GENERALISED HECKE ALGEBRAS AND C^*-COMPLETIONS

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Abstract. For a Hecke pair (G, H) and a character σ of H we consider a generalised Hecke algebra $\mathcal{H}_\sigma(G, H)$, which we study by embedding the given Hecke pair in a Schlichting completion (G_σ, H_σ) that comes equipped with a continuous extension σ on H_σ. The image of σ in $C_c(G_\sigma)$ is a non-full projection p_σ, and $\mathcal{H}_\sigma(G, H)$ is isomorphic to $p_\sigma C_c(G_\sigma)p_\sigma$. We study the structure and properties of C^*-completions of a generalised Hecke algebra arising from this corner realisation, and via Morita-Rieffel equivalence we identify, in some cases explicitely, the resulting proper ideals of $C^*(G_\sigma)$. By letting σ vary, we can compare these ideals. Applications include $ax + b$-groups and the Heisenberg group.

Introduction

A Hecke pair (G, H) consists of a group G with a subgroup H such that $L(x) := [H : H \cap xHx^{-1}] < \infty$ for all $x \in G$. Suppose that (G, H) is a Hecke pair and σ is a character of H such that $\sigma(H)$ is finite. Our interest lies in studying a *generalised Hecke algebra* $\mathcal{H}_\sigma(G, H)$ of the triple (G, H, σ). As a vector space, $\mathcal{H}_\sigma(G, H)$ consists of functions $f : G \to \mathbb{C}$ which have finite support in $H\backslash G/H$, and satisfy

$$f(hxk) = \sigma(h)f(x)\sigma(k) \text{ for all } h, k \in H, x \in G.$$

When the group is locally compact, totally disconnected, and the subgroup is compact and open, such algebras (allowing general representations σ), endowed with a natural convolution, play a fundamental role in the representation theory of reductive p-adic groups, see for example [13].

When σ is the trivial character, $\mathcal{H}_\sigma(G, H)$ is the *Hecke algebra* $\mathcal{H}(G, H)$ of the pair (G, H), see [16]. With an appropriate involution, $\mathcal{H}_\sigma(G, H)$ becomes a $*$-algebra. Our goal is to study existence and structure of C^*-completions of $\mathcal{H}_\sigma(G, H)$, and to identify conditions which ensure that a largest C^*-completion exists. That this issue is important and non-trivial was demonstrated by Hall, who in [11] gave an example of a Hecke pair (G, H) such that $\mathcal{H}(G, H)$ does not have a largest C^*-completion. We find it natural to investigate the existence of C^*-completions in a more general context. Our main objective is to show how the strategy developed in [15] to study C^*-completions of $\mathcal{H}(G, H)$ can be carried over to the generalised Hecke algebra $\mathcal{H}_\sigma(G, H)$ of a triple (G, H, σ).

The interesting structure and properties of the enveloping C^*-algebra of the Hecke algebra of a Hecke pair introduced by Bost and Connes in [4] have motivated intense research devoted to the study of Hecke C^*-algebras of large classes of Hecke pairs, see for example [20, 1, 5, 20, 11, 10, 29, 22, 18, 15, 6]. A powerful tool to analyse $\mathcal{H}(G, H)$ is the "Schlichting completion" $(\overline{G}, \overline{H})$ of (G, H): this is a new Hecke pair consisting of a locally compact group

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Date: September 1, 2006.

This research was supported by the Norwegian Research Council.

and a compact open subgroup [29]. Then $\mathcal{H}(G,H)$ is isomorphic to $C_c(\overline{H}\backslash\overline{G}/\overline{H})$, which is a corner of the group algebra $C_c(\overline{G})$, and this viewpoint facilitates the analysis of C^*-completions in the realm of Banach $*$-algebras. Tzanev's construction of $(\overline{G},\overline{H})$ was inspired by work of Schlichting, see for example [27], and was reviewed in [15], where it is employed to study C^*-completions in general terms, see [17, 10, 22] for other approaches.

In [8], Curtis considers a Hecke algebra of a triple (G,H,σ) where σ is a finite dimensional unitary representation of H, and studies a von Neumann algebra naturally associated to it. As a convolution space, her algebra is defined similarly to $\mathcal{H}_\sigma(G,H)$, but as a $*$-algebra, it is different. Curtis constructs a completion of (G,H,σ), but does not prove that it is unique.

We concentrate our attention to characters of H with finite range. One reason for so doing is that we can produce a Schlichting completion of (G,H,σ) entirely in terms of the Schlichting completion of a new Hecke pair, thus having the theory of [29, 15] at our disposal, and another is that many interesting examples fit this situation, cf. Section 5. With our extended theory we use both Fell's and Rieffel's versions of Morita equivalence to analyse the structure of the C^*-completions in relation with σ. It turns out that we compare corners of $C_c(L)$ for L a locally compact, totally disconnected group, and these corners are determined by projections that are only full when σ is the trivial character. So for non-trivial σ, which is the case of interest here, we obtain proper ideals, whose structure we are able to characterise in great detail in the case of the p-adic and the full rational "$ax+b$"-group, the Heisenberg group and the lamplighter group. For example, the rational "$ax+b$"-group with a character yields ideals built from the primitive ideals of the Bost-Connes Hecke C^*-algebra identified by Laca and Raeburn in [21].

The organisation of the paper is as follows. In section 1 we construct our Schlichting completion of a Hecke triple (G,H,σ), where σ is a character of H with finite range. We denote this completion by $(G_\sigma,H_\sigma,\sigma)$, and we prove in Theorem 1.4 that it has a universal property, which is essentially provided by the universal property of Schlichting completions of Hecke pairs, see [15, Theorem 3.8]. In section 2 we define the generalised Hecke algebra $\mathcal{H}_\sigma(G,H)$ of (G,H,σ), and we show in Proposition 2.6 that the generalised Hecke algebras of (G,H,σ) and $(G_\sigma,H_\sigma,\sigma)$ are isomorphic $*$-algebras. Similar to [29, 15] in the case of Hecke algebras we realise $\mathcal{H}_\sigma(G,H)$ as a corner of $C_c(G_\sigma)$, but unlike many situations in [15] we obtain a non-full corner for non-trivial σ.

One new ingredient that appears in the study of generalised as opposed to usual Hecke algebras is that not every double coset HxH for x in G supports a non-zero function in $\mathcal{H}_\sigma(G,H)$. The subset B of the x's in G which do support a non-zero function need not be a subgroup of G, but nevertheless it harmonises with the Schlichting completion, cf. Proposition 2.12. If B is a group, $\mathcal{H}_\sigma(G,H)$ coincides with $\mathcal{H}_\sigma(B,H)$.

Section 3 contains an analysis of the continuity properties of the induced representation from H to G with respect to the process of taking the Schlichting completion. To illustrate the point that our Schlichting completion of (G,H,σ) is a profitable alternative to studying $\mathcal{H}_\sigma(G,H)$, we employ it to give a short proof (see Theorem 3.5) of a classical result which asserts that the commutant of the induced representation $\operatorname{Ind}_H^G\sigma(G)$ is the weak closure of the "intertwining operators", cf. [2, Theorem 2.2] (which mends the apparently deficient proof of [7, Theorem 3]) or [9, Proposition 1.3.10]. As an immediate corollary, we describe the irreducibility of $\mathrm{Ind}_H^G \sigma$ in terms of the Hecke algebra, thus recovering Mackey's condition in [23].

In section 4 we study the C^*-completions of $\mathcal{H}_\sigma(G,H)$ that arise naturally from the identification as a corner of $C_c(G_\sigma)$, and by using Morita-Rieffel equivalence, we view these completions as ideals of $C^*(G_\sigma)$. In the presence of a normal subgroup N of G which contains H we describe their structure as twisted crossed products. If in addition H is normal in N and $N \subset B$, we can conclude that $\mathcal{H}_\sigma(G,H)$ has a largest C^*-completion. Using the approach of [15, §5] allows us to describe a category equivalence between appropriate families of representations of $\mathcal{H}_\sigma(G,H)$ and G, see Corollary 4.4. Finally, we study the special, but interesting instance where B is a group and (B,H) is directed in the sense of [15, §5]. A largest C^*-completion turns out to exist, and we can give a concrete description of the ideal in $C^*(G_\sigma)$, see Theorem 4.13.

The last section is devoted to applications. We show in Theorem 5.1 that $\mathcal{H}_\sigma(G,H)$ is isomorphic to $\mathcal{H}(G,H)$ when σ extends to a character of G. Criteria for when a unitary representation of H extends to a unitary representation of G on the same Hilbert space have been recently analysed in, for example, [14]. However, in the examples where we have been able to extend, it has been straightforward to write down a formula for the extended character. We illustrate in examples the multitude of possible outcomes of the construction of the Schlichting completion. In some cases, the Schlichting completion of (G,H,σ) is independent of σ, and yet $\mathcal{H}_\sigma(G,H)$ may or may not coincide with $\mathcal{H}(G,H)$, cf. Example 5.4 and Example 5.14. In Example 5.13 the generalised and the usual Hecke algebras are isomorphic, while the Schlichting completions are not. In Examples 5.5 and 5.17, both the Schlichting completion and $\mathcal{H}_\sigma(G,H)$ depend on σ.

Conventions. For a Hecke pair (G,H) and a subset X of G, the notation $y \in X/H$ (and $y \in H\backslash X/H$) means that y runs over a set of representatives for the left cosets X/H (and the double cosets $H\backslash X/H$).

All characters on topological groups are assumed to be continuous. If L is a locally compact group with a left invariant Haar measure μ and modular function Δ, then the space $C_c(L)$ of compactly supported continuous functions on L is a $*$-algebra with multiplication given by usual convolution $f * g(x) = \int_L f(y)g(y^{-1}x)d\mu(y)$, and involution given by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$. The group C^*-algebra $C^*(L)$ is generated by a universal unitary representation of L into the unitary group of the multiplier algebra $M(C^*(L))$, and $\{\int_L f(x)x d\mu(x) \mid f \in C_c(L)\}$ spans a dense subspace of $C^*(L)$, where we identify x in L with its image in $M(C^*(L))$.

1. Hecke pairs and Schlichting completions

We recall from [15, Definition 3.3] that if (G,M) is a Hecke pair, then the collection $\{xMx^{-1} \mid x \in G\}$ is a neighbourhood subbase for the Hecke topology on G from (G,M). If a Hecke pair (G,M) is such that

$$\bigcap_{x\in G} xMx^{-1} = \{e\}, \tag{1.1}$$

then it is called *reduced* [29], and the Hecke topology from (G,M) is Hausdorff. The *Schlichting completion* of a Hecke pair (G,M) was constructed in [29] to be an essentially unique Hecke pair consisting of a locally compact group with a compact open subgroup in which G and M embedd densely. In the terminology of [15, §3], the Schlichting completion of (G, M) consists of the closures of G and M in the Hecke topology from (G, M). The Schlichting completion of a Hecke pair is a *Schlichting pair*, which by [15, §3] is a reduced Hecke pair with the additional feature that the underlying subgroup is compact and open in its corresponding Hecke topology.

Suppose that (G, H) is a Hecke pair and σ is a character of H with finite range. Then $K := \ker\sigma$ is a normal subgroup of H of finite index, and hence (G, K) is also a Hecke pair. Let (G_σ, K_σ) denote the Schlichting completion of (G, K). We have the following lemma.

Lemma 1.1. (a) *The closure H_σ of H in the Hecke topology from (G, K) is a compact open subgroup of G_σ.*

(b) *σ is continuous for the Hecke topology from (G, K), and thus has a unique extension to a finite character σ of H_σ with kernel K_σ.*

Proof. We claim that $hK \to hK_\sigma$ for $h \in H$ is an isomorphism $H/K \xrightarrow{\cong} H_\sigma/K_\sigma$; indeed, an element $h \in H \setminus K$ is carried to the open set hK_σ which is disjoint from K, showing injectivity, and surjectivity follows because any given xK_σ in H_σ/K_σ is open, and hence meets the dense subset H of H_σ. Thus H_σ/K_σ is finite, so H_σ is compact because its quotient by a compact subgroup is again compact. This proves (a). For (b) it suffices to show continuity of σ at e, and this follows by inspection using that $\sigma(H)$ is finite and K is open. □

Definition 1.2. A *Hecke triple* (G, H, σ) consists of a Hecke pair (G, H) and a finite character σ of H. We say that (G, H, σ) is *reduced* if the Hecke pair $(G, K := \ker\sigma)$ is reduced. We call the Hecke triple $(G_\sigma, H_\sigma, \sigma)$ from Lemma 1.1 the *Schlichting completion* of (G, H, σ).

Remark 1.3. Suppose that L is a totally disconnected, locally compact group, M a compact, open subgroup, and χ a continuous character on M. Arguing as in, for example, the proof of [12, Theorem (24.17)(II)], shows that $\ker\chi$ is an open subgroup of M, and hence $\chi(M)$ is finite.

We next prove that the Schlichting completion of a Hecke triple has a universal property, and is unique up to topological isomorphism.

Theorem 1.4. *Let (G, H, σ) be a reduced Hecke triple. Then the Schlichting completion $(G_\sigma, H_\sigma, \sigma)$ has the following universal property: suppose that L is a locally compact, totally disconnected topological group, M a compact open subgroup, χ a non-trivial character on M and $\phi : G \to L$ is a homomorphism such that $(L, \ker\chi)$ is reduced, $\phi(G)$ is dense in L, $\phi(H) \subseteq M$ and $\sigma = \chi \circ \phi|_H$. Then there is a unique continuous homomorphism $\overline{\phi}$ from G_σ onto L, which extends ϕ, and satisfies the identity*

$$\chi \circ \overline{\phi}|_{H_\sigma} = \sigma. \tag{1.2}$$

If in addition $\phi^{-1}(M) = H$, then $\overline{\phi}$ will be a topological group isomorphism of G_σ onto L and of H_σ onto M.

Proof. Denote $N := \ker\chi$. Then $\phi(K) \subseteq N$. Applying the first half of [15, Theorem 3.8] to the Schlichting pair (L, N) gives a unique continuous homomorphism extension $\overline{\phi}$ from G_σ into L. Since

$$\chi \circ \overline{\phi}|_H(h) = \chi \circ \phi(h) = \sigma(h)$$

for all $h \in H$, (1.2) follows from the continuity of $\chi \circ \overline{\phi}$ and σ on H_σ.

If $\phi^{-1}(M) = H$, a straightforward verification then shows that $\phi^{-1}(N) = K$, and it follows from the second half of [15, Theorem 3.8] that $\overline{\phi}$ is a topological group isomorphism of G_σ onto L. The assumption $\phi^{-1}(M) = H$ implies that $\phi(H) = M \cap \phi(G)$. Since M is open and closed,

$$\overline{\phi(H)} = \overline{M \cap \phi(G)} = M \cap \overline{\phi(G)} = M. \tag{1.3}$$

The set $\overline{\phi}(H_\sigma)$ is compact, hence closed, and so it equals $\overline{\phi(H)}$. By invoking (1.3) we obtain the last claim of the theorem. □

Remark 1.5. Let (G, H, σ) be a reduced Hecke triple, $(G_\sigma, H_\sigma, \sigma)$ the Schlichting completion, and j_1 the dense embedding of G in G_σ. Denote by (G_0, H_0) the Schlichting completion of (G, H) and by j_0 the dense embedding $G \to G_0$. Since $K \subseteq H \subseteq H_0$, the first half of [15, Theorem 3.8] gives a continuous homomorphism $\iota : G_\sigma \to G_0$ such that $\iota \circ j_1 = j_0$. Since H is dense in both H_σ and H_0, $\iota(H_\sigma) = H_0$. We typically omit j_0 and j_1 from the notation.

Remark 1.6. In [8], for a Hecke pair (G, H) and a finite dimensional unitary representation σ of H, Curtis defines an equivalence relation $\sim$ on $G \times \sigma(H)$ by $(g, t) \sim (gh^{-1}, \sigma(h)t)$ for all $h \in H$. With $S_\sigma := (G \times \sigma(H))/\sim$ denoting the quotient space, G is endowed with the topology pulled back from the compact-open topology on the space of continuous functions $\{f : S_\sigma \to S_\sigma\}$. Then part of [8, Theorem 3] asserts that the closures of G and H in this topology and the unique extension of σ to the closure of H have the universal property. If σ is a finite character, note that the map $g \mapsto [g, 1]$ from G onto S_σ is a bijection from G/K onto S_σ, which is equivariant for the actions of G as permutations on G/K and on S_σ. Thus G_σ is the same as the completion constructed in [8].

2. The generalised Hecke algebra of (G, H, σ)

Given a group G, a subgroup H and a unitary representation $\sigma : H \to B(V)$, the Hecke algebra of (G, H, σ) is defined in [8] as a convolution algebra on the vector space of functions $f : G \to B(V)$ which have finite support on $H\backslash G$ and G/H, and satisfy $f(hxk) = \sigma(h)f(x)\sigma(k)$ for all $h, k \in H, x \in G$. However, for the purposes of using Schlichting completions, the interesting case (also in [8]) is that of a Hecke pair (G, H).

Definition 2.1. Given a Hecke pair (G, H) and a character σ of H, let $\mathcal{H}_\sigma(G, H)$ be the vector space of functions $f : G \to \mathbb{C}$ which have finite support in $H\backslash G/H$, and satisfy $f(hxk) = \sigma(h)f(x)\sigma(k)$ for all $h, k \in H, x \in G$. The *generalised Hecke algebra* associated with (G, H, σ) is $\mathcal{H}_\sigma(G, H)$ endowed with the convolution

$$f * g(x) = \sum_{yH \in G/H} f(y)g(y^{-1}x). \tag{2.1}$$

The identity element is the function ε_H defined by $\varepsilon_H(x) = \sigma(x)$ when $x \in H$ and $\varepsilon_H(x) = 0$ otherwise.

When $\sigma = 1$ the generalised Hecke algebra is the classical Hecke algebra $\mathcal{H}(G, H)$. We refer to Krieg's monography [16] for a presentation of the construction and properties of an abstract Hecke algebra, as formalised by Shimura, who built on work of Hecke.

The key observation that facilitates and motivates the study of $\mathcal{H}_\sigma(G, H)$ in terms of the Schlichting completion $(G_\sigma, H_\sigma, \sigma)$ of (G, H, σ) is the following result. We leave the standard proof to the reader.

Lemma 2.2. *Let L be a locally compact group, M a compact open subgroup, χ a non-trivial character on M, and choose the Haar measure μ on L normalised so that $\mu(M) = 1$. Then $\mathcal{H}_\chi(L, M)$ is equal to the subalgebra*

$$\{f \in C_c(L) \mid f(mxn) = \chi(m)f(x)\chi(n), \forall m, n \in M, x \in L\} \tag{2.2}$$

of $C_c(L)$, endowed with the convolution with respect to μ.

Proposition 2.3. *Suppose that (G, H, σ) is a reduced Hecke triple. Let $(G_\sigma, H_\sigma, \sigma)$ be the Schlichting completion of (G, H, σ), and choose the Haar measure μ on G_σ normalised so that $\mu(H_\sigma) = 1$. Then the map $\Psi : \mathcal{H}_\sigma(G_\sigma, H_\sigma) \to \mathcal{H}_\sigma(G, H)$ given by $\Psi(f) = f|_G$ is an algebra isomorphism.*

Proof. By adapting the argument in [15, Proposition 3.9 (iii)] to the reduced pair (G, K), it follows that $HxH \mapsto H_\sigma x H_\sigma$ for $x \in G$ is a bijection from $H\backslash G/H$ onto $H_\sigma\backslash G_\sigma/H_\sigma$. Since G_σ and H_σ contain dense copies of G and H, it follows that the map Ψ is well-defined.

Given f in $\mathcal{H}_\sigma(G, H)$, note that by the invariance property of f,

$$f((xKx^{-1})x) = f(xK) = f(x)\sigma(K) = f(x)$$

for all $x \in G$. Thus f is continuous for the Hecke topology from (G, K), and so extends to a function in $\mathcal{H}_\sigma(G_\sigma, H_\sigma)$. It follows that Ψ is bijective. A routine calculation shows that $\Psi(f * g) = \Psi(f) * \Psi(g)$, and the claim follows. □

From Proposition 2.3 and Lemma 2.2 it seems natural to define the involution on $\mathcal{H}_\sigma(G, H)$ as in $C_c(G_\sigma)$ by using the modular function of G_σ, so we investigate its meaning for the original Hecke triple. This is in fact answered by Schlichting in [27, Lemma 1(iii)]. We need some notation first.

Suppose that (G, H) is a Hecke pair. For x in G denote

$$H_x := H \cap xHx^{-1}, \tag{2.3}$$

$$L(x) := [H : H_x], \text{ and}$$

$$\Delta_H(x) := \frac{L(x)}{L(x^{-1})}. \tag{2.4}$$

Lemma 2.4. *Suppose H is a compact open subgroup of a locally compact group G. Then the modular function Δ of G satisfies $\Delta(x) = \Delta_H(x)$. In particular, $\Delta_H(x)$ does not depend on which compact open subgroup we use.*

Proof. See [27], or perform the following calculation:

$$\Delta_H(x) = \frac{\mu(H)\mu(H_{x^{-1}})}{\mu(H_x)\mu(H)} = \frac{\mu(xH \cap Hx)}{\mu(H_x)} = \Delta(x).$$

□

With the notation of Remark 1.5, Lemma 2.4 implies the following:

Corollary 2.5. *The modular functions Δ_σ of G_σ and Δ_0 of G_0 satisfy $\Delta_0 \circ \iota = \Delta_\sigma$.*

If (G, H, σ) is a reduced Hecke triple with Schlichting completion $(G_\sigma, H_\sigma, \sigma)$, then $[H : H_x] = [H_\sigma : (H_\sigma)_x]$, so we can supplement Proposition 2.3.

Proposition 2.6. *Let (G, H, σ) be a reduced Hecke triple with $K := \ker \sigma$, and define an involution on $\mathcal{H}_\sigma(G, H)$ by*

$$f^*(x) = \Delta_K(x^{-1})\overline{f(x^{-1})}, \text{ for } x \in G. \tag{2.5}$$

Then the map Ψ of Proposition 2.3 is an isomorphism of $*$-algebras.

Remark 2.7. Most authors do not include Δ in the definition of an involution on a (generalised) Hecke algebra. But we claim that this is more natural, for instance the l^1-norm on $\mathcal{H}_\sigma(G, H)$ defined by

$$\|f\|_1 = \sum_{y \in G/H} |f(y)| \text{ for } f \in \mathcal{H}_\sigma(G, H). \tag{2.6}$$

satisfies $\|f^*\|_1 = \|f\|_1$. We let $l^1(G, H, \sigma)$ be the completion of $\mathcal{H}_\sigma(G, H)$ in $\|\cdot\|_1$. As a consequence of Proposition 2.3 and Proposition 2.6 we get the following.

Proposition 2.8. *With the assumptions and notation from Proposition 2.3, the map Ψ extends to an isomorphism $L^1(G_\sigma, H_\sigma, \sigma) \cong l^1(G, H, \sigma)$ of Banach $*$-algebras.*

Proof. A computation shows that $\|\Psi(f)\|_1 = \|f\|_1$ for all $f \in \mathcal{H}_\sigma(G_\sigma, H_\sigma)$. Hence Ψ extends to the completions in the norm from (2.6).

Note that (2.6) on $\mathcal{H}_\sigma(G_\sigma, H_\sigma)$ is the usual L^1-norm on $C_c(G_\sigma)$. A routine calculation shows that $L^1(G_\sigma, H_\sigma, \sigma)$ is a closed $*$-subalgebra of $L^1(G_\sigma)$ for the involution on $L^1(G_\sigma)$ defined in terms of the chosen Haar measure. Hence $L^1(G_\sigma, H_\sigma, \sigma)$ is a Banach $*$-algebra and the claim follows. □

Theorem 2.9. *Given a reduced Hecke triple (G, H, σ), let $(G_\sigma, H_\sigma, \sigma)$ denote its Schlichting completion. Denote by μ the left invariant Haar measure on G_σ such that $\mu(H_\sigma) = 1$.*

Then the function $p_\sigma(x) := \chi_{H_\sigma}(x)\sigma(x)$ satisfies $p_\sigma^* = p_\sigma = p_\sigma * p_\sigma$ in $C_c(G_\sigma)$ (and we refer to it as being a self-adjoint projection), and we have isomorphisms between the $*$-algebras $p_\sigma C_c(G_\sigma)p_\sigma = \mathcal{H}_\sigma(G_\sigma, H_\sigma)$ and $\mathcal{H}_\sigma(G, H)$, and the Banach $*$-algebras $p_\sigma L^1(G_\sigma)p_\sigma$ and $l^1(G, H, \sigma)$.

Proof. It is straightforward that p_σ is a self-adjoint projection. Lemma 2.2 implies that $\mathcal{H}_\sigma(G_\sigma, H_\sigma) = p_\sigma C_c(G_\sigma)p_\sigma$, and then the claimed isomorphisms follow from Proposition 2.8 and Proposition 2.6. □

Similarly to [15, Lemma 4.2(iii)] we have that $p_\sigma C_c(G_\sigma)p_\sigma = \operatorname{span}\{p_\sigma x p_\sigma \mid x \in G_\sigma\}$. We proceed to identify functions in a spanning set for $\mathcal{H}_\sigma(G, H)$ which correspond by Theorem 2.9 to the products $p_\sigma x p_\sigma$.

It is known, see for instance [16], that the Hecke algebra of a pair (G, H) is linearly spanned by the collection $\{\chi_{HxH} \mid x \in H\backslash G/H\}$ of characteristic functions of double cosets. To account for non-zero functions in $\mathcal{H}_\sigma(G,H)$ supported on a given double coset, note that for $f\in\mathcal{H}_\sigma(G,H)$, $x\in G$ and $h\in H_x$ we have

$$\sigma(h)f(x)=f(hx)=f(xx^{-1}hx)=f(x)\sigma(x^{-1}hx).$$

Thus for f to be supported on HxH we need $\sigma(h)=\sigma(x^{-1}hx)$ for $h\in H_x$. This is condition (t_g) in [2, Proposition 1.2], and goes at least back to Mackey [23]. We denote

$$B:=\{x\in G\mid\sigma(h)=\sigma(x^{-1}hx)\text{ for }h\in H_x\}. \tag{2.7}$$

Remark 2.10. The set B contains H, is closed under inverses, and satisfies $BH=B=HB$. In general, B is not a group, see for example [9, Example 1.4.4]. However, B is a group in many cases such as, for instance, when σ extends to a character of G.

When B is a group, (B,H,σ) is a new Hecke triple, and it follows from the definitions that $\mathcal{H}_\sigma(G,H)=\mathcal{H}_\sigma(B,H)$. We point out that the Hecke topology from (G,K) may differ on B from the Hecke topology from (B,K), see example 5.14; thus the Schlichting completion of (B,H,σ) need not come from (G_σ,K_σ).

Suppose that (G,H,σ) is reduced. We aim to show that the closure of B in G_σ is precisely the set defined by (2.7) for the Schlichting completion. Towards this end we need a lemma.

Lemma 2.11. *We have* $\overline{H\cap xHx^{-1}}=H_\sigma\cap xH_\sigma x^{-1}$ *in* G_σ *for all* $x\in G$.

Proof. Since H_x is included in the closed set $(H_\sigma)_x$ for every $x\in G$, we obtain one inclusion. Suppose that $h\in(H_\sigma)_x$. Let F be a finite subset of G, and take $K_{\sigma,F}=\bigcap_{y\in F}yK_\sigma y^{-1}$, a neighbourhood of e. By restricting, if needed, to a smaller neighbourhood, we may assume that $x\in F$. Since H is dense in H_σ, it intersects the open neighbourhood $hK_{\sigma,F}$ of h. Thus we have

$$hk_1=h_1,hk_2=xh_2x^{-1}, \tag{2.8}$$

with $h_1,h_2\in H$ and $k_1,k_2\in K_{\sigma,F}$. Then $k_1^{-1}k_2=h_1^{-1}xh_2x^{-1}$ is an element of $G\cap j_1^{-1}(K_{\sigma,F})$, which is $\bigcap_{y\in F}yKy^{-1}$ because the Schlichting completion satisfies $j_1^{-1}(K_\sigma)=K$. Thus $k_1^{-1}k_2$ lies in xHx^{-1}, and so

$$h_1=xh_2x^{-1}k_2^{-1}k_1\in xHx^{-1}\cap H,$$

from which it follows via (2.8) that $h_1\in hK_{\sigma,F}\cap H_x$. Since this holds for all neighbourhoods $K_{\sigma,F}$, we have $h\in\overline{H_x}$, as claimed. ☐

Proposition 2.12. *Let* (G,H,σ) *be a reduced Hecke triple, and consider its Schlichting completion* $(G_\sigma,H_\sigma,\sigma)$. *Let* B_σ *denote*

$$\{x\in G_\sigma\mid\sigma(h)=\sigma(x^{-1}hx)\textit{ for }h\in H_\sigma\cap xH_\sigma x^{-1}\}. \tag{2.9}$$

Then B_σ *is equal to the closure of* B *in the Hecke topology from* (G,K).

Proof. If (x_i) is a net in B_σ converging to x, then eventually $H_\sigma\cap x_iH_\sigma x_i^{-1}$ coincides with $H_\sigma\cap xH_\sigma x^{-1}$, and so B_σ is closed. Since $B_\sigma H_\sigma=B_\sigma$, it is also open.

Lemma 2.11 implies that

$$B_\sigma\cap G=\{x\in G\mid\sigma(h)=\sigma(x^{-1}hx)\text{ for }h\in H_\sigma\cap xH_\sigma x^{-1}\}=B.$$

Hence the closure of B is included in B_σ. To show equality, take $x\in B_\sigma$ and $K_{\sigma,F}$ a neighbourhood of e. We must show that $xK_{\sigma,F}$ has non-empty intersection with B. By density of G in G_σ there is $k \in K_{\sigma,F}$ such that $xk \in G$. From $K_{\sigma,F} \subset H_\sigma$ and $B_\sigma H_\sigma = B_\sigma$ it follows that $xk \in B_\sigma \cap G = B$, as claimed. □

We can describe a linear basis for $\mathcal{H}_\sigma(G,H)$. For $x \in B$ the function

$$\varepsilon_x(y) = \begin{cases} \sigma(hk) & \text{if } y \in HxH \text{ and } y = hxk \\ 0 & \text{if } y \notin HxH. \end{cases} \tag{2.10}$$

is well defined and immediately seen to be in $\mathcal{H}_\sigma(G,H)$. These functions are essentially the elementary intertwining operators from [2]. Note that $\varepsilon_{h_0xk_0}(hxk) = \overline{\sigma(h_0)\sigma(k_0)}\varepsilon_x(hxk)$ for all $h_0, k_0 \in H$. It follows from the definitions that $\{\varepsilon_x \mid x \in H\backslash B/H\}$ form a linear basis for $\mathcal{H}_\sigma(G,H)$. When B is a group, this fact implies that $\mathcal{H}_\sigma(G,H)$ is the corner in $C_c(B_\sigma)$ determined by the projection p_σ, as shown below.

Proposition 2.13. *Let (G,H,σ) be a reduced Hecke pair, let $(G_\sigma,H_\sigma,\sigma)$ be its Schlichting completion, and assume that B is a subgroup of G. Then the isomorphism of Proposition 2.6 restricts to an isomorphism of $\mathcal{H}_\sigma(B_\sigma,H_\sigma)$ onto $\mathcal{H}_\sigma(B,H)$. In particular, $\mathcal{H}_\sigma(G,H)$ is isomorphic to $p_\sigma C_c(B_\sigma)p_\sigma$.*

Proof. Since B_σ is open in G_σ, it is locally compact. The subgroup H_σ is open and compact in B_σ, so on one hand Lemma 2.4 implies that B_σ and G_σ have the same modular function, equal to Δ_{K_σ} and Δ_{H_σ}, and on the other $\mathcal{H}_\sigma(B_\sigma,H_\sigma)$ is well-defined and equals $p_\sigma C_c(B_\sigma)p_\sigma$ by Lemma 2.2.

Given x in B_σ, it follows from Proposition 2.12 that we can pick b in B such that $xH_\sigma = j_1(b)H_\sigma = j_1(bH)$, and so $\Psi(\varepsilon_x) = \varepsilon_b$. Thus by the comments made before the proposition, Ψ carries the linear basis $\{\varepsilon_x \mid x \in H\backslash B_\sigma/H\}$ of $\mathcal{H}_\sigma(G_\sigma,H_\sigma)$, identified with $\mathcal{H}_\sigma(B_\sigma,H_\sigma)$, onto the linear basis $\{\varepsilon_b \mid b \in H\backslash B/H\}$ of $\mathcal{H}_\sigma(G,H)$, identified with $\mathcal{H}_\sigma(B,H)$, and the proposition follows. □

Lemma 2.14. *With the notation of Theorem 2.9 we have*

$$p_\sigma x p_\sigma = \begin{cases} \frac{1}{L(x)}\varepsilon_x & \textit{if } x \in B_\sigma \\ 0 & \textit{otherwise.} \end{cases} \tag{2.11}$$

In particular, $\|p_\sigma x p_\sigma\|_1 = 1$ for every x in B_σ.

Proof. Since p_σ is supported on H_σ, the product $p_\sigma x p_\sigma$ is supported on the double coset $H_\sigma x H_\sigma$. The claim then follows because

$$\begin{aligned} p_\sigma x p_\sigma(hxk) &= \sigma(h)\sigma(k)\int_{H_\sigma \cap xH_\sigma x^{-1}} \sigma(l)\overline{\sigma(x^{-1}lx)}dl \\ &= \begin{cases} \sigma(h)\sigma(k)\mu(H_\sigma \cap xH_\sigma x^{-1}) & \text{if } x \in B_\sigma \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

3. The von Neumann algebra of (G, H, σ) and induced representations

Irreducibility criteria for induced representations of locally compact groups have been studied by a number of authors, including Mackey [23], Corwin [7], Binder [2, 3]. In [2, Theorem 2.2], the commutant of the induced representation is realised as the weak closure of the algebra of so-called elementary intertwining operators (this result was obtained earlier by Corwin [7], but that proof is claimed to be incomplete [2]), see also [9, Proposition 1.3.10]. In this section we study the continuity properties of the induced representation arising from a Hecke triple, and obtain as a direct consequence a new proof of Binder's theorem for such representations.

Suppose that (G, H, σ) is a Hecke triple. The induced representation $\lambda_\sigma := \operatorname{Ind}_H^G \sigma$ acts by the formula $\lambda_\sigma(x)(f)(y) := f(x^{-1}y)$ in the space $l^2(G, H, \sigma)$ defined as

$$\{f : G \to \mathbb{C} \mid f(xh) = \sigma(h^{-1})f(x), \forall x \in G, h \in H, \text{ and } \sum_{x\in G/H} |f(x)|^2 < \infty\}. \tag{3.1}$$

The function $\delta_y : G \to \mathbb{C}$ defined by $\delta_y(z) = \chi_H(z^{-1}y)\sigma(z^{-1}y)$ for $y \in G$ lies in $l^2(G, H, \sigma)$. If we choose a set of coset representatives $y \in G/H$, then $\{\delta_y \mid y \in G/H\}$ is an orthonormal basis for $l^2(G, H, \sigma)$.

Lemma 3.1. *Suppose that (G, H, σ) is a reduced Hecke triple. Let $K = \ker \sigma$ and $(G_\sigma, H_\sigma, \sigma)$ be the Schlichting completion. Then λ_σ is a homeomorphism from G with the Hecke topology from (G, K) into $B(l^2(G, H, \sigma))$ with its weak topology.*

Proof. For $x, y, w \in G$ we have

$$\langle \lambda_\sigma(w)\delta_x, \delta_y\rangle = \chi_H(y^{-1}wx)\sigma(y^{-1}wx). \tag{3.2}$$

Since $\sigma(H)$ is finite, there is $0 < \varepsilon_0 < 1$ such that $K = \{h \in H \mid |\sigma(h) - 1| \le \varepsilon_0\}$. Let $F \subseteq G$ be finite. A set of the form

$$\mathcal{V} = \{T \in B(l^2(G, H, \sigma)) \mid |\langle T\delta_x, \delta_y\rangle - \langle \delta_x, \delta_y\rangle| < \varepsilon, \forall x, y \in F\}$$

is a typical neighbourhood of $\lambda_\sigma(e)$ for the weak topology. We claim that $\lambda_\sigma^{-1}(\mathcal{V}) = \bigcap_{x\in F} xKx^{-1}$ if $\varepsilon \le \varepsilon_0$. Let $x, y \in F$ and $w \in xKx^{-1}$. Then $y^{-1}x \in H$ if and only if $y^{-1}wx \in H$, and (3.2) shows that $\lambda_\sigma(w) \in \mathcal{V}$, proving one inclusion. In particular, λ_σ is continuous at e, hence everywhere. Suppose now that $\lambda_\sigma(w) \in \mathcal{V}$. Inserting $y = x$ in (3.2) forces $|\chi_H(x^{-1}wx)\sigma(x^{-1}wx) - \langle \delta_x, \delta_x\rangle| < \varepsilon < \varepsilon_0$, so $w \in xKx^{-1}$, and thus λ_σ carries a neighbourhood subbase at e for the Hecke topology into a neighbourhood subbase at $\lambda_\sigma(e)$ for the weak topology. □

Denote by $\overline{\lambda_\sigma}$ the continous extension of λ_σ to G_σ. The next result shows that the induced representation of σ from H_σ to G_σ is, up to unitary equivalence, just $\overline{\lambda_\sigma}$.

Proposition 3.2. *With the assumptions of Lemma 3.1, let*

$$L^2(G_\sigma, H_\sigma, \sigma) := \{f \in L^2(G_\sigma) \mid f(wk) = \sigma(k^{-1})f(w), \forall w \in G_\sigma, k \in H_\sigma\}.$$

Then $f \mapsto f|_G$ defines a unitary $U : L^2(G_\sigma, H_\sigma, \sigma) \to l^2(G, H, \sigma)$, and $U^*\overline{\lambda_\sigma}(w)U$, $w \in G_\sigma$, is the induced representation of σ from H_σ to G_σ.

Proof. With (G_σ, K_σ) denoting the Schlichting completion of (G,K), take $w \in G_\sigma$ and note that since wK_σ is open, there is y in G such that $y \in wK_\sigma$. If $z \in G$, then $w^{-1}z \in H_\sigma$ if and only if $y^{-1}z \in H_\sigma \cap G = H$, and so $\delta_w = \delta_y$. Since $G_\sigma/H_\sigma \cong G/H$, U carries the orthonormal basis $\{\delta_w \mid w \in G_\sigma/H_\sigma\}$ onto the orthonormal basis $\{\delta_y \mid y \in G/H\}$. Finally, a routine calculation shows that $U^*\overline{\lambda_\sigma}(w)U$ acts as the induced representation, and the proposition follows. $\square$

With the same assumptions, let L and R denote the left, respectively the right regular representation of G_σ. A consequence of Proposition 3.2 is that $L^2(G_\sigma, H_\sigma, \sigma)$ is a closed subspace of $L^2(G_\sigma)$ which is invariant under L, and that $\overline{\lambda_\sigma}$ is the restriction of L to this subspace. With R also denoting the integrated form of the right regular representation, the projection onto $L^2(G_\sigma, H_\sigma, \sigma)$ is $P_\sigma := R_{p_\sigma}$. Since $\mathcal{H}_\sigma(G_\sigma, H_\sigma) = p_\sigma C_c(G_\sigma)p_\sigma$, it follows that R is a $*$-representation of $\mathcal{H}_\sigma(G_\sigma, H_\sigma)$ on $L^2(G_\sigma, H_\sigma, \sigma)$. Hence we have the following consequence of Proposition 3.2:

Lemma 3.3. *Let (G,H,σ) be a reduced Hecke triple. Then*

$$(R_\phi\xi)(y) = \sum_{z\in G/H} \Delta_K(z)^{-1/2}\xi(yz)\phi(z), \tag{3.3}$$

for $\phi \in \mathcal{H}_\sigma(G,H)$ and $\xi \in l^2(G,H,\sigma)$, is a $*$-representation of $\mathcal{H}_\sigma(G,H)$.

In [9], the Hecke von Neumann algebra of (G,H,σ) is defined as the von Neumann algebra generated by the image of the generalised Hecke algebra in a left regular representation on the space of the induced representation. Similar to this we let $\mathcal{R}(G,H,\sigma)$ be the von Neumann algebra generated by $R(\mathcal{H}_\sigma(G,H))$ in $B(l^2(G,H,\sigma))$. Since Proposition 3.2 and Proposition 2.3 imply that $UR(f) = R(\Psi(f))U$ for all $f \in \mathcal{H}_\sigma(G_\sigma, H_\sigma)$, we recover a result analogue to [8, Theorem 3].

Proposition 3.4. *With U defined in Proposition 3.2, the map $a \mapsto \operatorname{Ad} U(a)$ implements an isomorphism of $\mathcal{R}(G_\sigma, H_\sigma, \sigma)$ onto $\mathcal{R}(G,H,\sigma)$.*

The spaces $\mathcal{R}(G_\sigma) := \{R_x \mid x \in G_\sigma\}''$ and $\mathcal{L}(G_\sigma) := \{L_x \mid x \in G_\sigma\}''$ are known to be each others commutant inside $B(L^2(G_\sigma))$. Take $P_\sigma = R_{p_\sigma}$ as above, and note that $P_\sigma \in \mathcal{R}(G_\sigma)$ and $L_xP_\sigma = \overline{\lambda_\sigma}(x)$, for all $x \in G_\sigma$.

Theorem 3.5. *Suppose that (G,H,σ) is a reduced Hecke triple. Then $\lambda_\sigma(G)'$ equals $\mathcal{R}(G,H,\sigma)$.*

Proof. Let $(G_\sigma, H_\sigma, \sigma)$ be the Schlichting completion of (G,H,σ). Proposition 3.2 implies that $\operatorname{Ad} U^*$ carries the subset $\lambda_\sigma(G)''$ of $B(l^2(G,H,\sigma))$ into $\overline{\lambda_\sigma}(G_\sigma)''$ inside $B(L^2(G_\sigma))$. Then

$$\overline{\lambda_\sigma}(G_\sigma)'' = \{L_xP_\sigma \mid x \in G_\sigma\}'' = (P_\sigma\{L_x \mid x \in G_\sigma\}'P_\sigma)' = \mathcal{L}(G_\sigma)P_\sigma.$$

So by the Double Commutant Theorem we have $\overline{\lambda_\sigma}(G_\sigma)' = P_\sigma\mathcal{L}(G_\sigma)'P_\sigma = P_\sigma\mathcal{R}(G_\sigma)P_\sigma = R(G_\sigma, H_\sigma, \sigma)$, and by applying $\operatorname{Ad} U$ we have back in $B(l^2(G,H,\sigma))$ that $R(G,H,\sigma)$ is equal to $\lambda_\sigma(G)'$. $\square$

As an immediate consequence of Theorem 3.5 and Proposition 2.13 we obtain the following classical result, see [23, Theorem 6'].

Corollary 3.6. *λ_σ is irreducible if and only if $B = H$.*

4. C^*-COMPLETIONS OF GENERALISED HECKE ALGEBRAS

A consequence of Theorem 2.9 is that $p_\sigma C^*(G_\sigma)p_\sigma$ is a C^*-completion of $\mathcal{H}_\sigma(G,H)$. In this section we analyse this completion, and identify conditions which ensure that it is the universal one. We denote by $C^*(\mathcal{H}_\sigma(G,H))$ the enveloping C^*-algebra of $\mathcal{H}_\sigma(G,H)$, when it exists. Note that $p_\sigma C^*(G_\sigma)p_\sigma$ is Morita-Rieffel equivalent to the ideal

$$I := \overline{C^*(G_\sigma)p_\sigma C^*(G_\sigma)} \tag{4.1}$$

of $C^*(G_\sigma)$. Next we introduce the analogue for (G,H,σ) of the H-smooth representations on G that arise from (G,H).

Definition 4.1. Suppose that (G,H) is a Hecke pair and σ is a character of H. Given a unitary representation π of G on a Hilbert space V, let

$$V_{H,\pi} := \{\xi \in V \mid \pi(h)\xi = \sigma(h)\xi, \forall h \in H\}. \tag{4.2}$$

We say that π is (H,σ)-smooth if $\overline{\operatorname{span}}(\pi(G)V_{H,\pi}) = V$.

Note that λ_σ is (H,σ)-smooth (and "smooth" in the sense of [28, §1.7]), as is every representation of G that is unitarily equivalent to λ_σ. We have the following generalisation of [15, Proposition 5.18].

Proposition 4.2. *Let (G,H,σ) be a reduced Hecke triple, and $(G_\sigma,H_\sigma,\sigma)$ be its Schlichting completion. Then a representation of G is (H,σ)-smooth if and only if it extends to a continuous (H_σ,σ)-smooth representation of G_σ.*

Proof. The proof of [15, Proposition 5.18] carries over to this situation with one modification: we need to show that an (H,σ)-smooth representation π of G is continuous from G with the Hecke topology from (G,K) into $B(V)$ with the strong topology. So suppose that $x \to e$ in G, and pick $\xi \in V$, which by the assumption on smoothness we may take of the form $\pi(y)\eta$ with $\eta \in V_{H,\pi}$. Then eventually x belongs to the neighbourhood yKy^{-1}, and

$$\pi(x)\pi(y)\eta = \pi(y)\pi(y^{-1}xy)\eta = \pi(y)\sigma(y^{-1}xy)\eta = \pi(y)\eta,$$

so $\|\pi(x)\pi(y)\eta - \pi(y)\eta\| \to 0$, proving the desired continuity. □

However, the generalisation to Hecke triples of [15, Corollary 5.10] fails for $\sigma \neq 1$.

Proposition 4.3. *The trivial representation of G_σ is not (H_σ,σ)-smooth, and p_σ is not full in $C^*(G_\sigma)$.*

Proof. It suffices to note, first, that the integrated form of the trivial representation of G_σ carries p_σ into 0, and second, that $V_{H_\sigma,\pi} = \pi(p_{\overline{\sigma}})V$ for any representation π of G_σ on V. □

We show next how the strategy developed in [15, §5], based on Fell's imprimitivity bimodules for $*$-algebras, for studying the representations of $\mathcal{H}(G,H)$ can be carried over to tie up the representations of $\mathcal{H}_\sigma(G,H)$ and the (H,σ)-smooth representations of G.

We recall that if ${}_EX_D$ is an imprimitivity bimodule of $*$-algebras E and D in the sense of [15, Definition 5.1], then a representation π of D is *positive* with respect to the right inner product $\langle\rangle_R$ provided that $\pi(\langle f,f\rangle_R) \geq 0$ for all $f \in X$.

Let L be a locally compact group, M a compact open subgroup, and χ a non-trivial character on M. Choose a Haar measure on L such that χ becomes a projection p_χ in

$C_c(L)$. Consider the $*$-algebras $E = C_c(L)p_\chi C_c(L)$, $D = \mathcal{H}_\chi(L, M)$, $B = \overline{L^1(L)p_\chi L^1(L)}$ (closure taken in $L^1(L)$) and $C = p_\chi L^1(L)p_\chi$. Then we have an inclusion of bimodules ${}_E X_D \subset {}_B Y_C$, where $Y := C_c(L)p_\chi$ and $X := L^1(L)p_\chi$ with bimodule operations inherited from $L^1(L)$, and right-inner product $\langle f, g\rangle_R = f^*g$ for f, g in Y and X respectively. We claim that the C^*-completions $C^*(E)$ and $C^*(B)$ coincide with $\overline{C^*(L)p_\chi C^*(L)}$. Indeed, the proof of [15, Theorem 5.7] carries through with the following alterations: given a nondegenerate representation π of E on a Hilbert space V, the formula

$$\tilde{\pi}(x)\pi(f)\xi := \pi(xf)\xi \text{ for } x \in L, f \in E, \xi \in V,$$

defines a representation of L on V which is (M, χ)-smooth because $mp_{\overline{\chi}} = \chi(m)p_{\overline{\chi}}$ for $m \in M$ implies that $\pi(p_{\overline{\chi}})V = V_{M,\chi}$. The integrated form of $\tilde{\pi}$ will then be a nondegenerate extension of π to $\overline{C^*(L)p_{\overline{\chi}}C^*(L)}$, from which the claim follows. So we obtain the analogue of the category equivalences from [15, Corollaries 5.12 and 5.20].

Corollary 4.4. *Let (G, H, σ) be a reduced Hecke triple. Then there are category equivalences between the following:*

(a) *the (H, σ)-smooth representations of G and the $\langle\rangle_R$-positive representations of $\mathcal{H}_\sigma(G, H)$;*

(b) *the nondegenerate representations of $\overline{C^*(G_\sigma)p_\sigma C^*(G_\sigma)}$ and the $\langle\rangle_R$-positive representations of $p_\sigma L^1(G_\sigma)p_\sigma$.*

4.1. **The case $H \subseteq N \trianglelefteq G$.** Let (G, H, σ) be a reduced Hecke triple with Schlichting completion $(G_\sigma, H_\sigma, \sigma)$, and suppose that H is contained in a normal subgroup N of G. We will show like in [15, §8] that I defined in (4.1) is a (twisted) crossed product, see [26] (and [24]) for definitions. Let N_σ denote the closure of N in G_σ, and let Ad be the action of G_σ by conjugation on N_σ (and on $C^*(N_\sigma)$). The universal covariant representation (π, u) of $(C^*(N_\sigma), G_\sigma)$ into the twisted crossed product $C^*(N_\sigma) \rtimes G_\sigma/N_\sigma$ determines an isomorphism $\pi \times u : \pi(b)u(f) \mapsto bf$ of $C^*(N_\sigma) \rtimes G_\sigma/N_\sigma$ onto $C^*(G_\sigma)$. Note that the twist disappears when G is a semi-direct product of N by a group Q, see also [22].

Theorem 4.5. *Suppose that (G, H, σ) is a reduced Hecke triple such that H is contained in a normal subgroup N of G. Let N_σ denote the closure of N in the Schlichting completion $(G_\sigma, H_\sigma, \sigma)$ of (G, H, σ).*

(a) *Then $I_\sigma := \overline{\operatorname{span}}\{xp_\sigma x^{-1}n \mid x \in G_\sigma, n \in N_\sigma\}$ is an* Ad*-invariant ideal of $C^*(N_\sigma)$, and the isomorphism $\pi \times u$ carries $I_\sigma \rtimes G_\sigma/N_\sigma$ onto I defined in* (4.1)*.*

(b) *$I_\sigma \rtimes G/N$ is Morita-Rieffel equivalent to $p_\sigma C^*(G_\sigma)p_\sigma$.*

Proof. Since $xH \mapsto xH_\sigma$ is a bijection from G/H onto G_σ/H_σ (essentially by [15, Proposition 3.9]), it follows that

$$\operatorname{span}\{xp_\sigma x^{-1}n \mid x \in G_\sigma, n \in N_\sigma\} = \operatorname{span}\{xp_\sigma x^{-1}n \mid x \in G, n \in N\}.$$

Note that $xp_\sigma x^{-1} \in C_c(N_\sigma)$, so $I_\sigma \subset C^*(N_\sigma)$. Since $mI_\sigma = I_\sigma m = I_\sigma$ for $m \in N$, we get as in the proof of [15, Theorem 8.1] that I_σ is a closed, Ad-invariant ideal of $C^*(N_\sigma)$. Since $xp_\sigma y = xp_\sigma x^{-1}(xp_\sigma y) \subset \operatorname{span}\{xp_\sigma x^{-1}f \mid f \in C_c(G_\sigma)\}$, it follows that

$$\begin{aligned}(\pi \times u)(I_\sigma \rtimes G_\sigma/N_\sigma) &= \overline{\operatorname{span}}\{xp_\sigma x^{-1}nf \mid x \in G, n \in N, f \in C_c(G_\sigma)\}\\ &= \overline{\operatorname{span}}\{xp_\sigma x^{-1}f \mid x \in G, f \in C_c(G_\sigma)\}\\ &= \overline{\operatorname{span}}\{xp_\sigma y \mid x, y \in G\} = I,\end{aligned}$$

as claimed in (a).

For (b), it suffices by (a) to establish that $I_\sigma \rtimes G/N \cong I_\sigma \rtimes G_\sigma/N_\sigma$. Since $hp_\sigma = \overline{\sigma(h)}p_\sigma$ for all $h \in H$, the argument in the proof of [15, Theorem 8.2] shows that the canonical homomorphism $\omega : G \to M(I_\sigma \rtimes G/N)$ is (H,σ)-smooth. Thus ω has a continuous extension $\overline{\omega}$ to G_σ by Proposition 4.2, and then $\overline{\omega}$ forms a covariant pair together with the canonical homomorphism $I_\sigma \to M(I_\sigma \rtimes G/N)$, from which the claimed isomorphism follows. $\square$

Corollary 4.6. *With the notation from Theorem 4.5, assume that B is a subgroup of G such that N is a normal subgroup of B. Then $I_{\sigma,B} = \overline{\operatorname{span}}\{xp_\sigma x^{-1}n \mid x \in B_\sigma, n \in N_\sigma\}$ is an* Ad*-invariant ideal of $C^*(N_\sigma)$, and the closed ideal generated by p_σ in $C^*(B_\sigma)$ is Morita-Rieffel equivalent to the twisted crossed products $I_{\sigma,B} \rtimes B_\sigma/N_\sigma$ and $I_{\sigma,B} \rtimes B/N$.*

With the hypotheses of Theorem 4.5, we assume that N is abelian, and we consider the Fourier transform $f \mapsto \hat{f}$ from $C^*(N_\sigma)$ onto $C_0(\widehat{N_\sigma})$. We let

$$\sigma + H_\sigma^\perp := \{\alpha \in \widehat{N_\sigma} \mid \alpha|_{H_\sigma} = \sigma\} \tag{4.3}$$

be the set of all continuous extensions of σ to N_σ. One can verify that $\widehat{p_\sigma} = \chi_{\sigma+H_\sigma^\perp}$. The dual action of G_σ on $\widehat{N_\sigma}$ is characterised by

$$\langle n, x\cdot\alpha\rangle = \langle x^{-1}nx, \alpha\rangle \text{ for } n \in N_\sigma, \alpha \in \widehat{N_\sigma}, x \in G_\sigma,$$

and then we have for all $x \in G$ that

$$\begin{aligned}(xp_\sigma x^{-1})^\wedge(\alpha) &= \int_{N_\sigma} \overline{\langle m,\alpha\rangle}(xp_\sigma x^{-1})(m)d\mu(m)\\ &= \Delta(x)\int_{N_\sigma} \overline{\langle m,\alpha\rangle}\langle x^{-1}mx,\sigma\rangle\chi_{xH_\sigma x^{-1}}(m)d\mu(m)\\ &= \Delta(x)\int_{xH_\sigma x^{-1}} \langle m, x\cdot\sigma - \alpha\rangle d\mu(m)\\ &= \Delta(x)\mu(xH_\sigma x^{-1})\chi_{x\cdot(\sigma+H_\sigma^\perp)}(\alpha)\\ &= \chi_{x\cdot(\sigma+H_\sigma^\perp)}(\alpha). \qquad (4.4)\end{aligned}$$

Therefore $\widehat{I_\sigma}$ is the ideal in $C_0(\widehat{N_\sigma})$ generated by $\{\chi_{x\cdot(\sigma+H_\sigma^\perp)} \mid x \in G\}$, so if we let

$$\Omega_\sigma := \bigcup_{x\in G} x\cdot(\sigma + H_\sigma^\perp), \tag{4.5}$$

then we have proved the following specialisation of Theorem 4.5:

Theorem 4.7. *With the hypotheses of Theorem 4.5, if N is moreover abelian, then $\widehat{I_\sigma} = C_0(\Omega_\sigma)$. If σ is non-trivial, then $0 \notin \Omega_\sigma$, so p_σ is not full.*

Remark 4.8. If B is a group and N is normal in B, then $\widehat{I_{\sigma,B}}$ from Corollary 4.6 equals $C_0(\Omega_{\sigma,B})$, where

$$\Omega_{\sigma,B} := \bigcup_{x\in B} x\cdot(\sigma + H_\sigma^\perp). \tag{4.6}$$

4.2. **The case $H \trianglelefteq N \trianglelefteq G$.** Suppose that (G,H) is a reduced Hecke pair and N is such that $H \trianglelefteq N \trianglelefteq G$. Let σ be a finite character of H such that

$$\sigma(nhn^{-1}) = \sigma(h) \text{ for all } n \in N, h \in H, \tag{4.7}$$

(so $N \subseteq B$ in the notation of (2.7)). Then p_σ is central in $C^*(N_\sigma)$, and so are $xp_\sigma x^{-1}$ for all x in G_σ. On one hand, this shows that the ideal I_σ defined in Theorem 4.5 will satisfy $I_\sigma = C^*(N_\sigma)p_1$, where $p_1 := \sup\{xp_\sigma x^{-1} \mid x \in G\}$.

On the other hand, $xp_\sigma x^{-1}p_\sigma$ is a projection, and $p_\sigma xp_\sigma$ is a partial isometry, for all $x \in G_\sigma$. Thus for every $*$-representation π of $\mathcal{H}_\sigma(G,H)$ and for all $x \in B$ we have

$$\|\pi(p_\sigma xp_\sigma)\| \le 1 = \|p_\sigma xp_\sigma\|_1,$$

where the equality is from Lemma 2.14. This establishes that $C^*(\mathcal{H}_\sigma(G,H))$ exists and is equal to the enveloping C^*-algebra of $l^1(G,H,\sigma)$.

But more is true. We show next as in [15, Theorem 5.13] that the right inner product $\langle\rangle_R$ on $X := C_c(G_\sigma)p_\sigma$ is positive in the following sense: given f in X, there are g_i in $\mathcal{H}_\sigma(G_\sigma,H_\sigma)$, $i = 1,\dots,n$, such that $\langle f,f\rangle_R = \sum_{i=1}^n g_i^*g_i$. From this we deduce that $C^*(\mathcal{H}_\sigma(G,H))$ is precisely $p_\sigma C^*(G_\sigma)p_\sigma$.

Theorem 4.9. *Let L be a locally compact, totally disconnected group, M a compact open subgroup, and χ a non-trivial character on M. Suppose that M is normal in a closed normal subgroup N of L, and choose a Haar measure on L such that χ becomes a self-adjoint projection p in $C_c(L)$. If*

$$\chi(nmn^{-1}) = \chi(m) \text{ for all } n \in N, m \in M,$$

then the left–$C_c(L)pC_c(L)$ and right–$\mathcal{H}_\chi(L,M)$ bimodule of $*$-algebras $Y := C_c(L)p$ has positive right inner product, and hence $C^*(\mathcal{H}_\chi(L,M)) = pC^*(L)p$.

We have the following consequence of this theorem and of Proposition 2.13.

Corollary 4.10. *Let (G,H,σ) be a reduced Hecke triple, N a normal subgroup of G such that H is normal in N, and suppose that σ satisfies* (4.7). *Let $(G_\sigma,H_\sigma,\sigma)$ be the Schlichting completion of (G,H,σ). Then $C^*(\mathcal{H}_\sigma(G_\sigma,H_\sigma)) = p_\sigma C^*(G_\sigma)p_\sigma$.*

If B is a group, then $C^*(\mathcal{H}_\sigma(G_\sigma,H_\sigma)) = p_\sigma C^*(B_\sigma)p_\sigma$.

Proof of Theorem 4.9. Like in the proof of [15, Theorem 5.13], the bimodule Y has a spanning set, which in our case is $\{xp \mid x \in L\}$. The extra hypothesis on χ implies that p is a central projection in $C_c(N)$ and hence, by normality of N in L, so is xpx^{-1} for every $x \in L$. Then $\{xpx^{-1} \mid x \in L\}$ are commuting projections in $C^*(L)$, and by following verbatim the proof of [15, Theorem 5.13] we conclude that for any element $f = \sum_1^n c_ix_ip$ of Y, the product f^*f is a finite sum of elements h^*h with $h \in pC_c(L)p$. Positivity being established, the last claim follows from [15, Proposition 5.5(iii)]. □

4.3. **The case when (B,H) is directed.** Let (G,H,σ) be a reduced Hecke triple with Schlichting completion $(G_\sigma,H_\sigma,\sigma)$. Let B be the subset of G defined in (2.7), and consider its closure B_σ from Proposition 2.12.

Lemma 4.11. *If $x \in B$ and $xHx^{-1} \supset H$, then $x^{-1}p_\sigma x \ge p_\sigma$ in $C^*(G_\sigma)$.*

Proof. It suffices to note that $x^{-1}p_\sigma x = \mu(x^{-1}H_\sigma x)^{-1}\sigma|_{x^{-1}H_\sigma x}$. □

Assume that B is a group. Consider the semigroup $B^+ := \{x \in B \mid xHx^{-1} \supset H\}$, and recall from [15, Definition 6.1] that (B, H) is called directed if B^+ is an Ore semigroup in B, i.e. $B = \{x^{-1}y \mid x, y \in B^+\}$. In general, (B, H) need not be directed when (G, H) is. In examples 5.5 and 5.14 we prove that (B, H) is directed. For x, y in B with $yx \in B^+$ we have $x^{-1}y^{-1}p_\sigma yx \geq p_\sigma$ by Lemma 4.11, so $p_\sigma yxp_\sigma = yxp_\sigma$, and the proof of [15, Theorem 6.4] can be used here to give that the bimodule $C_c(B_\sigma)p_\sigma$ has positive right inner product. Therefore [15, Proposition 5.5 (iii)] gives the following.

Proposition 4.12. *Let (G, H, σ) be a reduced Hecke triple such that B is a subgroup of G and (B, H) is directed. Then $C^*(\mathcal{H}_\sigma(G, H)) = C^*(\mathcal{H}_\sigma(B, H)) = p_\sigma C^*(B_\sigma)p_\sigma$.*

In fact we have a precise description of the ideal in $C^*(B_\sigma)$ generated by p_σ.

Theorem 4.13. *Let (G, H, σ) be a reduced Hecke triple such that B is a subgroup of G and (B, H) is directed. Denote by H_∞ the subgroup $\bigcap_{x\in B^+} x^{-1}H_\sigma x$ of H_σ, and let μ_∞ be normalised Haar measure on H_∞. Then the closed ideal generated by p_σ in $C^*(B_\sigma)$ is equal to $C^*(B_\sigma)p_{\sigma,\infty}$, where*

$$p_{\sigma,\infty} = \int_{H_\infty} \sigma(h)h d\mu_\infty(h). \tag{4.8}$$

To prove this theorem we need a general lemma.

Lemma 4.14. *Suppose that G is a locally compact group, H a compact open subgroup, $\{H_i\}_{i\in I}$ a family of open subgroups of H over a directed set I with $H_j \subset H_i$ for $i < j$, and σ a character of H with finite range. Denote $H_\infty = \cap_{i\in I} H_i$, let μ_i be normalised Haar measure on H_i for $i \in I \cup \{\infty\}$, and let*

$$p_{\sigma,i} = \int_{H_i} \sigma(h)h d\mu_i(h)$$

for $i \in I \cup \{\infty\}$. Then $p_{\sigma,i} \to p_{\sigma,\infty}$ in $M(C^*(G))$.

Proof. Let $K_i = H_i \cap \ker\sigma$ and ν_i be normalised Haar measure on K_i for $i \in I \cup \{\infty\}$. Define $q_i = \int_{K_i} h d\nu_i(h)$ for $i \in I \cup \{\infty\}$.

We claim first that $q_i \to q_\infty$ in $M(C^*(G))$. Given $a \in C^*(G)$ and $\epsilon > 0$, take $b = q_\infty a$ and let $U = \{x \in G \mid \|xb - b\| < \epsilon\}$. Then U is open and contains K_∞. We want to show that $K_i \subset U$ eventually. If not, for every $i \in I$ there is $k_i \in K_i \setminus U$, and then by compactness of H there is a subnet (k_i) converging to an element $k \in H \setminus U$. For given i, the set $K_i k$ is open and contains k, and thus there is $j > i$ such that $k_j \in K_i k$. Therefore $k \in K_i k_j \subset K_i K_j = K_i$, and it follows that $k \in \cap K_i = K_\infty$, contradicting $k \notin U$. Since $q_i q_\infty = q_i$ we have

$$q_i a - q_\infty a = q_i b - b = \int_{K_i} (hb - b)\, d\nu_i(h).$$

But $K_i \subset U$ eventually, and so $\|q_i a - q_\infty a\| < \epsilon$ for sufficiently large i, proving the claim.

We next claim that $p_{\sigma,i} = q_i p_{\sigma,\infty}$ for sufficiently large i. Indeed, since the finite sets $\sigma(H_i)$ form a decreasing family, there is $i_0 \in I$ such that $\sigma(H_i) = \sigma(H_{i_0})$ for $i > i_0$. Then

$\sigma(H_\infty) = \cap_j \sigma(H_j) = \sigma(H_{i_0}) = \sigma(H_i)$ for $i > i_0$, and hence $H_i = K_i H_\infty$ $i > i_0$. Then for each $i > i_0$, the Haar measure μ_i on H_i is given by

$$\int_{H_i} f(h)\, d\mu_i(h) = \int_{K_i} \int_{H_\infty} f(kl)\, d\nu_i(k)\, d\mu_\infty(l),$$

and the claim follows because

$$p_{\sigma,i} = \int_{H_\infty} \int_{K_i} k\sigma(l)l\, d\nu_i(k)\, d\mu_\infty(l) = q_i p_{\sigma,\infty}.$$

Using the two claims we conclude the proof of the lemma by observing that

$$p_{\sigma,i} = q_i p_{\sigma,\infty} \to q_\infty p_{\sigma,\infty} = p_{\sigma,\infty} \text{ in } M(C^*(G)).$$

□

Proof of Theorem 4.13. Since (B, H) is directed, Lemma 4.11 shows that $x^{-1}p_\sigma x \geq p_\sigma$ for $x \in B^+$. The projections $p_{\sigma,x} := x^{-1}p_\sigma x$ have support in $x^{-1}H_\sigma x$, and then Lemma 4.14 implies that $p_{\sigma,x} \nearrow p_{\sigma,\infty}$ in $M(C^*(B_\sigma))$. Hence the ideal generated by p_σ in $C^*(B_\sigma)$ is $C^*(B_\sigma)p_{\sigma,\infty}$. □

Suppose in addition that H is normal in a normal abelian subgroup N of G and N is normal in B. Then $x^{-1}p_\sigma x$ is a projection in $C^*(N_\sigma)$ for all $x \in B^+$, and the Fourier transform applied to both sides of the inequality $x^{-1}p_\sigma x \geq p_\sigma$ gives $\widehat{p_{\sigma,x^{-1}}} \geq \widehat{p_\sigma}$. Hence (4.4) implies that $x \cdot (\sigma + H_\sigma^\perp) \subset \sigma + H_\sigma^\perp$ for $x \in B^+$. Since $\widehat{p_{\sigma,x^{-1}}}$ converges in $C_0(\widehat{N_\sigma})$ to $\widehat{p_{\sigma,\infty}}$, and since $\widehat{p_{\sigma,\infty}}$ equals the characteristic function of the set

$$\sigma + H_\infty^\perp := \{\alpha \in \widehat{N_\sigma} \mid \alpha|_{H_\infty} = \sigma|_{H_\infty}\},$$

we obtain the following strengthening of Corollary 4.6 and (4.6):

Corollary 4.15. *With the notation of the previous paragraph, the Hecke algebra $C^*(\mathcal{H}_\sigma(G, H))$ is Morita-Rieffel equivalent to*

$$\overline{C^*(B_\sigma)p_\sigma C^*(B_\sigma)} = C^*(N_\sigma)p_{\sigma,\infty} \rtimes B/N \cong C_0(\sigma + H_\infty^\perp) \rtimes B/N.$$

5. Applications

We begin this section by analysing a simple situation when the generalised Hecke algebra of a reduced Hecke triple (G, H, σ) does not depend on σ.

Proposition 5.1. *Let (G, H, σ) be a Hecke triple, and suppose that σ extends to a character of G. Then the map*

$$\Phi(f)(x) = \overline{\sigma(x)}f(x)$$

for $f \in \mathcal{H}_\sigma(G, H)$ and $x \in G$, is a $*$-isomorphism of $\mathcal{H}_\sigma(G, H)$ onto $\mathcal{H}(G, H)$.

Proof. The definition of Φ implies that $\Phi(f)(hxk) = \Phi(f)(x)$ for $f \in \mathcal{H}_\sigma(G, H)$, $x \in G$ and $h, k \in H$, so $\Phi(f)$ is H-biinvariant and thus Φ is well-defined. Using that $\Delta_H = \Delta_K$ shows that

$$\Phi(f)^*(x) = \Delta_H(x^{-1})\overline{\Phi(f)(x^{-1})} = \overline{\sigma(x)}\Delta_K(x^{-1})\overline{f(x^{-1})} = \Phi(f^*)(x),$$

so Φ is adjoint preserving. Finally, since σ is defined everywhere on G, a routine verification shows that $\Phi(f * g) = \Phi(f) * \Phi(g)$ for all $f, g \in \mathcal{H}_\sigma(G, H)$, as wanted. □

Example 5.2. Suppose that H and N are subgroups of a group G such that H is finite, $N \trianglelefteq G$, $G = HN$, and (G, H) is reduced. Suppose that σ is a character on H. Then $\sigma(hn) := \sigma(h)$ for $h \in H$ and $n \in N$ is a well-defined extension to a character on G. Hence $\mathcal{H}_\sigma(G, H)$ is isomorphic to $\mathcal{H}(G, H)$ by Proposition 5.1.

As a concrete example of this set-up we can take G to be the infinite dihedral group $\mathbb{Z} \rtimes_\psi \mathbb{Z}_2$ with generators a for $\mathbb{Z}$ and b for $\mathbb{Z}_2$, where $\psi_b(a) = a^{-1}$. Alternatively, G has presentation $\langle a, b \mid b^2 = 1, bab = a^{-1}\rangle$. Let $H = \langle b\rangle \cong \mathbb{Z}_2$, $N = \mathbb{Z}$, and consider the character $\sigma : H \to \mathbb{T}$, $\sigma(b) = -1$. Using either [29, Example 3.4] or [15, Example 10.1] we conclude that $\mathcal{H}_\sigma(G, H)$ does not have a largest C^*-norm because $\mathcal{H}(G, H)$ fails to have one.

Suppose that (G, H, σ) is a reduced Hecke triple. Let $(G_\sigma, H_\sigma, \sigma)$ and (G_0, H_0) be the Schlichting completions of (G, H, σ) and (G, H), respectively. Choose left invariant Haar measures μ on G_σ and ν on G_0, normalised so that $\mu(H_\sigma) = 1$ and $\nu(H_0) = 1$. Let p_σ be the projection defined in Theorem 2.9, and p_0 the projection χ_{H_0} in $C_c(G_0)$. In certain cases we can identify $p_\sigma C^*(G_\sigma)p_\sigma$ and $p_0 C^*(G_0)p_0$ inside the same algebra, as shown in the next proposition. In concrete examples, it suffices to verify whether $K \supseteq x_0 H x_0^{-1}$ for some x_0 in G, because then the continuity hypothesis is automatic.

Proposition 5.3. *Given a reduced Hecke triple (G, H, σ), suppose that σ extends to a character of G, and is continuous with respect to the Hecke topology from (G, H). Then $\iota : G_\sigma \to G_0$ is a topological isomorphism, and $\Phi(x) = \overline{\sigma(x)}x$ for $x \in G$ extends to an automorphism of $C^*(G_\sigma)$ which carries p_σ into p_0.*

Proof. The hypothesis implies that σ has a continuous extension σ_0 to H_0. By the second part of Theorem 1.4, the map ι is a topological isomorphism of G_σ onto G_0 and of H_σ onto H_0. Since the modular functions of G_0 and G_σ coincide on G by Corollary 2.5, the involution is preserved by Φ. □

Example 5.4. (The rational Heisenberg group.) We analyse now the Hecke pair studied in [15, Example 10.7]. We use the same notation, so

$$[u, v, w] := \begin{pmatrix} 1 & v & w \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } u, v, w \in \mathbb{Q}.$$

Then $G = \{ [u, v, w] \mid u, v \in \mathbb{Q}, w \in \mathbb{Q}/\mathbb{Z} \}$ and $H = \{ [u, v, 0] \mid u, v \in \mathbb{Z} \}$ form a reduced Hecke pair. We let N be the (abelian) subgroup of G with $u, v \in \mathbb{Z}$, and then $H \trianglelefteq N \trianglelefteq G$. Fix s, t in $\mathbb{Q}$ and let σ be the character of H given by

$$\sigma([m, n, 0]) = \exp(2\pi i(sm + tn)) \text{ for } m, n \in \mathbb{Z}. \tag{5.1}$$

Since $[u_1, v_1, w_1][u_2, v_2, w_2] = [u_1 + u_2, v_1 + v_2, w_1 + w_2 + v_1 u_2]$ in G, the equation (5.1) extends to a character σ on G given by $\sigma([u, v, w]) = \exp(2\pi i(su + tv))$. Thus $\mathcal{H}_\sigma(G, H)$ is isomorphic to $\mathcal{H}(G, H)$ by Proposition 5.1. We know from [15, Example 10.7] that the collection of sets

$$H_{x,y} = \{[u, v, 0] \mid u \in \mathbb{Z} \cap y\mathbb{Z}, v \in \mathbb{Z} \cap x\mathbb{Z}\}$$

forms a neighbourhood base at e when $x, y \in \mathbb{Z} \setminus \{0\}$. If we denote by b and d the denominators of s and t respectively, then

$$K = \{[m, n, 0] \mid sm + tn \in \mathbb{Z}\} \supset H_{d,b}.$$

Thus σ is continuous at e, hence everywhere, for the Hecke topology from (G, H), and so Proposition 5.3 implies that the Schlichting completion (G_0, H_0) of (G, H) is also the Schlichting completion of (G, H, σ). Then Corollary 4.10 or [15, Theorem 5.13] imply that $p_0 C^*(G_0) p_0$ is the largest C^*-completion of $\mathcal{H}_\sigma(G, H)$.

For different choices of σ, the ideals $\widehat{I_\sigma} \rtimes G/N$ from Theorem 4.5 are all isomorphic to $\widehat{I_0} \rtimes G/N$, where I_0 corresponds to the trivial character $\sigma \equiv 1$. Let $\mathcal{A}_f$ and $\mathcal{Z}$ respectively denote the ring of finite adeles and its compact open subring of integral adeles. To describe the sets Ω_σ defined in (4.5), we recall from [15] that $G_0 = \{[u, v, w] \mid u, v \in \mathcal{A}_f, w \in Q/Z\}$, with N_0 the subgroup with components $u, v \in \mathcal{Z}$, and hereby with $H_0 = \{[u, v, 0] \mid u, v \in \mathcal{Z}\}$. Then a computation shows that $\Omega_\sigma = [s, t, 0] + \Omega_0$, where Ω_0 corresponding to the trivial character σ is described in [15, Example 10.7]. The isomorphism Φ is obtained from translation by $[s, t, 0]$.

Example 5.5. (The p-adic $ax + b$-group.) Let p be a prime and denote

$$N := \mathbb{Z}[p^{-1}] = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z}, n \geq 0\}.$$

Let $G := N \rtimes \mathbb{Z}$ with $m \cdot b = bp^m$ for $m \in \mathbb{Z}$ and $b \in N$. Then G and $H := \mathbb{Z}$ form a reduced Hecke pair. Let q be a positive non-zero integer that is co-prime with p, and σ the character of H given by $\sigma(n) = \exp(\frac{2\pi i n}{q})$. Then $K = \ker \sigma = q\mathbb{Z}$, and (G, K) is also reduced. For $g = (x, p^k) \in G$ we have that $gHg^{-1} = p^{-k}\mathbb{Z}$, and hence deduce that σ is not continuous in the Hecke topology from (G, H). We will study $\mathcal{H}_\sigma(G, H)$ using Corollary 4.15.

In order to describe the Schlichting completion of (G, H, σ) we recall some facts about $\mathbf{q}$-adic integers and numbers, where $\mathbf{q}$ is a doubly infinite sequence of integers greater than one. We refer to [25, §12.3.35] for details (see also [12, §25.1]). We are interested in the particular sequence $\mathbf{q}$ in which $\mathbf{q}_0 = q$ and $\mathbf{q}_n = p$ for all other $n \in \mathbb{Z}$, and we view an element a of $\Omega_{\mathbf{q}}$ as a formal sum

$$a = \sum_{i=-N}^{-1} a_i p^i + a_* + q \sum_{i=0}^{\infty} a_i p^i$$

with $0 \leq a_i < p$ and $0 \leq a_* < q$. By denoting $a_- := \sum_{i=-N}^{-1} a_i p^i$ and $a_+ := \sum_{i=0}^{\infty} a_i p^i$ we have

$$a = a_- + a_* + q a_+ \tag{5.2}$$

with $a_+ \in \mathbb{Z}_p$. Elements a in $\Omega_{\mathbf{q}}^0$ are characterised by the condition that $a_- = 0$. The first theorem in [25, §12.3.35] says that there is an injective group homomorphism ϕ from $\mathbb{Z}[p^{-1}]$ into the locally compact, totally disconnected (additive) abelian group $\Omega_{\mathbf{q}}$, such that ϕ has dense range, and restricts to a bijection of $\mathbb{Z}$ onto a dense subgroup of the compact, totally disconnected subgroup $\Omega_{\mathbf{q}}^0$ of $\Omega_{\mathbf{q}}$.

Multiplication by p (with carry-over) is a continuous action of $\mathbb{Z}$ on our $\mathbf{q}$-adic numbers (because it is clearly continuous on the dense subset of elements a with a_+ finite), and so

ϕ extends to a group homomorphism from G into the locally compact, totally disconnected group $L := \Omega_{\mathbf{q}} \rtimes \mathbb{Z}$. The range of ϕ is still dense, and $M := \Omega_{\mathbf{q}}^0$ is compact, open in L. If $y \in G$ is such that $\phi(y) \in M$, then $\phi(y)_+$ is finite and $\phi(y)_- = 0$, and it follows that $\phi^{-1}(M) = H$. The formula

$$\chi(a_* + qa_+) := \exp(2\pi i a_*/q)$$

defines a character of M. Note that the kernel K of χ consists of the formal sums $\{qa_+ \mid a_+ \in \mathbb{Z}_p\}$. Now (L, K) being reduced is the same as $\bigcap_{m\in\mathbb{Z}} p^m K = \{0\}$, and this last identity can be verified directly using (5.2). From the definition we have $\chi \circ \phi|_H = \sigma$, and Theorem 1.4 gives the following.

Proposition 5.6. *The Schlichting completion of* (G, H, σ) *is* $(\Omega_{\mathbf{q}} \rtimes \mathbb{Z}, \Omega_{\mathbf{q}}^0, \chi)$.

It is also possible to identify $\Omega_{\mathbf{q}}$ as the topological limit $\varprojlim N/qp^n\mathbb{Z}$, where the bonding maps are reductions modulo $qp^n\mathbb{Z}$, see for example [15, Proposition 3.10]; then $\Omega_{\mathbf{q}}^0$ is the profinite group $\varprojlim \mathbb{Z}/qp^n\mathbb{Z}$.

The Schlichting completion of (G, H) is $(\mathbb{Q}_p \rtimes \mathbb{Z}, \mathbb{Z}_p)$, where $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are respectively the p-adic numbers and p-adic integers. Let $(a, k) \in N_\sigma \rtimes \mathbb{Z}$ with a as in (5.2). Then the homomorphism ι from Remark 1.5 sends $(a_- + a_* + qa_+, k)$ into $(a_- + a_* + a_+, k)$, and so is not a topological isomorphism.

Lemma 5.7. *Let* n_0 *be the smallest integer* $n > 0$ *such that* $p^n \equiv 1 \mod q$. *Then* $B = \mathbb{Z}[p^{-1}] \rtimes n_0\mathbb{Z}$.

Proof. It suffices to note that $xHx^{-1} \cap H$ is either $\mathbb{Z}$ or of the form $p^n\mathbb{Z}$ for $n \geq 0$, in which case $\sigma(p^n h) = \sigma(h)$ is equivalent to $p^n \equiv 1 \bmod q$. □

Lemma 5.7 implies that $\{ngn^{-1}H \mid n \in N, g \in H\backslash B/H\}$ contains infinitely many disjoint right cosets, and so we infer the following from [3, Corollary 1.10] and Theorem 3.5:

Corollary 5.8. $\mathcal{R}(G, H, \sigma)$ *is a factor.*

Since $B^+ = \{x \in B \mid xHx^{-1} \supset H\} = \mathbb{Z}[p^{-1}] \rtimes n_0\mathbb{N}$, the pair (B, H) is directed. Then $C^*(\mathcal{H}_\sigma(G, H))$ is equal to $p_\sigma C^*(B_\sigma)p_\sigma$ by Proposition 4.12, and is Morita-Rieffel equivalent to $C_0(\sigma + H_\infty^\perp) \rtimes B/N$ by Corollary 4.15. Towards describing the last crossed product we dwell a little longer on the structure of $N_\sigma = \Omega_{\mathbf{q}}$. Note that for a and b as in (5.2) with the sums in a_- and b_- starting from $-N$ and $-M$ respectively, $\frac{1}{q}(a \cdot b)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$, because in the product there are only finitely many terms not in $\mathbb{Z}$. With $e(x) := \exp(2\pi i x)$, we claim that

$$\langle a, b\rangle = e(\frac{1}{q}(a \cdot b)) \text{ for } a, b \in N_\sigma \tag{5.3}$$

is a well-defined duality[1] pairing. To prove this claim is essentially an argument similar to the proof of the second theorem in [25, §12.3.35], and we leave the details to the reader.

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[1] We can also appeal to the second theorem in [25, §12.3.35], which shows that $\langle a, b\rangle = \exp\Big[2\pi i \sum_{n=-M}^{N} b_n \Big(\sum_{m=n}^{N} \frac{a_m}{\mathbf{q}_n \cdots \mathbf{q}_m}\Big)\Big]$ implements a self-duality of N_σ. The third theorem in [25, §12.3.35] gives necessary and sufficient conditions on the $\mathbf{q}$-numbers that admit a multiplication, and our choice of $\mathbf{q}$ certainly fulfills those conditions.

One can check from (5.3) that the annihilator of $\Omega_{\mathbf{q}}^0$ can be identified as the set of sequences $\{q\sum_0^\infty a_i p^i\}$ with $0 \leq a_i < p$, i.e. as $q\mathbb{Z}_p$. Since $\sigma(x) = e(\frac{1}{q}x)$, the set defined in (4.3) of all extensions to N_σ is equal to

$$\sigma + H_\sigma^\perp = \{a \in N_\sigma \mid \langle a, b\rangle = \sigma(b), \forall b \in \Omega_{\mathbf{q}}^0\} = 1 + q\mathbb{Z}_p.$$

Since q is a unit in $\mathbb{Z}_p$, the element $w_0 := -q^{-1}$ belongs to $\mathbb{Z}_p$. We let

$$z_0 := 1 + qw_0 \in \Omega_{\mathbf{q}}^0 \setminus \{0\}. \tag{5.4}$$

Lemma 5.9. *We have* $p^{n_0} z_0 = z_0$ *and, in the notation of Theorem 4.13,* $H_\infty = \{jz_0 \mid 0 \leq j < q\}$.

Proof. By the choice of n_0, for each $k \in \mathbb{N}$ there is $s_k \in \mathbb{Z}$ such that $p^{kn_0} = 1 + qs_k$. Since $s_1 = q^{-1}(p^{n_0} - 1) = (1 - p^{n_0})w_0$ in $\mathbb{Z}_p$, we have

$$p^{n_0} z_0 = p^{n_0} + qp^{n_0} w_0 = 1 + q(s_1 + p^{n_0} w_0) = z_0,$$

as claimed. For the second claim, we have by definition that $H_\infty = \bigcap_{k\in\mathbb{N}} p^{kn_0}\Omega_{\mathbf{q}}^0$, Clearly $z_0 \in H_\infty$, and since $qz_0 = q(1 + qw_0) = 0$ we get $jz_0 \in H_\infty$ for $0 \leq j < q$.

To prove the other inclusion, we claim first that $q\mathbb{Z}_p \cap H_\infty = \{0\}$. Indeed, if $a \in \mathbb{Z}_p$ and $qa \in H_\infty$, then $p^{-n_0}qa \in \Omega_{\mathbf{q}}^0$, so there are $0 \leq b_* < q$ and $b_+ \in \mathbb{Z}_p$ such that

$$\begin{aligned} qa &= p^{n_0}(b_* + qb_+) = (1 + qs_1)b_* + qp^{n_0}b_+ \\ &= b_* + q(s_1 b_* + p^{n_0} b_+). \end{aligned}$$

It follows that $b_* = 0$ and $p^{-n_0}a \in \mathbb{Z}_p$. By repeating the argument, we conclude that $a \in \cap_{k\geq 0} p^{kn_0}\mathbb{Z}_p = \{0\}$. To finish off, suppose that $a = a_* + qa_+$ is in H_∞, where $0 \leq a_* < q$ and $a_+ \in \mathbb{Z}_p$. Then $a - a_* z_0 \in q\mathbb{Z}_p \cap H_\infty = \{0\}$, and so $a = a_* z_0$, as needed. ☐

Lemma 5.10. *The annihilator* $H_\infty^\perp$ *of* H_∞ *in* $\Omega_{\mathbf{q}}$ *is equal to* $\bigcup_{k\geq 0} p^{-kn_0}(q\mathbb{Z}_p)$ *and to the set* Y *of elements* $p^{-kn_0}\alpha_- + \alpha_* + q\alpha_+$ *such that* $k > 0$, $0 \leq \alpha_- < p^{kn_0}$, $\alpha_- + \alpha_* \in q\mathbb{Z}$, *and* $\alpha_+ \in \mathbb{Z}_p$.

Proof. Since by Lemma 5.9 H_∞ is a cyclic group generated by z_0,

$$H_\infty^\perp = \{z_0\}^\perp = \{a \in N_\sigma \mid az_0 \in q\mathbb{Z}_p\}.$$

Given a in $\Omega_{\mathbf{q}}$, we can write $a = p^{-kn_0}a_- + a_* + qa_+$ for $k > 0$, $0 \leq a_- < p^{kn_0}$, $0 \leq a_* < q$, and $a_+ \in \mathbb{Z}_p$. By Lemma 5.9, $az_0 = a(p^{kn_0}z_0) = (p^{kn_0}a)z_0$, and so az_0 has form $a_- + a_* + qa'_+$ for a'_+ in $\mathbb{Z}_p$. Thus $az_0 \in q\mathbb{Z}_p$ if and only if $a_- + a_* \in q\mathbb{Z}$, showing that $\{z_0\}^\perp = Y$.

Since $p^{n_0}z_0 = z_0$ and $q\mathbb{Z}_p \subset \{z_0\}^\perp$, we also have $\bigcup_{k\geq 0} p^{-kn_0}(q\mathbb{Z}_p) \subset \{z_0\}^\perp$. If now $a \in \{z_0\}^\perp$, we have seen that $a_- + a_* \in q\mathbb{Z}$, and then a calculation shows that $a \in p^{-kn_0}(q\mathbb{Z}_p)$, finishing the proof. ☐

Corollary 5.11. *Let* X_0 *denote the open and closed subset* $(1 + q\mathbb{Z}_p) \setminus p^{n_0}(1 + q\mathbb{Z}_p)$ *of* $\Omega_{\mathbf{q}}^0$. *Then the set* $\sigma + H_\infty^\perp$ *from Corollary 4.15 is the disjoint union of* p^{n_0}*-invariant sets*

$$\{z_0\} \cup \bigcup_{k\in\mathbb{Z}} p^{kn_0} X_0. \tag{5.5}$$

Proof. From its definition, the set $\sigma + H_\infty^\perp$ is equal to $1 + H_\infty^\perp$. Since

$$1 + p^{-kn_0}(q\mathbb{Z}_p) = p^{-kn_0}(1 + qs_k + q\mathbb{Z}_p) = p^{-kn_0}(1 + q\mathbb{Z}_p),$$

Lemma 5.10 implies that $\sigma + H_\infty^\perp = \bigcup_{k\geq 0} p^{-kn_0}(1 + q\mathbb{Z}_p)$. The inclusion $p^{ln_0}(1 + q\mathbb{Z}_p) \subset 1 + q\mathbb{Z}_p$ for all $l \geq 0$ implies that

$$\bigcup_{k\geq 0} p^{-kn_0}(1 + q\mathbb{Z}_p) = \bigcap_{l\geq 0} p^{ln_0}(1 + q\mathbb{Z}_p) \cup \bigcup_{k\in\mathbb{Z}} p^{kn_0} X_0.$$

To see that this decomposition is exactly (5.5), we need to verify that in the right hand side the union is over disjoint sets, and that

$$\bigcap_{l\geq 0} p^{ln_0}(1 + q\mathbb{Z}_p) = \{z_0\}. \tag{5.6}$$

First, the inclusions $p^{kn_0}X_0 \subset p^{kn_0}(1 + q\mathbb{Z}_p) \subset p^{n_0}(1 + q\mathbb{Z}_p)$ for $k > 0$ show that $p^{kn_0}X_0$ and X_0 are disjoint, and since $p^{-kn_0}X_0$ is disjoint from $1 + q\mathbb{Z}_p$, it is also disjoint from X_0. Next, suppose that z is in the left hand side of (5.6). Then there is $a_l \in \mathbb{Z}_p$ for each $l \geq 0$ such that, with s_l as in the proof of Lemma 5.9, we have

$$z = p^{ln_0}(1 + qa_l) = 1 + q(s_l + p^{ln_0}a_l).$$

Since $p^{ln_0}a_l$ is in $p^{ln_0}\mathbb{Z}_p$, it converges to $0 \in \mathbb{Z}_p$ as $l \to \infty$. A computation shows that $s_{l+1} = s_1 + s_l + qs_1s_l$ for all $l \geq 0$, so by passing to a subnet we may assume that s_l converges to an element s_* that satisfies $s_* = s_1 + s_* + qs_1s_*$. This equation has solution $s_* = -1/q = w_0$ in $\mathbb{Z}_p$, and hence $z = \lim(1 + q(s_l + p^{ln_0}a_l)) = 1 + qw_0 = z_0$, as claimed. To finish, note that if $p^{kn_0}z_0 \in X_0$ for some $k \in \mathbb{Z}$, then $z_0 \in X_0$, a falsehood. □

The main result concerning the structure of the Hecke algebra is the following.

Theorem 5.12. *The generalised Hecke C^*-algebra $C^*(\mathcal{H}_\sigma(G, H))$ is Morita-Rieffel equivalent to $C(\mathbb{T}) \oplus (C(X_0) \otimes \mathcal{K}(l^2(\mathbb{Z})))$.*

Proof. By Corollary 4.15, $\mathcal{H}_\sigma(G, H)$ is Morita-Rieffel equivalent to $C_0(\sigma + H_\infty^\perp) \rtimes B/N$. Applying Corollary 5.11 gives that $C_0(\sigma + H_\infty^\perp)$ is the direct sum of $C(\{z_0\})$ and $C_0(\bigcup_{k\in\mathbb{Z}} p^{kn_0}X_0)$. But $\bigcup_{k\in\mathbb{Z}} p^{kn_0}X_0$ is homeomorphic to $X_0 \times \mathbb{Z}$ via a map which is equivariant for the action of $B/N = n_0\mathbb{Z}$, and the result follows. □

To conclude, we recollect that for $\mathcal{H}(G, H)$, [18, Theorem 1.9] and [19] show that the universal C^*-completion $A := C^*(\mathcal{H}(N \rtimes \mathbb{Z}, \mathbb{Z}))$ is canonically isomorphic to a semigroup crossed product $C^*(N/\mathbb{Z}) \rtimes \mathbb{N}$. Then [6, Theorem 2.1] says that A has an ideal $C(\mathcal{U}(\mathbb{Z}_p)) \otimes \mathcal{K}(l^2(\mathbb{N}))$ such that the resulting quotient is $C(\mathbb{T})$ (here $\mathcal{U}(\mathbb{Z}_p)$ is the group of units in the ring of p-adic integers). The Schlichting completion of (G, H) is, as noted already, $(\mathbb{Q}_p \rtimes \mathbb{Z}, \mathbb{Z}_p)$. Then the Morita-Rieffel equivalence implemented by the full projection $\chi_{\mathbb{Z}_p}$ carries the ideal $C(\mathcal{U}(\mathbb{Z}_p)) \otimes \mathcal{K}(l^2(\mathbb{N}))$ of A to the ideal $C(\mathcal{U}(\mathbb{Z}_p)) \otimes \mathcal{K}(l^2(\mathbb{Z}))$ of $C^*(\mathbb{Q}_p \rtimes \mathbb{Z})$.

Example 5.13. (p-adic version of the Heisenberg group.) Suppose that p is a prime and q, r are integers co-prime with p and co-prime with each other. Let

$$G = \{\, [u, v, w] \mid u, v \in \mathbb{Z}[p^{-1}], w \in \mathbb{Z}[p^{-1}]/\mathbb{Z} \,\},$$

$H = \{\, [m,n,0] \mid m,n \in \mathbb{Z} \,\} \cong \mathbb{Z} \times \mathbb{Z}$, and $\sigma(m,n,0) = \exp(2\pi i(\frac{m}{q} + \frac{n}{r}))$. This is a reduced Hecke triple, and $K = \ker\sigma = q\mathbb{Z} \times r\mathbb{Z}$. It follows as in examples 5.4 and 5.5 that the Schlichting completion of (G,H,σ) consists of

$$G_\sigma = \{\, [a,b,w] \mid a \in \Omega_{\mathbf{q}}, b \in \Omega_{\mathbf{r}}, w \in \mathbb{Z}[p^{-1}]/\mathbb{Z} \,\},$$

the compact open subgroup $H_\sigma = \{\, [a,b,0] \mid a \in \Omega^0_{\mathbf{q}}, b \in \Omega^0_{\mathbf{r}}, \}$, and the natural extension of σ to H_σ. Since σ extends to a character of G, Proposition 5.1 implies that $\mathcal{H}_\sigma(G,H)$ is isomorphic to $\mathcal{H}(G,H)$. However, we claim that σ is not continuous for the Hecke topology from (G,H). Indeed, one can verify that for $x = [p^{-m}, p^{-n}, 0]$ in G, where $m,n \geq 0$, $K \cap xKx^{-1}$ is $p^m q\mathbb{Z} \times p^n r\mathbb{Z}$. The latter set can contain no $H \cap yHy^{-1}$, which has form $p^k\mathbb{Z} \times p^l\mathbb{Z}$ for $y = [p^{-k}, p^{-l}, w]$ in G. Thus the continuous map $\iota : G_\sigma \to G_0$ is not open.

Example 5.14. (The full $ax+b$-group of $\mathbb{Q}$.) Let $N = (\mathbb{Q}, +)$ and consider $Q = (\mathbb{Q}^*_+, \cdot)$ acting by multiplication $(x,k) \mapsto xk$ for $x \in \mathbb{Q}^*_+$, $k \in \mathbb{Q}$. Then $G := N \rtimes Q$ and $H = \mathbb{Z}$ form the reduced Hecke pair from [4]. We can identify G as a matrix group in the form

$$G = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \,\middle|\, a \in \mathbb{Q}^*_+, \, b \in \mathbb{Q} \right\},$$

and then N is the subgroup with $a = 1$ in which H is the subgroup with $b \in \mathbb{Z}$.

Let n be a non-zero positive integer and σ the character of H defined by $\sigma(m) = \exp(2\pi i m/n)$ for $m \in \mathbb{Z}$. Taking $x_0 = (1, n^{-1})$ in G shows that $K := \ker\sigma$ contains $x_0 H x_0^{-1}$, and so σ is continuous with respect to the Hecke topology from (G,H). It is known, see for example [17], that $(G_0 := \mathcal{A}_f \rtimes Q, H_0 := \mathcal{Z})$ is the Schlichting completion of (G,H), and then Theorem 1.4 implies that $G_\sigma = G_0$ and $H_\sigma = H_0$. As a consequence of the description of B below, we see that a neighbourhood subbase at e for the Hecke topology from (B,K) consists of the groups $\{mn\mathbb{Z} \mid m \equiv \pm 1 \bmod n\}$, and so does not contain the open set $n^2\mathbb{Z}$ in G_0. It was computed in [9, Lemma 3.2.3] that B consists of pairs $(r, a/b)$ with $r \in \mathbb{Q}, a, b \in \mathbb{N}, b > 0$, and such that $\gcd(a,b) = 1$, and $a - b \in n\mathbb{Z}$. We claim (without proof) that the following description of B is valid:

Lemma 5.15. *Let T_q be the subsemigroup $\{m \in \mathbb{N}^* \mid m-1 \in n\mathbb{N}$ or $m+1 \in n\mathbb{N}\}$ of $\mathbb{N}^*$. Then B is the subgroup $\mathbb{Q} \rtimes T_q T_q^{-1}$ of G, and (B,H) is directed.*

Here G_σ is the same for all σ and we want to study the relation between the different ideals $\overline{C^*(G_\sigma) p_\sigma C^*(G_\sigma)}$. By Theorem 4.5 this is the same as studying the ideals I_σ in $C^*(N_\sigma)$, or by Theorem 4.7 the sets Ω_σ defined in (4.5). If σ corresponds to $n \in \mathbb{N}^*$ then, since $N_\sigma = \mathcal{A}_f$ and $\mathcal{A}_f$ is self-dual with duality carrying $\mathcal{Z}$ into $\mathcal{Z}^\perp$, we have that $\sigma + H_\sigma^\perp = \{a \in \mathcal{A}_f \mid a - \sigma \in \mathcal{Z}^\perp\} = 1/n + \mathcal{Z}$. Thus the sets are given by

$$\Omega_n = \bigcup_{t \in Q} t(\frac{1}{n} + \mathcal{Z}).$$

We can then describe exactly the ideals $C_0(\Omega_n) \rtimes Q$ inside $C_0(\mathcal{A}_f) \rtimes Q$ corresponding to different choices of σ, and link to the results of [21] and [6].

Lemma 5.16. (a) *If q is a prime and $m > 0$, then $\Omega_{q^m} = \{x \in \mathcal{A}_f \mid x_q \neq 0\}$. In particular, Ω_{q^m} is independent of m.*

(b) *If* $n = q_1^{i_1}\cdots q_l^{i_l}$ *with the* q_j*'s different primes and each* $i_j > 0$*, then*

$$\Omega_n = \{x \in \mathcal{A}_f \mid x_{q_j} \neq 0,\, j = 1, \cdots, l\}.$$

Proof. We only prove part (b) for $l = 2$, as the rest follows by similar arguments. Thus we assume that $n = q^j r^i$ with q and r distinct primes.

The forward inclusion is obvious. To prove the other, we take $x \in \mathcal{A}_f$ with $x_q \neq 0$ and $x_r \neq 0$. We can write $x_q = q^l(y + q^j y_q)$ and $x_r = r^m(w + r^i w_r)$ with $0 < y < q^j$, q not dividing y, $y_q \in \mathbb{Z}_q$, $0 < w < r^i$, r not dividing w, and $w_r \in \mathbb{Z}_r$. Pick $s, t, c, d \in \mathbb{Z}$ such that $sy - tq^j = 1$ and $cw - dr^i = 1$. By the Chinese Remainder Theorem there is an integer k such that

$$k \equiv sr^m \mod q^j \text{ and } k \equiv cq^l \mod r^i.$$

Then $ky \equiv r^m \mod q^j$ and $kw \equiv q^l \mod r^i$, and so there are $y'_q \in \mathbb{Z}_q$ and $w'_r \in \mathbb{Z}_r$ such that

$$x_q = q^l r^m k^{-1}(1 + q^j r^i y'_q) \text{ and } x_r = q^l r^m k^{-1}(1 + q^j r^i w'_r).$$

Since q and r are units in $\mathbb{Z}_p$ for every prime p different from q and r, there is $z_p \in \mathbb{Z}_p$ such that $x_p = q^l r^m k^{-1}(1 + q^j r^i z_p)$. Hence with $z := (z_p) \in \mathcal{Z}$, and replacing (if necessary) k with $-k$, we have $x \in \Omega_a$, as claimed. □

We can now resume our description of $\mathcal{H}_\sigma(G, H)$. When p is a prime, [21, Proposition 2.5] says that $J_p := C_0(\mathcal{A}_f \setminus \{x \in \mathcal{A}_f \mid x_p = 0\}) \rtimes Q$ is one of the primitive ideals of the dilation $C_0(\mathcal{A}_f) \rtimes Q$ cf. [17] of the Hecke C^*-algebra of Bost and Connes, see also [6, §4]. Suppose that n is a non-zero positive integer and let S be the set of primes in the decomposition of n. Using Lemma 5.16 shows that $\mathcal{H}_\sigma(G, H)$ is Morita-Rieffel equivalent to

$$C_0(\Omega_n) \rtimes Q = C_0(\bigcap_{p \in S} \Omega_p) \rtimes Q = \bigcap_{p \in S} J_p.$$

Example 5.17. (The lamplighter group, cf. [9, §3.1]).

Suppose that F is a finite abelian group with identity e. Let

$$N_- = \bigoplus_{-\infty}^{0} F, \quad H = \bigoplus_{1}^{\infty} F,$$

and set $N = N_- \oplus H$. The forward shift α on N acts as $\alpha((x_k)_{k\in\mathbb{Z}}) = (y_k)_{k\in\mathbb{Z}}$, with $y_k = x_{k-1}$ for all $k \in \mathbb{Z}$.

It is proved in [9, Lemma 3.1.1] that $(G := N \rtimes_\alpha \mathbb{Z}, H)$ is a Hecke pair, and we note that (G, H) is reduced. Let H_0 be the profinite, hence compact, abelian group $\varprojlim_{n\geq 1} F^n$, identified as $\prod_{n=1}^{\infty} F$, and set $N_0 := N_- \oplus H_0$. We regard an element of N_0 as a sequence $(x_k)_{k\in\mathbb{Z}}$ in $\prod_{k\in Z} F$ such that for some integer n_0 we have $x_k = 0$ for $k < n_0$. Let β be the natural continuous extension of α to N_0. The inclusion of H in H_0 is equivariant for the actions of $\mathbb{Z}$, and so gives rise to a homomorphism $\phi : N \rtimes_\alpha \mathbb{Z} \to N_0 \rtimes_\beta \mathbb{Z}$. By construction, ϕ has dense range and $\phi^{-1}(H_0) = H$. Hence [15, Theorem 3.8] implies that $(G_0 := N_0 \rtimes_\beta \mathbb{Z}, H_0)$ is the Schlichting completion of (G, H).

The subset $T = \{(y, k) \in G \mid (y, k)H(y, k)^{-1} \supseteq H\}$ is a subsemigroup of G, and $T = \{(y, k) \in G \mid k \geq 0\}$. Thus $G = T^{-1}T$, so (G, H) is directed in the sense of [15, §6]. By

[15, Theorems 6.4 and 6.5], or by [18, Theorem 1.9] and [19], the universal C^*-completion $C^*(G,H)$ of $\mathcal{H}(G,H)$ is isomorphic to the corner in $C^*(G_0)$ determined by the full projection χ_{H_0}. Moreover, [15, Corollary 8.3] (or [22, Theorem 2.5]) imply that $C^*(G,H)$ is Morita-Rieffel equivalent to $C_0(\widehat{N_0}) \rtimes_{\widehat{\beta}} \mathbb{Z}$, where $\widehat{N_0} = \bigcup_{n\in\mathbb{Z}} \widehat{\beta}_n(H_0^\perp)$.

We now assume that σ is a character of H. Thus $\sigma = (\sigma_n)_{n\geq 1}$, where σ_n is a character of F for every $n \geq 1$. Note that $\sigma(H)$ is included in the finite set $\{\langle \pi, f\rangle_F \mid \pi \in \widehat{F}, f \in F\}$, where $(f, f') \mapsto \langle f, f'\rangle_F$ is a fixed self-duality of F. However, (G,K) will not be directed for arbitrary σ, and to proceed we specialise further.

When σ is not periodic, $B = N$ by [9]. Hence [9, Proposition 1.5.2] or Lemma 2.13 imply that $\mathcal{H}_\sigma(G,H) \cong \mathbb{C}(N/H)$. Therefore $C^*(\mathcal{H}_\sigma(G,H))$ is the group algebra $C^*(N_-)$.

Next we restrict the attention to 1-periodic characters, so we assume that $\sigma_n = \sigma_1$, $\forall n \geq 2$. We have that $K = \ker\sigma = \{f = (f_k)_k \in H \mid \sigma_1(\sum_k f_k) = 1\}$, and (G,K) is reduced.

Let M be the profinite group $\varprojlim_{n\geq 1}(F^n \oplus \sigma(H))$ with bonding homomorphisms

$$(f_1, \dots, f_n, f_{n+1}, \sigma(h)) \mapsto (f_1, \dots, f_n, \sigma(h))$$

from $F^{n+1} \oplus \sigma(H)$ onto $F^n \oplus \sigma(H)$ and canonical homomorphisms π^n from M onto $F^n \oplus \sigma(H)$. By viewing M as the subset of $\prod_{n\geq 1} F^n \oplus \sigma(H)$ of sequences compatible under all bonding maps, we see that for $n \geq 1$ and $h = (h_k)_{k\geq 0} \in H$, the formula $\pi_n(h) := (h_1, \dots, h_n, \sigma(h))$ defines an element of M. This gives a homomorphism $\pi : H \to M$ such that $\pi(h) := (\pi_n(h))_{n\geq 1}$ for $h \in H$. Since the cylinder sets $\{x \mid \pi^n(x) = \pi_n(h)\}$ form a basis for the topology on M, π has dense range. Let L be the locally compact group $L := (N_- \oplus \prod_{n\geq 1} F \oplus \sigma(H)) \rtimes_\beta \mathbb{Z}$, with the compact, open subgroup M, and define a homomorphism $\phi : G \to L$ and a continuous character $\chi : M \to \mathbb{T}$ by $\phi(y,h,k) = (y, \pi(h), k)$, and $\chi(x,s) = s$ for $(x,s) \in H_0 \oplus \sigma(H)$. The map ϕ has dense range because π does, and clearly $\phi^{-1}(M) = H$. For $h = (h_1, \dots, h_n, e, \dots)$ in H we have

$$\chi \circ \phi(h) = \chi \circ \pi(h) = \chi(h, \sigma(h)) = \sigma(h).$$

Hence Theorem 1.4 yields the following result, which includes [8, Example 4] (or [9, §3.1]).

Corollary 5.18. *The triple* (L, M, χ) *is the Schlichting completion* $(G_\sigma, H_\sigma, \sigma)$ *of* (G, H, σ). *Moreover, the closure of* N *in the Hecke topology from* (G,K) *is*

$$N_\sigma = N_- \oplus H_\sigma.$$

So G_σ and G_0 are different in this example. Since $B = G$ by [9, §3.1] and $H \trianglelefteq N \trianglelefteq G$, Corollary 4.10 implies that $p_\sigma C^*(G_\sigma) p_\sigma$ is the enveloping C^*-algebra of $\mathcal{H}_\sigma(G,H)$. By Theorem 4.7, this C^*-completion is Morita-Rieffel equivalent to $C_0(\Omega_\sigma) \rtimes_\beta \mathbb{Z}$, where Ω_σ is defined in (4.5). Using results from [9, §3.1], we will identify this set. The self-duality $(f, f') \mapsto \langle f, f'\rangle_F$ of F implements a self-duality

$$\langle (y_k)_{k\in\mathbb{Z}}, (y'_k)_{k\in\mathbb{Z}}\rangle = \prod_{n\geq 1} \langle y_{1-n}, y'_n\rangle_F \prod_{n\geq 1} \langle y'_{1-n}, y_n\rangle_F$$

of $N_0 = N_- \oplus H_0$. Note that under this identification ${H_0}^\perp$ is carried into H_0. The group $\sigma(H)$ is also self-dual with duality expressed in terms of a fixed generator ω as $\langle \omega^j, \omega^k\rangle = \omega^{jk}$

for $j, k \in \mathbb{Z}$. Hence $N_\sigma = N_0 \oplus \sigma(H)$ is self-dual. Under the described duality pairings, an element (x, ω^j) of N_σ is in $\sigma + H_\sigma^\perp$ precisely when $j = 1$ and $x \in {H_0}^\perp$. Thus $\sigma + H_\sigma^\perp = H_0 \oplus \omega$.

The action of $\mathbb{Z}$ on N_σ is β on N_0 and is trivial on $\sigma(H)$. The formulas in [9, Lemma 3.1.6] show that the same is true for the dual action $\hat{\beta}$, and then

$$\Omega_\sigma = \bigcup_{n\in\mathbb{Z}} \hat{\beta}_n(\sigma + H_\sigma^\perp) = (\bigcup_{n\in\mathbb{Z}} \hat{\beta}_n(H_0)) \oplus \{\omega\} = N_0 + \{\omega\}.$$

Finally, we get an isomorphism $C_0(\Omega_\sigma) \rtimes_{\hat{\beta}} \mathbb{Z} \cong (C_0(N_0) \rtimes_{\hat{\beta}} \mathbb{Z}) \oplus C(\mathbb{T})$; in other words, the ideal $\overline{C^*(G_\sigma)p_\sigma C^*(G_\sigma)}$ is isomorphic to $C^*(G_0) \oplus C(\mathbb{T})$.

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