by

$$C_1 = \{x \in A \mid xa \in C \text{ and } ax \in C \text{ for all } a \in C\}.$$

In general, it is not clear if C_1 is actually contained in C. We say a little more about this after the next proposition.

In the following proposition, we show that we always have local units in a left invariant subalgebra.

A.15 Proposition Let (A, Δ) be an algebraic quantum group and let C be a (non-trivial) left invariant *-subalgebra of A. Given elements $a_1, a_2, \ldots, a_n$ in A, there exists an element $e \in C$ so that $a_i e = a_i$ for all i.

Proof: Take any element b in C such that $\psi(b) \neq 0$ where, as before, ψ is a positive right integral. This is possible because C is a (non-trivial) *-subalgebra and ψ is a faithful positive linear functional. Next choose $p \in A$ so that

$$(1 \otimes a_i)\Delta(b)(p \otimes 1) = (1 \otimes a_i)\Delta(b)$$

for all i. Then apply the right invariant integral ψ on the first leg of this equation. This will give $a_i e = a_i$ for all i with $e = (\psi \otimes \iota)(\Delta(b)(p \otimes 1))$. Then $e \in C$ and the result is proven.

A consequence of this result is that $AC = A$. Similarly $CA = A$. In other words, A is a *unital* C-bimodule, as defined in [Dr-VD-Z]. It follows that the imbedding of C in A extends to a *-homomorphism of $M(C)$ to $M(A)$. This extension will be injective and so we may view $M(C)$ as sitting inside $M(A)$.

We now come back to a problem discussed earlier (preceding Proposition A.15). If C is a *-subalgebra satisfying $\Delta(C) \subseteq M(A \otimes C)$ and if A has local units in C, then C will be left invariant in the sense of Definition A.13. Indeed, we have seen that the right leg of $\Delta(C)$ will consist of elements $x \in A$ satisfying $xC \subseteq C$. Now we can choose $e \in C$ such that $xe = x$. So $x \in C$.

In Section 3 of this paper, we work with such left invariant *-subalgebras. Because nowhere else in the literature, this concept is defined properly for multiplier Hopf algebras, we have included such a precise definition here in this appendix.

References

[A] E. Abe: *Hopf algebras.* Cambridge University Press (1977).

[C-V] Y.A. Chapovsky & L.I. Vainerman: *Compact quantum hypergroups.* J. Operator Theory 41 (1999), 261–289.

[DC-VD] K. De Commer & A. Van Daele. Multiplier Hopf algebras imbedded in C*-algebraic quantum groups. Preprint K.U. Leuven (2006). Arxiv math.OA/0611872.

[De-VD1] L. Delvaux & A. Van Daele: *Algebraic quantum hypergroups.* Preprint University of Hasselt and K.U. Leuven (2006). Arxiv math.RA/0606466.

[De-VD2] L. Delvaux & A. Van Daele: *Algebraic quantum hypergroups. Examples and special cases.* Preprint University of Hasselt and K.U. Leuven (in preparation).

[Dr-VD] B. Drabant & A. Van Daele: *Pairing and the quantum double of multiplier Hopf algebras.* Algebras and representation theory 4 (2001), 109-132.

[Dr-VD-Z] B. Drabant, A. Van Daele & Y. Zhang: *Actions of Multiplier Hopf Algebras.* Commun. in Alg. 27(9) (1999), 4117-4127.

[G-H-J] F.M. Goodman, P. de la Harpe & V.F.R Jones: *Coxeter graphs and towers of algebras.* MSRI publications 14 (1989) Springer, New York.

[Kr] A. Krieg: *Hecke algebras.* Mem. Amer. Math. Soc. **87** (1990) no.435.

[Ku] J. Kustermans: *The analytic structure of algebraic quantum groups.* J. of Alg. **259** (2003), 415–450.

[K-V1] J. Kustermans & S. Vaes: *A simple definition for locally compact quantum groups.* C.R. Acad. Sci., Paris, Sér. I 328 (10) (1999), 871-876.

[K-V2] J. Kustermans & S. Vaes: *Locally compact quantum groups.* Ann. Sci. Ec. Norm. Sup. 33 (2000), 837-934.

[K-V3] J. Kustermans & S. Vaes: *Locally compact quantum groups in the von Neumann algebra setting.* Math. Scand. **92** (2003),68–92.

[K-VD] J. Kustermans & A. Van Daele: *C*-algebraic quantum groups arising from algebraic quantum groups.* Int. J. Math. 8 (1997), 1067-1139.

[L] M. B. Landstad: *Duality for covariant systems.* Trans. Amer. Math. Soc. 248 (1979) 223-267.

[L-VD1] M. B. Landstad & A. Van Daele: *Groups with compact open subgroups and multiplier Hopf *-algebras.* Preprint University of Trondheim and K.U. Leuven (2007). Arxiv math.OA/0701525

[L-VD2] M. B. Landstad & A. Van Daele: *Compact and discrete subgroups of algebraic quantum groups II.* Preprint University of Trondheim and K.U. Leuven (2007).

[S] M.E. Sweedler: *Hopf algebras.* Mathematical Lecture Note Series. Benjamin (1969).

[V-V] S. Vaes & L. Vainerman: *Extensions of locally compact quantum groups and the bicrossed product construction.* Adv. in Math. 17 (2003), 1–101 .

[VD1] A. Van Daele: *Multiplier Hopf algebras.* Trans. Amer. Math. Soc. 342 (1994), 917-932.

[VD2] A. Van Daele: *Discrete quantum groups.* J. of Alg. 180 (1996), 431-444.

[VD3] A. Van Daele: *An algebraic framework for group duality.* Adv. in Math. 140 (1998), 323–366.

[VD4] A. Van Daele: *The Haar measure on some locally compact quantum groups.* Preprint K.U. Leuven (2001). Arxiv math.OA/0109004.

[VD5] A. Van Daele: *Multiplier Hopf *-algebras with positive integrals: A laboratory for locally compact quantum groups.* Proceedings of the Meeting of Theoretical Physicists and Mathematicians, Strasbourg, February 21 - 23, 2002., Ed. L. Vainerman, IRMA Lectures on Mathematics and Mathematical Physics, Walter de Gruyter, Berlin, New York (2003), pp. 229–247.

[VD6] A. Van Daele: *Locally compact quantum groups: The von Neumann algebras versus the C^*-algebra approach.* Bulletin of Kerala Mathematics Association. Special issue (2006), 153–177 .

[VD7] A. Van Daele: *Locally compact quantum groups. A von Neumann algebra approach.* Preprint K.U. Leuven (2006). Arxiv math.OA/0602212

[VD-Z] A. Van Daele & Y. Zhang: *A survey on multiplier Hopf algebras.* Proceedings of the conference in Brussels on Hopf algebras. Hopf Algebras and Quantum Groups, eds. Caenepeel/Van Oystaeyen (2000), 269-309. Marcel Dekker (New York).

[W] S.L. Woronowicz: *Compact Quantum Groups.* Quantum symmetries/Symmétries quantiques. Proceedings of the Les Houches summer school 1995, North-Holland, Amsterdam (1998), 845–884.