HECKE C^*-ALGEBRAS AND SEMIDIRECT PRODUCTS

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Abstract.

Introduction

A *Hecke pair* (G,H) comprises a group G and a subgroup H for which every double coset is a finite union of left cosets, and the associated *Hecke algebra*, generated by the characteristic functions of double cosets, reduces to the group $*$-algebra of G/H when H is normal.

In [4] we introduced the *Schlichting completion* as a tool for analyzing Hecke algebras, based in part upon work of Tzanev [13]. (A slight variation on this construction appears in [14].) The idea is that $\overline{H}$ is a compact open subgroup of $\overline{G}$ such that the Hecke algebra of $(\overline{G},\overline{H})$ is naturally identified with the Hecke algebra $\mathcal{H}$ of (G,H). The characteristic function p of $\overline{H}$ is a projection in the group C^*-algebra $A := C^*(\overline{G})$, and $\mathcal{H}$ can be identified with $pC_c(\overline{G})p \subseteq A$; thus closure of $\mathcal{H}$ in A coincides with the corner pAp, which is Morita-Rieffel equivalent to the ideal $\overline{ApA}$.

In [4] we were mainly interested in studying when pAp is the enveloping C^*-algebra of the Hecke algebra $\mathcal{H}$, and when the projection p is full in A, making the C^*-completion pAp of $\mathcal{H}$ Morita-Rieffel equivalent to the group C^*-algebra A. We had most success when $G = N \rtimes Q$ was a semidirect product with H contained in the normal subgroup N. In this paper we again consider $G = N \rtimes Q$, but now we allow the Hecke subgroup H to be spread across both N and Q. This leads to a refinement of the Morita-Rieffel equivalence $ApA \underset{\text{MR}}{\sim} pAp$ (see Theorem 4.1).

We begin in Section 1 by recalling our conventions from [4] regarding Hecke algebras. In Section 2 we describe the main properties of our

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group-theoretic setup for Hecke semidirect products. In order to effectively analyze how our semidirect-product decomposition affects the Hecke topology, we need to go into somewhat more detail than might be expected. In particular, we must exercise some care to obtain the semidirect-product decomposition for the Schlichting completion (see Corollary 2.8), and to describe various bits of this completion as inverse limits of groups (see Theorem 2.14).

In Section 4 we prove a Morita-Rieffel equivalence (Theorem 4.1) which takes full advantage of the semidirect-product decomposition, and we look briefly at the special case where the normal subgroup is abelian. In Section 3, to prepare the way for Section 4, we prove a few abstract results concerning crossed products, projections, and compact normal subgroups. **(ML: I think we maybe should put in some more details about section 3+4 in the introduction. After all, this is what most (all) will read anyway.) (JQ: Ok, I'll think about this.)**

Finally, in Section 5 we give some examples to illustrate our results. Classical Hecke algebras have most commonly treated pairs of semi-simple groups such as $(\mathrm{GL}(n,\mathbb{Q}),\mathrm{SL}(n,\mathbb{Z}))$. The work of Bost and Connes [1] showed the importance of also studying Hecke pairs of solvable groups. In the examples we mostly deal with the following situation:

Let K be either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{Z}[p^{-1}]$ of rational numbers with denominators of the form p^n. We take $N = K^2$, $M = \mathbb{Z}^2$, Q to be a subgroup of $\mathrm{GL}(2,K)$ containing the diagonal subgroup, and $R = Q \cap \mathrm{SL}_{\pm}(2,\mathbb{Z})$. Q acts on N in the obvious way. Then it is not so difficult to see that the Schlichting completions are p-adic or adelic versions of the same groups.

As to specific examples we look at the algebra studied by Connes-Marcoli in [2], see also [7]. Here R is not normal in Q, so the full results of Section 4 do not apply. On the other hand, if R is normal in Q then Corollary 4.7 does apply, and as in [8] one can use the Mackey orbit method to study the ideal structure of the C^*-algebras involved. A particular example of this is the $ax+b$-group over a quadratic extension $K[\sqrt{d}]$ treated in [6], and we shall see that this example raises some interesting questions. We also look at a nilpotent example, *i.e.*, one version of the Heisenberg group over the rationals.

After we had completed the research for this paper, we became aware of the recent preprint [7], which treats semidirect-product Hecke pairs in a way quite similar to ours. [7] and the present paper were written independently, and the techniques have only incidental overlap. We should mention that we treat only the case where M normal in N, while the context in [7] seems to be more general. Thus, for example, it would be difficult to adapt our results on inverse limits (see Subsection 2.4 to the context of [7]. **(JQ: I'll have to insert a little more detail earlier in this intro introducing M, N, *etc.*.)**

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1. Preliminaries

We adopt the conventions of [4], which contains more references. A *Hecke pair* (G, H) comprises a group G and a *Hecke subgroup* H, *i.e.*, one for which every double coset HxH is a finite union of left cosets $\{y_1H, \dots, y_{L(x)}H\}$. A good reference for the basic theory of Hecke pairs is [5]. A Hecke pair (G, H) is *reduced* if $\bigcap_{x\in G} xHx^{-1} = \{e\}$, and a reduced Hecke pair (G, H) is a *Schlichting pair* if G is locally compact Hausdorff and H is compact and open in G. In [4, Theorem 3.8], we gave a new proof of [13, Proposition 4.1], which says that every reduced Hecke pair (G, H) can be embedded in an essentially unique Schlichting pair pair $(\overline{G}, \overline{H})$, which we call the *Schlichting completion* of (G, H). Specifically, $\overline{G}$ is the completion of G in the (two-sided uniformity defined by the) *Hecke topology* having a local subbase $\{xHx^{-1} : x \in G\}$ of neighborhoods of e, and $(\overline{G}, \overline{H})$ is unique in the sense that if (L, K) is any Schlichting pair and $\sigma : G \to L$ is a homomorphism such that $\sigma(G)$ is dense and $H = \sigma^{-1}(K)$, then σ extends uniquely to a topological isomorphism $\overline{\sigma} : \overline{G} \to L$, and moreover $\overline{\sigma}(\overline{H}) = K$.

The associated *Hecke algebra* is the vector subspace $\mathcal{H}$ of $\mathbb{C}^G$ spanned by the characteristic functions of double H-cosets, with operations defined by

$$f * g(x) = \sum_{yH\in G/H} f(y)g(y^{-1}x)$$
$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}),$$

where $\Delta(x) = L(x)/L(x^{-1})$. Warning: some authors do not include the factor of Δ in the involution; for us it arises naturally when we embed $\mathcal{H}$ in $C_c(\overline{G})$. One way to see how this embedding goes is the following: let $p = \chi_{\overline{H}}$, which is a projection in $C_c(\overline{G})$ when the Haar measure on $\overline{G}$ has been normalized so that $\overline{H}$ has measure 1. Then the restriction map $f \mapsto f|G$ gives a $*$-isomorphism of the convolution algebra $pC_c(\overline{G})p$ onto $\mathcal{H}$.

(JQ: What else to put here? Probably should give a summary of the prelims in [4], plus a summary of semi-direct product set-up, with references to [7] and [2]?)

Notation. $H < G$ means H is a subgroup of G. $H \triangleleft G$ means H is a normal subgroup of G.

2. Groups

Here we describe the main properties of our group-theoretic setup for Hecke semidirect products. *Throughout this paper*, G is a group which is an internal semidirect product of $N \triangleleft G$ by $Q < G$; so $N \cap Q = \{e\}$ and $G = NQ$. We write $G = N \rtimes Q$. Further, $M \triangleleft N$ and R is a subgroup of Q which normalizes M, so that the internal semidirect product $H = MR = M \rtimes R$ is a subgroup of G. Briefly:

$$\begin{array}{ccccc} G & = & N & \rtimes & Q \\ \vee & & \triangledown & & \vee \\ H & = & M & \rtimes & R. \end{array} \tag{2.1}$$

In many examples we will have $R \triangleleft Q$, and some of the proofs are easier in this case.

(The following questions should also appear in the Introduction.) In this section we answer the following questions in terms of N, Q, M, and R: When is (G, H) a Hecke pair? When is (G, H) reduced? What is $(\overline{G}, \overline{H})$?

We need to establish many elementary facts from groups theory which are not standard, so we will give more detail than might seem necessary.

2.1. **Generalities.** We will be interested in subgroups of H of the form LS, where $L < M$ and $S < R$. Note that $LS < MR$ if and only if S normalizes L.

Lemma 2.1. *If A, B, C are subgroups of G with:*

(i) $A \supseteq B$*;*
(ii) $A \cap C = \{e\}$*;*
(iii) $AC = CA$*;*
(iv) $BC = CB$*,*

then

$$[AC : BC] = [A : B].$$

Proof. The map $aB \mapsto aBC : A/B \to AC/BC$ is obviously well-defined and surjective, and is injective because

$$a_1BC = a_2BC \implies a_2^{-1}a_1 \in BC \cap A = B. \qquad \square$$

Corollary 2.2. *Suppose $L < M$ and $S < R$, and suppose S normalizes L, so that LS is a subgroup of MR. Then*

$$[M : L][R : S] = [MR : LS].$$

Proof. We have

$$[MR : LS] = [MR : MS][MS : LS],$$

so the result follows from the above lemma. □

Notation and Terminology. For any subgroup K of G and $x \in G$, we define

$$K_x = K \cap xKx^{-1}.$$

Thus K_x is precisely the subgroup of stabilizers of the coset xK under the action of K on G/K by *left translation*, and

$$[K : K_x] = |KxK/K|\,. \tag{2.2}$$

If T is another subgroup of G, we let

$$T_{x,K} = \{t \in T \mid txKt^{-1} = xK\}$$

denote the subgroup of stabilizers of xK under the action of T by *conjugation* on the set of all subsets of G; thus

$$[T : T_{x,K}] = \left|\{txKt^{-1} \mid t \in T\}\right|. \tag{2.3}$$

Note that if T normalizes K, then the conjugation action of T descends to G/K.

For $E \subseteq G$, we further define

$$K_E = \bigcap_{x\in E} K_x \quad \text{and} \quad T_{E,K} = \bigcap_{x\in E} T_{x,K}.$$

It will also be useful to observe that if $\{M_i\}_{i\in I}$ is a family of subgroups of N and $\{R_i\}_{i\in I}$ is a family of subgroups of Q such that R_i normalizes M_i for each $i \in I$, then, because $N \cap Q = \{e\}$, we have

$$\bigcap_{i\in I} M_iR_i = \Big(\bigcap_{i\in I} M_i\Big)\Big(\bigcap_{i\in I} R_i\Big) \tag{2.4}$$

Lemma 2.3. *Let L be a subgroup of N which is normalized by R. For any $r \in R$ and $n \in N$, the following are equivalent*:

(i) *$r \in R_{n,L}$;*

(ii) *$rnr^{-1} \in nL$;*

(iii) $r \in nLRn^{-1}$.

Sketch of Proof. (i) $\implies$ (ii) $\implies$ (iii) are clear. (iii) $\implies$ (ii) uses $N \cap Q = \{e\}$. (ii) $\implies$ (i) uses normalizedness of L by R. $\square$

Taking $L = M$ in Lemma 2.3 and using $H = MR$, we have

$$R_{n,M} = R \cap nMRn^{-1} = R \cap nHn^{-1} \supseteq R \cap nRn^{-1} = R_n. \tag{2.5}$$

From this we deduce:

Lemma 2.4. *For any $n \in N$ and $q \in Q$,*

(i) $H_n = MR_{n,M}$;
(ii) $H_q = M_qR_q$;
(iii) $H_{qn} \cap H_q = M_q(qR_{n,M}q^{-1} \cap R)$.

Proof. (i) Suppose $h = mr \in H_n$ for $m \in M$ and $r \in R$. Then

$$r \in m^{-1}nMRn^{-1} = n(n^{-1}m^{-1}nMR)n^{-1} = nMRn^{-1} = nHn^{-1},$$

so (using (2.5))

$$mr \in m(R \cap nHn^{-1}) \subseteq MR_{n.M}.$$

Thus $H_n \subseteq MR_{n.M}$. Conversely, also using (2.5),

$$MR_{n,M} = M(R \cap nHn^{-1}) \subseteq MR \cap MnHn^{-1} = H \cap nHn^{-1} = H_n.$$

(ii) We have

$$\begin{aligned} H_q &= H \cap qHq^{-1} = MR \cap (qMq^{-1})(qRq^{-1}) \\ &= (M \cap qMq^{-1})(R \cap qRq^{-1}) = M_qR_q, \end{aligned}$$

by (2.4).

(iii) We have

$$\begin{aligned} H_{qn} \cap H_q &= H \cap qHq^{-1} \cap qnHn^{-1}q^{-1} = H \cap q(H_n)q^{-1} \\ &= MR \cap (qMq^{-1})(qR_{n,M}q^{-1}) = M_q(R \cap qR_{n,M}q^{-1}), \end{aligned}$$

using part (i) and (2.4). $\square$

2.2. Hecke pairs.

Since $[H : H_x] = |HxH/H|$ for any $x \in G$, the pair (G, H) is Hecke if and only if each subgroup H_x has finite index in H. Applying this to the pair $(N \rtimes Q, Q)$, we see that $(N \rtimes Q, Q)$ is Hecke if and only if $[Q : Q_n] = [Q : Q_{n,\{e\}}] < \infty$ for each $n \in N$. The next lemma extends this observation to our more general context.

Proposition 2.5. *The following are equivalent*:

(i) *(G, H) is a Hecke pair*
(ii) *$[R : R_q]$, $[M : M_q]$ and $[R : R_{n,M}]$ are all finite for each $q \in Q$ and $n \in N$*

(iii) *(Q, R), (G, M) and $(N/M \rtimes R, R)$ are Hecke pairs*
(iv) *(Q, R), (G, M) and (NR, H) are Hecke pairs.*

Proof. If (G, H) is a Hecke pair, then for all $q \in Q$ and $n \in N$ we have

$$[M : M_q][R : R_q] = [MR : M_q R_q] = [H : H_q] < \infty$$

and

$$[R : R_{n,M}] = [MR : MR_{n,M}] = [H : H_n] < \infty,$$

so (i) implies (ii). Conversely, assuming (ii), for any $q \in Q$ and $n \in N$, Lemma 2.4 gives

$$\begin{aligned}[H : H_{qn}] \leq [H : H_{qn} \cap H_q] &= [MR : M_q(qR_{n,M}q^{-1} \cap R)] \\ &= [M : M_q][R : qR_{n,M}q^{-1} \cap R] \\ &= [M : M_q][R : R_q][R_q : qR_{n,M}q^{-1} \cap R],\end{aligned}$$

which is finite because for any subgroups $S \supset T$ of G we have $[R \cap S : R \cap T] \leq [S : T]$ and $[qSq^{-1} : qTq^{-1}] = [S : T]$. Thus (ii) implies (i).

If $q \in Q$ and $n \in N$ then $qnMn^{-1}q^{-1} = qMq^{-1}$, so $[M : M_q] < \infty$ for all $q \in Q$ if and only if (G, M) is Hecke. As observed above, R is a Hecke subgroup of $N/M \rtimes R$ if and only if, for each $nM \in N/M$, the stability subgroup (for conjugation) of R at nM has finite index in R. Since this subgroup is precisely $R_{n,M}$, we have $[R : R_{n,M}] < \infty$ for all $n \in N$ if and only if $(N/M \rtimes R, R)$ is Hecke. Therefore (ii) if and only if (iii).

Finally, if $n \in N$ and $r \in R$ then $nrHr^{-1}n^{-1} = nHn^{-1}$, so

$$[H : H_{nr}] = [H : H_n] = [R : R_{n,M}],$$

therefore (iii) if and only if (iv). □

Proposition 2.6. *Suppose (G, H) is a Hecke pair. Then the following are equivalent*:

(i) *(G, H) is reduced;*
(ii) *$M_Q = \{e\}$ and $R_{N,\{e\}} \cap R_Q = \{e\}$.*

Proof. Since (G, H) is reduced if and only if $H_G = \{e\}$, the lemma will follow easily from the identity

$$H_G = M_Q(R_{N,M_Q} \cap R_Q). \tag{2.6}$$

To establish (2.6), we first use Lemma 2.4 (iii) and Corollary 2.2 to get

$$\begin{aligned}H_G = \bigcap_{x\in G} H_x = \bigcap_{q\in Q, n\in N} H_{qn} &= \bigcap_{q\in Q, n\in N} (H_q \cap H_{qn})\\ &= \bigcap_{q\in Q, n\in N} M_q(qR_{n,M}q^{-1} \cap R)\\ &= \Big(\bigcap_{q\in Q} M_q\Big)\Big(\bigcap_{q\in Q, n\in N} qR_{n,M}q^{-1} \cap R\Big).\end{aligned}$$

Further,

$$\begin{aligned}\bigcap_{q\in Q, n\in N} qR_{n,M}q^{-1} \cap R &= \bigcap_{q\in Q, n\in N} R_{qnq^{-1},qMq^{-1}} \cap R_q\\ &= \bigcap_{q\in Q, n\in N} R_{n,qMq^{-1}} \cap \bigcap_{q\in Q} R_q\\ &= R_{N,M_Q} \cap R_Q.\end{aligned}$$

□

Note that $R_{N,\{e\}}$ consists of those elements of R which commute element-wise with N.

2.3. **Hecke topology.** In addition to our standing assumptions (2.1), now assume that (G, H) is a reduced Hecke pair. Let $(\overline{G}, \overline{H})$ denote its Schlichting completion.

Proposition 2.7. *The relative Hecke topologies of the important subgroups have the following subbases at the identity:*

(i) *for both* N *and* M: $\{M_q \mid q \in Q\}$*;*
(ii) *for* Q: $\{qR_{n,M}q^{-1} \mid q \in Q, n \in N\}$*;*
(iii) *for* R: $\{R \cap qR_{n,M}q^{-1} \mid q \in Q, n \in N\}$.

Proof. (i) follows from the computations

$$\begin{aligned}N \cap qnHn^{-1}q^{-1} &= qn(N \cap H)n^{-1}q^{-1}\\ &= qnMn^{-1}q^{-1} = qMq^{-1}\end{aligned}$$

and

$$\begin{aligned}M \cap qnHn^{-1}q^{-1} &= M \cap N \cap qnHn^{-1}q^{-1}\\ &= M \cap qMq^{-1} = M_q.\end{aligned}$$

For (ii), we have

$$Q \cap qnHn^{-1}q^{-1} = q(Q \cap nMRn^{-1})q^{-1},$$

and

$$\begin{aligned} Q \cap nMRn^{-1} &\subset Q \cap MNRN \\ &= Q \cap NR = R, \end{aligned}$$

so

$$Q \cap nMRn^{-1} = R \cap nHn^{-1} = R_{n,M},$$

hence

$$Q \cap qnHn^{-1}q^{-1} = qR_{n,M}q^{-1}.$$

Finally (iii) follows from

$$R \cap qnHn^{-1}q^{-1} = R \cap Q \cap qnHn^{-1}q^{-1} = R \cap qR_{n,M}q^{-1}. \qquad \square$$

Corollary 2.8. *The Schlichting completions are semidirect products:*

$$\overline{G} = \overline{N} \rtimes \overline{Q} \quad \text{and} \quad \overline{H} = \overline{M} \rtimes \overline{R},$$

where the closures are all taken in $\overline{G}$.

Proof. First of all, to show that $\overline{G}$ is the semidirect product $\overline{N} \rtimes \overline{Q}$ of its subgroups $\overline{N}$ and $\overline{Q}$ requires:

(i) $\overline{N} \lhd \overline{G}$;
(ii) $\overline{G} = \overline{N}\,\overline{Q}$;
(iii) $\overline{N} \cap \overline{Q} = \{e\}$;
(iv) $\overline{G}$ has the product topology of $\overline{N} \times \overline{Q}$.

Item (i) is obvious. To see (ii), note that the subgroup $\overline{N}\,\overline{Q}$ contains both $G = NQ$ and $\overline{M}\,\overline{R}$. Since $\overline{M}$ is compact, the subgroup $\overline{M}\,\overline{R}$ is closed, and it follows that $\overline{H} = \overline{M}\,\overline{R}$. This implies (ii), since every coset in $\overline{G}/\overline{H}$ can be expressed in the form $x\overline{H}$ for $x \in G$.

For (iii), note that the quotient map $\psi : G \to Q \subseteq \overline{Q}$ is continuous for the Hecke topology of G and the relative Hecke topology of Q, because a typical subbasic neighborhood of e in Q is of the form $qR_{n,M}q^{-1}$ for $q \in Q$ and $n \in N$, and

$$\psi^{-1}(qR_{n,M}q^{-1}) = NqR_{n,M}q^{-1}$$

contains the neighborhood

$$H_{qn} \cap H_q = M_q(R \cap qR_{n,M}q^{-1})$$

of e in G. Since $\overline{Q}$ is a complete topological group, ψ extends uniquely to a continuous homomorphism $\overline{\psi} : \overline{G} \to \overline{Q}$. Because ψ takes N to e and agrees with the inclusion map on Q, by density and continuity $\overline{\psi}$ takes $\overline{N}$ to e and agrees with the inclusion map on $\overline{Q}$. Therefore $\overline{N} \cap \overline{Q} = \{e\}$.

To see how (iv) follows, first note that the product topology is stronger than the Hecke topology, because if $n_i \to n$ in $\overline{N}$ and $q_i \to q$ in

$\overline{Q}$ then $n_i q_i \to nq$ in $\overline{G}$ by continuity of multiplication in a topological group. On the other hand, since the quotient map has been shown to be continuous, so is the map

$$nq \mapsto n = (nq)q^{-1},$$

i.e., the map $x \mapsto x\psi(x)^{-1}$, so the Hecke topology is stronger than the product topology.

It only remains to show that $\overline{H} = \overline{M} \rtimes \overline{R}$, but this follows immediately: we have $\overline{M} \cap \overline{R} = \{e\}$, and the subgroup $\overline{M}\,\overline{R}$ has the product topology since $\overline{N}\,\overline{Q}$ does. □

2.4. **Inverse limits.** Here we again assume that (G, H) is a reduced Hecke pair. For each of our groups M, N, R, H, and Q we want to describe the closure as an inverse limit of groups, so that we capture both the algebraic and the topological structure. From [4, Proposition 3.10], we know that the closure is topologically the inverse limit of the coset spaces of finite intersections of stability subgroups. To get the algebraic structure we need enough of these intersections to be normal subgroups. In the case of M and N, we already have what we need, since each M_q is normal in N, and hence also in M. However, for R we need to do more work. Recall from Lemma 2.4 that for each $q \in Q$ and $n \in N$ we have

$$H_{qn} \supseteq H_q \cap H_{qn} = M_q(qR_{n,M}q^{-1} \cap R).$$

Lemma 2.9. *Suppose $L < M$ and $S < R$. Then $LS \lhd MR$ if and only if*

(i) $L \lhd MR$,
(ii) $S \lhd R$, *and*
(iii) $S \subseteq R_{M,L}$.

Moreover, in this case

$$MR/LS \cong (M/L) \rtimes (R/S).$$

Proof. First assume $LS \lhd MR$. Then

$$S = R \cap LS \lhd R \cap MR = R,$$

and since $M \lhd MR$, we also have

$$L = M \cap LS \lhd MR.$$

For (iii), fix $s \in S$ and $m \in M$. Then $m^{-1}sm \in LS$ because $LS \lhd MR$, so $m^{-1}sms^{-1} \in LS$. On the other hand, $m^{-1}sms^{-1} \in M$ because $S \subseteq R$ and R normalizes M. Thus

$$m^{-1}sms^{-1} \in LS \cap M = L,$$

so $s \in R_{m.L}$.

Conversely, assume (i)–(iii). Then it suffices to show that M conjugates S into LS: for $m \in M$ and $s \in S$ we have $m^{-1}sm \in LS$ by Lemma 2.3 (iii), because

$$s \in S \cap R_{m,L} = S_{m,L}.$$

For the last statement, it is routine to verify that the map

$$mrLS \mapsto (mL, rS) \qquad \text{for } m \in M, r \in R$$

gives a well-defined isomorphism. □

Notation. For $E \subset Q$ and $F \subset N$ put

$$R_F^E = \bigcap_{q\in E} qR_{F,M}q^{-1} \cap R.$$

Recall that $R_{F,M} = \bigcap_{n\in F} R_{n,M}$.

Note that the families

$$\{M_E : E \subset Q \text{ finite}\} \quad \text{and} \quad \{R_F^E : E \subset Q \text{ and } F \subset N \text{ finite}\}$$

are neighborhood bases at e in the relative Hecke topology of M and R, respectively.

Notation. Let $\mathcal{E}$ be the family of all subsets $E \subset Q$ such that:

(i) E is a finite union of cosets in Q/R;
(ii) $e \in E$;
(iii) $RE = E$,

and let $\mathcal{F}$ be the family of all pairs (E, F) such that:

(iv) $E \in \mathcal{E}$;
(v) F is a finite union of cosets in N/M;
(vi) $q^{-1}Mq \subset F$ for all $q \in E$.

Lemma 2.10. *For all* $(E, F) \in \mathcal{F}$*:*

(i) $R_F^E \lhd R$;
(ii) $[R : R_F^E] < \infty$.

Proof. R_F^E is a subgroup of R because $R_{F,M}$ is. For $r \in R$ we have

$$rR_F^E r^{-1} = \bigcap_{q\in E} r(qR_{F,M}q^{-1} \cap R)r^{-1} = \bigcap_{q\in E} rqR_{F,M}q^{-1}r^{-1} \cap R = R_F^E$$

since $rE = E$. This proves (i).

For (ii), first note that $[R : R_{F,M}] < \infty$ because $|F/M| < \infty$ and $R_{n,M}$ only depends upon the coset nM. Thus

$$R_0 := \bigcap_{r\in R} rR_{F,M}r^{-1}$$

is cofinite in R. For each coset tR contained in E we have

$$\bigcap_{q\in tR} qR_{F,M}q^{-1} = \bigcap_{r\in R} trR_{R,M}r^{-1}t^{-1} = tR_0t^{-1}.$$

Thus

$$\bigcap_{q\in tR} qR_{F,M}q^{-1} \cap R$$

is cofinite in R. Letting $E = \{t_1R, \dots, t_kR\}$, it follows that

$$\bigcap_{q\in E} qR_{F,M}q^{-1} \cap R = \bigcap_{i=1}^{k} \left(\bigcap_{q\in t_iR} qR_{F,M}q^{-1} \cap R \right)$$

is cofinite in R. □

Lemma 2.11. *For all $E \in \mathcal{E}$:*

(i) $M_E \lhd N$;
(ii) $M_E \lhd M$;
(iii) $M_E \lhd H$;
(iv) $[M : M_E] < \infty$.

Proof. (i) follows since $M_q \lhd N$ for each q.

(ii). $M_E \subset M$ since $e \in E$, so $M_E \lhd M$ by (i).

(iii). For $r \in R$ we have

$$rM_Er^{-1} = \bigcap_{q\in E} r(qMq^{-1} \cap M)r^{-1} = \bigcap_{q\in E} rqMq^{-1}r^{-1} \cap M = M_E$$

since $rE = E$. Thus $M_E \lhd MR = H$ by (ii).

(iv). For each coset tR contained in E we have

$$\bigcap_{q\in tR} qMq^{-1} = \bigcap_{r\in R} trMr^{-1}t^{-1} = tMt^{-1}.$$

Thus $\bigcap_{q\in tR} M_q = M_t$ is cofinite in M. Letting $E = \{t_1R, \dots, t_kR\}$, it follows that

$$\bigcap_{q\in E} M_q = \bigcap_{i=1}^{k} \bigcap_{q\in t_iR} M_q = \bigcap_{i=1}^{k} M_{t_i}$$

is cofinite in M. □

Lemma 2.12. *For all $(E, F) \in \mathcal{F}$ we have*

$$R_F^E \subset R_{M,M_E}.$$

Proof. Fix $s \in R_F^E$ and $m \in M$; we need to show that $s \in R_{m,M_E}$. Thus, for $q \in E$, we must show

$$m^{-1}sms^{-1} \in qMq^{-1}.$$

We have $q^{-1}mq \in F$, so

$$s \in qR_{q^{-1}mq,M}q^{-1}.$$

It follows that

$$qm^{-1}sms^{-1}q^{-1} = (qm^{-1}q^{-1})(qsq^{-1})(qmq^{-1})(qs^{-1}q^{-1}) \in M,$$

hence $m^{-1}sms^{-1} \in qMq^{-1}$, as desired. □

Corollary 2.13. *For all* $(E, F) \in \mathcal{F}$ *we have*

$$M_E R_F^E \lhd H \quad \text{and} \quad [H : M_E R_F^E] < \infty.$$

Theorem 2.14. *With the above notation, we have:*

(i) $\overline{M} = \varprojlim_{E\in\mathcal{E}} M/M_E$;
(ii) $\overline{N} = \varprojlim_{E\in\mathcal{E}} N/M_E$;
(iii) $\overline{R} = \varprojlim_{(E,F)\in\mathcal{F}} R/R_F^E$;
(iv) $\overline{H} = \varprojlim_{(E,F)\in\mathcal{F}} M/M_E \rtimes R/R_F^E$,

all as topological groups.

Proof. By the preceding results, it suffices to show that for all finite subsets

$$E' \subset Q \quad \text{and} \quad F' \subset N$$

there exists $(E, F) \in \mathcal{F}$ such that

$$M_E \subset M_{E'} \quad \text{and} \quad R_F^E \subset R_{F'}^{E'}.$$

Put

$$E'' = \big(E' \cup \{e\}\big)R \quad \text{and} \quad F'' = \big(F' \cup \{e\}\big)M.$$

Since (Q, R) is Hecke, $E := RE''$ is a finite union of cosets in Q/R, and it follows that $E \in \mathcal{E}$. We have

$$M_E \subset M_{E'} \quad \text{since} \quad E \supset E'.$$

Let M_0 be the subgroup of N generated by the conjugates $q^{-1}Mq$ for $q \in E$. Then $M_0 \lhd N$ since $q^{-1}Mq \lhd N$ for each q. Since E is a finite union of double cosets of R in Q, and since (Q, R) is Hecke, E is a finite union of right cosets of R in Q. Thus the family $\{q^{-1}Mq : q \in E\}$ is finite. Since M_0 is the product of the subgroups $q^{-1}Mq$ (because they are normal in N), it follows that $[M_0 : M] < \infty$. Thus, putting $F = M_0F''$, we have $(E, F) \in \mathcal{F}$, and moreover

$$R_F^E \subset R_{F'}^{E'} \quad \text{since} \quad E \supset E' \quad \text{and} \quad F \supset F'.$$ □

As a topological space, $\overline{Q} = \varprojlim_{E,F} Q/R_F^E$, but since the subgroups R_F^E are not in general normal in Q, the group structure of $\overline{Q}$ is more complicated. For details on this, we refer to [4, Remark 3.11]. In the special case where Q is abelian, we do have $R_F^E \lhd Q$, so

$$\overline{Q} = \varprojlim_{E,F} Q/R_F^E$$

as topological groups.

3. Crossed products

In this section we prove a few results concerning crossed products, compact normal subgroups, and projections. We state these results somewhat greater generality than we require, since they might be useful elsewhere and no extra work is required.

Compact subgroups. Let R be a compact normal subgroup of a locally compact group Q. We identify Q and $C_c(Q)$ with their canonical images in $M(C^*(Q))$ and $C^*(Q)$, respectively. Normalize the Haar measure on Q so that R has measure 1. Then $q := \chi_R$ is a central projection in $M(C^*(Q))$, and the map $\tau\, Q/R \to M(C^*(Q))$ defined by

$$\tau(sR) = sq \qquad \text{for } s \in Q \tag{3.1}$$

integrates to give an isomorphism of $C^*(Q/R)$ with the ideal $C^*(Q)q$ of $C^*(Q)$.

Let α be an action of Q on a C^*-algebra B. We identify B and $C^*(Q)$ with their canonical images in $M(B \times_\alpha Q)$. Thus q is a projection in $M(B \times_\alpha Q)$, and we may regard τ as a homomorphism of Q/R into $M(B \times_\alpha Q)$.

Let $\Phi(b) = \int_R \alpha_r(b)\, dr$ be the faithful conditional expectation of B onto the fixed-point algebra B^R. Then an elementary calculation shows that

$$qbq = \Phi(b)q = q\Phi(b) \qquad \text{for } b \in B$$

Thus $qBq = B^R q$, and q commutes with every element of B^R. Thus the formula

$$\sigma(b) = bq \tag{3.2}$$

defines a homomorphism σ of B^R onto the C^*-subalgebra $B^R q$ of $B \times_\alpha Q$. We will deduce from Proposition 3.1 below that σ is in fact an isomorphism.

Let β be the action of Q/R on B^R obtained from α. It is easy to see that the maps σ and τ from Equations 3.2 and 3.1 combine to form a covariant homomorphism (σ, τ) of the action $(B^R, Q/R, \beta)$, and that the integrated form

$$\theta := \sigma \times \tau\ B^R \times_\beta Q/R \to q(B \times_\alpha Q)q \tag{3.3}$$

is surjective.

In the special case $R = Q$, the following is the main result of [12]:

Proposition 3.1. *Let* (B, Q, α) *be an action, let* R *be a compact normal subgroup of* Q*, let* $(B^R, Q/R, \beta)$ *be the associated action, and let* $q = \chi_R$*. Then the map* $\theta\ B^R \times_\beta Q/R \to q(B \times_\alpha Q)q$ *from* (3.3) *is an isomorphism.*

Proof. By the discussion preceding the statement of the proposition, it remains to verify that θ is injective, and we do this by showing that for every covariant representation (π, U) of $(B^R, Q/R, \beta)$ on a Hilbert space V there exists a representation ρ of $q(B \times_\alpha Q)q$ on V such that $\rho \circ \theta = \pi \times U$.

Recall from the theory of Rieffel induction [11] that the conditional expectation $\Phi\ B \to B^R$ gives rise to a B^R-valued inner product

$$\langle b, c\rangle_{B^R} = \Phi(b^* c)$$

on B, so the completion X is a Hilbert B^R-module. Moreover, B acts on the left of X by adjointable operators, so we can use X to induce π to a representation $\widetilde{\pi}$ of B on $\widetilde{V} := X \otimes_{B^R} V$. An easy computation shows that the formula

$$\widetilde{U}_s(b \otimes \xi) = \alpha_s(b) \otimes U_{sR}\xi \qquad \text{for } s \in Q, b \in B, \xi \in V$$

determines a representation $\widetilde{U}$ of Q on $\widetilde{V}$ such that $(\widetilde{\pi}, \widetilde{U})$ is a covariant representation of (B, Q, α).

Thus $\widetilde{\pi} \times \widetilde{U}$ is a representation of the crossed product $B \times_\alpha Q$ on $\widetilde{V}$; let ρ_1 be its restriction to the corner $q(B \times_\alpha Q)q$. We have $\rho_1(q)\widetilde{V} = B^R \otimes_{B^R} V$, because if $b \in B$ and $\xi \in V$ then

$$\begin{aligned}
\rho_1(q)(b \otimes \xi) &= \int_R \widetilde{U}_r(b \otimes \xi)\, dr \\
&= \int_R \big(\alpha_r(b) \otimes U_{rR}\xi\big)\, dr \\
&= \int_R \alpha_r(b)\, dr \otimes \xi.
\end{aligned}$$

The subspace $B^R \otimes_{B^R} V$ is invariant for the representation ρ_1; let ρ_2 denote the associated subrepresentation of $q(B \times_\alpha Q)q$. A routine computation shows that

$$W(b \otimes \xi) = \pi(b)\xi \qquad \text{for } b \in B^R, \xi \in V$$

determines a unitary map W of $B^R \otimes_{B^R} V$ onto V which implements an equivalence between the representations $\rho_2 \circ \theta$ and $\pi \times U$. Thus we can take $\rho = \operatorname{Ad} W \circ \rho_2$. □

Corollary 3.2. *Let* (B, Q, α) *be an action, let* R *be a compact normal subgroup of* Q*, let* $(B^R, Q/R, \beta)$ *be the associated action, and let* $q = \chi_R$*. Then the map* $\sigma : B^R \to B^R q$ *from* (3.2) *is an isomorphism.*

Proof. It remains to observe that σ is faithful, being the composition of the injective homomorphism θ with the canonical embedding of B^R into $M(B^R \times_\beta Q/R)$. □

Two projections. If A is a C^*-algebra and p is a projection in $M(A)$, then one of the most basic applications of Rieffel's theory [11] is that the ideal $\overline{ApA}$ is Morita-Rieffel equivalent to the corner pAp via the $\overline{ApA} - pAp$ imprimitivity bimodule Ap. For later purposes, we will need a slightly more subtle variant:

Lemma 3.3. *Let* A *be a* C^**-algebra, and let* $p, q \in M(A)$ *be projections with* $p \leq q$*. Then* $q\overline{ApA}q$ *is Morita-Rieffel equivalent to* pAp*.*

Proof. Just apply the above Morita-Rieffel equivalence $\overline{ApA} \sim pAp$ with A replaced by qAq. □

Central projection. Let β be an action of a locally compact group T on a C^*-algebra C, and let $d \in M(C)$ be a central projection. Then d may also be regarded as a multiplier of the crossed product $C \times_\beta T$, and it generates the ideal

$$\overline{(C \times_\beta T)d(C \times_\beta T)}.$$

Proposition 3.4. *With the above notation, we have:*

(i) $\overline{(C \times_\beta T)d(C \times_\beta T)} = I \times_\beta T$*, where* I *is the* T*-invariant ideal of* C *generated by* d*.*

(ii) $I = \overline{\operatorname{span}}\{\beta_t(d)C : t \in T\} = \{c \in C : p_\infty c = c\}$*, where* $p_\infty = \sup\{\beta_t(d) : t \in T\}$*.*

Proof. (i) follows from [3, Propositions 11 (ii) and 12 (i)].

(ii) The first equality holds because d is a central projection. For the second, note that the projections $\{\beta_t(d) : t \in T\}$ are central, so their supremum p_∞ is an open central projection in C^{**}, and the desired equality follows from, *e.g.*, [10, Proposition 3.11.9]. To make this part of the proof self-contained, we include the argument: put

$$J = \{c \in C : p_\infty c = c\}.$$

For any $t \in T$ and $c \in C$ we have $\beta_t(d) \le p_\infty$, so

$$p_\infty \beta_t(d)c = \beta_t(d)c.$$

Thus $I \subset J$. Suppose $a \in J$ but $a \notin I$. Then there exists a nondegenerate representation π of C such that $\pi(a) \neq 0$ but $I \subset \ker \pi$. Extend π to a weak*-weak-operator continuous representation $\overline{\pi}$ of C^{**}. Enlarge the set $\{\beta_t(d) : t \in T\}$ to an upward-directed set P of central projections in $M(C)$, so that there is an increasing net $\{p_i\}$ in P converging weak* to p_∞. Then $p_i a \to p_\infty a$ weak*, so $\pi(p_i a) \to \overline{\pi}(p_\infty a)$. We have $\overline{\pi}(p_\infty a) = \pi(a)$ because $a \in J$, and $\pi(p_i a) = 0$ for all i, so we deduce that $\pi(a) = 0$, a contradiction. ☐

Question 3.5. When will p_∞ be a multiplier of B^R? (Example 5.10 shows that it is not always so.)

Combine. We will now combine the above results. Let α be an action of a locally compact group Q on a C^*-algebra B. Put

$$A = B \times_\alpha Q.$$

Let R be a compact normal subgroup of Q. Let $q = \chi_R$, a central projection in $M(C^*(Q))$ (assuming the Haar measure is suitably normalized). Let β be the associated action of Q/R on B^R. Let

$$\theta = \sigma \times \tau : B^R \times_\beta Q/R \xrightarrow{\cong} qAq$$

be the isomorphism of Proposition 3.1. Also let $d \in M(B)$ be an R-invariant central projection, so that d is also a central projection in $M(B^R)$. Put

$$p_\infty = \sup\{\alpha_s(d) : s \in Q\}.$$

Then p_∞ is an open central projection in $(B^R)^{**}$. Let I be the Q/R-invariant ideal of B^R generated by d. We have $dq = qd \in M(A)$, and we denote this projection by p.

The following theorem combines the previous results in this section:

Theorem 3.6. *With the above notation, we have:*

(i) $\theta(I \times_\beta Q/R) = q\overline{ApA}q$.
(ii) $I = \overline{\operatorname{span}}\{\alpha_s(d)B^R : s \in Q\} = \{b \in B^R : p_\infty b = b\}$.
(iii) $\sigma(I) = \overline{\operatorname{span}}\{sds^{-1}qBq : s \in Q\} = \overline{\operatorname{span}}\{sqdBqs^{-1} : s \in Q\}$.
(iv) *pAp is Morita-Rieffel equivalent to $I \times_\beta Q/R$.*

Proof. The only part that still requires proof is (iii). We have

$$\sigma(I) = \overline{\operatorname{span}}_{s \in Q}\, \theta \circ \alpha_s(dB^R)$$

because $\alpha_s(B^R) = B^R$. For each $s \in Q$ we have

$$\begin{aligned}
\sigma \circ \alpha_s(dB^R) &= \sigma \circ \beta_{sR}(dB^R) \\
&= \tau_{sR}\sigma(dB^R)\tau_{sR}^* \qquad \text{(covariance)} \\
&= (sq)dB^R q(sq)^* = sqdB^R qs^{-1} = sqdqBqs^{-1} \\
&= sqdBqs^{-1} \qquad (dq = qd) \\
&= sds^{-1}qBq \qquad (sq = qs, sB = Bs)\ ,
\end{aligned}$$

and (iii) follows. □

4. HECKE CROSSED PRODUCTS

In this section our main object of study is a Schlichting pair (G, H) which has (a slightly more restricted version of) the semidirect-product decomposition of Section 2. Recall that to say (G, H) is a *Schlichting pair* means that not only is it a reduced Hecke pair, but also G is a locally compact group and H is compact and open in G. Thus (G, H) coincides with its Schlichting completion.

We obtain crossed-product C^*-algebras which are Morita-Rieffel equivalent to the completion of the Hecke algebra inside $C^*(G)$, analogously to certain results of [4]. At the end of the section we briefly indicate how our results can be applied if the Hecke pair is incomplete.

Standing Hypotheses. We assume throughout this section that G is a locally compact group with closed subgroups N, Q, M, R such that:

- $G = N \rtimes Q$ is a semidirect product (with N normal);
- M is normal in N;
- R is normal in Q;
- R normalizes M;
- $H := M \rtimes R$ is compact open in G;
- $\bigcap_{x \in G} xHx^{-1} = \{e\}$.

Thus M and R are compact, and are open in N and Q, respectively. The two main differences between the above hypotheses and those of Section 2 are: (1) we assume that the Hecke pair (G, H) is complete, and (2) we assume that the subgroup R of Q is in fact normal rather than merely Hecke.

Put $A = C^*(G)$ and $B = C^*(N)$, and let α denote the canonical action of Q on B determined by conjugation of Q on N. Then A is isomorphic to the crossed product $B \times_\alpha Q$, and we identify these two C^*-algebras.

Normalize the Haar measures on N and Q so that M and R each have measure 1. Then the product measure is a Haar measure on G, and

H has measure 1. Thus $p_M := \chi_M$ is a central projection in B, hence is a projection in $M(A)$. Similarly, $p_R := \chi_R$ is a central projection in $C^*(Q)$, hence also a projection in $M(A)$, and we have

$$p_H := \chi_H = p_M p_R = p_R p_M \in A.$$

Note that the Hecke algebra of the pair (G, H) is $\mathcal{H} = p_H C_c(G) p_H$, whose closure in A is the corner $p_H A p_H$. From Section 3 we get isomorphisms

$$\begin{aligned} &\theta = \sigma \times \tau : B^R \times_\beta Q/R \xrightarrow{\cong} p_R A p_R \\ &\sigma : B^R \xrightarrow{\cong} B^R p_R \\ &\tau : C^*(Q/R) \xrightarrow{\cong} C^*(Q) p_R, \end{aligned}$$

and an ideal

$$I = \{b \in B^R : p_\infty b = b\} \lhd B^R,$$

where

$$p_\infty = \sup\{\alpha_s(p_M) : s \in Q\} \in (B^R)^{**}.$$

Theorem 3.6 quickly gives the following analogue of [4, Theorem 8.3]:

Theorem 4.1. *With the above notation:*

(i) $\theta(I \times_\beta Q/R) = p_R \overline{A p_H A} p_R$.

(ii) $I = \overline{\operatorname{span}}\{\alpha_s(p_M) B^R : s \in Q\}$.

(iii)

$$\begin{aligned} \sigma(I) &= \overline{\operatorname{span}}\{s p_M s^{-1} p_R B p_R : s \in Q\} \\ &= \overline{\operatorname{span}}\{s p_R p_M B p_R s^{-1} : s \in Q\} \\ &= \overline{\operatorname{span}}\{s p_M s^{-1} p_R n p_R : s \in Q, n \in N\}. \end{aligned}$$

(iv) *$p_H A p_H$ is Morita-Rieffel equivalent to $I \times_\beta Q/R$.*

Proof. The only thing left to prove is the last equality of part (iii), and this follows from Theorem 3.6, because M is compact open in N, hence

$$dB = \overline{\operatorname{span}}\{dn : n \in N\}. \qquad \square$$

Remark 4.2. Note that if R is nontrivial then p_H is never full in A: Since N is normal in G with $Q = G/N$, there is a natural homomorphism $C^*(G) \to C^*(Q)$ which maps p_H to p_R. Thus p_R is a nontrivial projection, which, being central, is not full in $C^*(Q)$.

We say that the family $\{sMs^{-1} : s \in Q\}$ of conjugates of M is *downward-directed* if the intersection of any two of them contains a third.

Proposition 4.3. *If* $\{sMs^{-1} : s \in Q\}$ *is downward-directed, then*

$$p_R\overline{Ap_HA}p_R = p_RAp_R \cong B^R \times_\beta Q/R.$$

Proof. Because the pair (G, H) is reduced we have

$$\bigcap_{s\in Q} sMs^{-1} = \{e\},$$

so the upward-directed set $\{sp_Ms^{-1} : s \in Q\}$ of projections has supremum $p_\infty = 1$ in $(B^R)^{**}$. Therefore the ideal I from Theorem 4.1 coincides with B^R, and the result follows. $\square$

Remark 4.4. In the above proposition, we have

$$p_R\overline{Ap_HA}p_R = p_RAp_R$$

although the ideal $\overline{ApA}$ of A is proper if R is nontrivial.

As in [4, Section 7], we specialize to the case where N is abelian. Taking Fourier transforms, the action α of Q on B becomes an action α' on $C_0(\widehat{N})$:

$$\alpha'_s(f)(\phi) = f(\phi \circ \alpha_s) \qquad \text{for } s \in Q, f \in C_0(\widehat{N}), \phi \in \widehat{N}.$$

The smallest Q-invariant subset of $\widehat{N}$ containing $M^\perp$ is

$$\Omega = \bigcup_{s\in Q} (sMs^{-1})^\perp.$$

The Fourier transform of the fixed-point algebra B^R is isomorphic to $C_0(\widehat{N}/R)$, where $\widehat{N}/R$ is the orbit space under the action of R. The smallest Q/R-invariant subset of $\widehat{N}/R$ containing $M^\perp/R$ is Ω/R. Thus the Fourier transform of the ideal I of B^R is $C_0(\Omega/R)$. Let γ be the associated action of Q/R on $C_0(\Omega/R)$. The following corollary is analogous to [4, Corollary 7.2].

Corollary 4.5. *With the above assumptions and notation, if* N *is abelian then* p_HAp_H *is Morita-Rieffel equivalent to the crossed product* $C_0(\Omega/R) \times_\gamma Q/R$.

We finish this section with a brief indication of how the above general theory can be used when (G, H) is the Schlichting completion of a reduced Hecke pair (G_0, H_0). More precisely, we assume that $G_0 = N_0 \rtimes Q_0$, $M_0 \lhd N_0$, $R_0 \lhd Q_0$, R_0 normalizes M_0, and that (G_0, H_0) is a reduced Hecke pair (and Propositions 2.5–2.6 give conditions under which the latter happens). By Corollary 2.8, the closures N, Q, M, and R of N_0, Q_0, M_0, and R_0, respectively, satisfy the conditions of the current section, namely $G = N \rtimes Q$, $M \lhd N$, $R \lhd Q$, and $H =$

$M \rtimes R$. The action (B, Q, α) restricts to an action (B, Q_0, α_0), and by density we have $B^R = B^{R_0}$. The map $sR_0 \mapsto sR$ for $s \in R_0$ gives an isomorphism $Q_0/R_0 \cong Q/R$ of discrete groups, and the action β of Q/R on B^R corresponds to an action β_0 of Q_0/R_0 on B^{R_0}. Thus we have a natural isomorphism

$$B^R \times_\beta Q/R \cong B^{R_0} \times_{\beta_0} Q_0/R_0.$$

Again by density, for all $s \in Q$ there exists $s_0 \in Q_0$ such that $p_R s = p_R s_0$, and similarly for all $n \in N$ there exists $n_0 \in N$ such that $np_M = n_0 p_M$. We deduce:

Corollary 4.6. *Using the above isomorphisms and identifications:*

(i) *I is the Q_0/R_0-invariant ideal of B^{R_0} generated by p_M;*
(ii) *$I \times_{\beta_0} Q_0/R_0 \cong p_R\overline{Ap_HA}p_R$;*
(iii) *$p_\infty = \sup\{sp_M s^{-1} : s \in Q_0\}$;*
(iv) *$I \cong \overline{\operatorname{span}}\{sp_M s^{-1} p_R n p_R : s \in Q_0, n \in N_0\}$;*
(v) *$p_H A p_H$ is Morita-Rieffel equivalent to $I \times_{\beta_0} Q_0/R_0$.*

As explained in [4], many of the nice properties of the Hecke algebra in [1] hold because the family $\{xHx^{-1} \mid x \in G\}$ of conjugates of H is downward directed; in particular this implies that the projection p is full. In our situation we can only have p full if $R = \{e\}$, but we do have the following:

Corollary 4.7. *Suppose the conjugates $\{qMq^{-1} \mid q \in Q\}$ of M are downward directed. Then $I = B^{R_0}$ and $p_H A p_H$ is Morita-Rieffel equivalent to $B^{R_0} \times_{\beta_0} Q_0/R_0$.*

Proof. We have $sp_M s^{-1} = p_{sMs^{-1}}$, so by the assumptions $p_\infty = 1$. □

Continuing with (G, H) being the Schlichting completion of (G_0, H_0) as above, we again consider the special case where N, equivalently N_0, is abelian. Fourier transforming, by density we have

$$\Omega = \bigcup_{s \in Q_0} (sMs^{-1})^\perp,$$

and there is an associated action γ_0 of Q_0/R_0 on $C_0(\Omega/R)$, giving:

Corollary 4.8. *With the above notation, $p_H A p_H$ is Morita-Rieffel equivalent to $C_0(\Omega/R) \times_{\gamma_0} Q_0/R_0$.*

5. Examples

We shall here illustrate the results from the preceding sections with a number of examples. **(Give more detail here about what**

we'll illustrate in the examples.) [1] Some arguments are only sketched, and we leave many details to the reader.

First note that the case $R = \{e\}$ is treated in [4, Sections 7–8].

Example 5.1. The situation with $M = \{e\}$ and $R \lhd Q$ is also interesting. From Section 2 we see that (NQ, R) is Hecke if and only if $R_{n,\{e\}} = \{r \in R \mid rnr^{-1} = n\}$ is cofinite in R for all n. It is reduced if and only if $\bigcap_n R_{n,\{e\}} = \{e\}$, *i.e.*, if the map $R \to \operatorname{Aut} N$ is injective. Here $\overline{N} = N$, $p := p_H = p_R$, and Theorem 4.1 gives a Morita-Rieffel equivalences between ApA, pAp and $C^*(N)^R \times Q/R$. [4, Example 10.1] is a special case of this situation.

We shall next study 2×2 matrix groups (and leave it to the reader to see how this generalizes to $n \times n$ matrices) acting on K^2 where K is either the field $\mathbb{Q}$ of rational numbers, or the field $\mathbb{Z}[p^{-1}]$ of rational numbers with denominators of the form p^n. Here p is a prime number, not to be confused with the projection p. We shall only give details in the first case.

We shall assume that

- $N = K^2$, $M = \mathbb{Z}^2$;
- Q is a subgroup of $\mathrm{GL}(2, K)$ containing the diagonal subgroup $D = \{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right) \mid k, k^{-1} \in K^\times\}$;
- $R = Q \cap \mathrm{SL}_\pm(2, \mathbb{Z})$,

with the obvious action of Q on K^2. So in particular R is the subgroup of elements in Q such that both r and r^{-1} are integer matrices.

We shall now see that Proposition 2.5 (ii) holds in this case. Given $q \in Q$ there is an integer matrix $k \in D$ such that kq^{-1} is an integer matrix. From this it follows that $kq^{-1}\mathbb{Z}^2 \subset \mathbb{Z}^2$ and therefore $kMk^{-1} \subset qMq^{-1}$. This means that the sets $\{k\mathbb{Z}^2\}$ are downward-directed and form a base at e for the Hecke topologies of M and N, by Proposition 2.7. We also note that $\bigcap_k kMk^{-1} = \bigcap_k k\mathbb{Z}^2 = \{e\}$, by Proposition 2.6. Thus $\overline{N} = \mathcal{A}_f^2$ and $\overline{M} = \mathcal{Z}^2$, with $\mathcal{A}_f$ the finite adeles and $\mathcal{Z}$ the integers in $\mathcal{A}_f$.

Next, if $n \in N = \mathbb{Q}^2$ there exists $k \in \mathbb{Z}$ such that $kn \in M = \mathbb{Z}^2$. Take $n_1 = \left(\begin{smallmatrix} 1/k \\ 0 \end{smallmatrix}\right)$ and $n_2 = \left(\begin{smallmatrix} 0 \\ 1/k \end{smallmatrix}\right)$. By definition $r \in R_{n,M}$ if and only if $(r - I)n \in \mathbb{Z}^2$. One checks that $R_{n_1,M} \cap R_{n_2,M} \subset R_{n,M}$ and that

$$R_{n_1,M} \cap R_{n_2,M} = \{r \in R \mid r - I \in M(2, k\mathbb{Z})\}.$$

Call this subgroup $R(k)$; it is clearly a normal subgroup of finite index in R.

---

[1] jq: Job for Magnus!

Finally, we need to look at $[R : R_q]$. Suppose $q = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Q$, and without loss of generality we may assume $q \in M(2, \mathbb{Z})$. Putting $t = \det(q) = ad - bc$, for $r \in R(t)$ we have

$$q^{-1}(r - I)q = t^{-1}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)(r - I)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(2, \mathbb{Z}),$$

and it follows that $q^{-1}rq \in M(2, \mathbb{Z})$. The same argument holds for r^{-1}, so both $q^{-1}rq$ and $q^{-1}r^{-1}q$ are integer matrices in Q. Thus

$$q^{-1}rq \in Q \cap \mathrm{SL}_{\pm}(2, \mathbb{Z}) = R.$$

From this it follows that

$$R(t) \subset R \cap qRq^{-1} \quad \text{for} \quad t = \det(q),$$

and we have just observed that this is cofinite in R.

The same argument also shows that $R(st) \subset R \cap qR(s)q^{-1}$, and therefore to any given finite sets $E \subset Q$ and $F \subset N$ there exists $s \in \mathbb{N}$ such that $R(s) \subset R_F^E$. Combining all this with Proposition 2.7 we see that the family $\{R(s) \mid s \in \mathbb{N}\}$ is a base at e for the Hecke topology restricted to R or Q.

Also note that $\bigcap_s R(s) = \{e\}$.

To sum up, we have:

Proposition 5.2. *Suppose* $N = \mathbb{Q}^2$, $M = \mathbb{Z}^2$, Q *is a subgroup of* $\mathrm{GL}(2, \mathbb{Q})$ *containing the diagonal subgroup* $D = \{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right) \mid k \in \mathbb{Q}^\times\}$, *and* $R = Q \cap \mathrm{SL}_{\pm}(2, \mathbb{Z})$. *Then* (NQ, MR) *is a reduced Hecke pair, and the Schlichting completion is given by*

$$\overline{N} = \mathcal{A}_f^2, \quad \overline{M} = \mathcal{Z}^2, \quad \overline{R} = \varprojlim R/R(s), \qquad \text{and} \qquad \overline{Q} = \bigcup_{q \in Q/R} q\overline{R},$$

where $\overline{Q}$ *has the topology from* $\overline{R}$, *i.e.,* $q_i \to e$ *if and only if* $q_i \in \overline{R}$ *eventually and* $q_i \to e$ *in* $\overline{R}$.

A similar result holds when $\mathbb{Q}$ is replaced by other number fields, *e.g.*, $\mathbb{Z}[p^{-1}]$ for a prime p. We state it without proof:

Proposition 5.3. *Suppose* $N = \mathbb{Z}[p^{-1}]^2$, $M = \mathbb{Z}^2$, Q *is a subgroup of* $\mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ *containing the diagonal subgroup* $D = \{\left(\begin{smallmatrix} p^n & 0 \\ 0 & p^n \end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$, *and* $R = Q \cap \mathrm{SL}_{\pm}(2, \mathbb{Z})$. *Then* (NQ, MR) *is a reduced Hecke pair, and the Schlichting completion is given by*

$$\overline{N} = \mathbb{Q}_p^2, \quad \overline{M} = \mathcal{Z}_p^2, \quad \overline{R} = \varprojlim R/R(p^n), \qquad \text{and} \qquad \overline{Q} = \bigcup_{q \in Q/R} q\overline{R},$$

where as above $\overline{Q}$ *has the topology from* $\overline{R}$.

Example 5.4. **(I put the p-adic case back in, it seems that the general case depends on it. In addition I do not quite understand the [LLN]-proof.)** [2]

Let us first consider the maximal p-adic case with $Q = \mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ and $R = \mathrm{SL}_{\pm}(2, \mathbb{Z})$.

Proposition 5.5. *Let* $T = \{ \left(\begin{smallmatrix} p^m & 0 \\ c & p^n \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}, c \in \mathbb{Z}[p^{-1}] \}$. *Then* $T\,\mathrm{SL}(2, \mathcal{Z}_p) = \{ g \in \mathrm{GL}(2, \mathbb{Q}_p) \mid \det(g) \in \pm p^{\mathbb{Z}} \}$.

Proof. Clearly the left hand side is included in the right hand side. For the opposite it suffices to show that every $g \in M(2, \mathcal{Z}_p)$ with $\det(g) \in \pm p^{\mathbb{N}}$ is a member of the left hand side. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Case 1. Suppose $b = 0$ and $ad = p^m$. If $a = p^n u$ with u a unit in $\mathcal{Z}_p$, we must have $d = u^{-1} p^{m-n}$. So

$$g = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ 0 & p^{m-n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

with $x = cu^{-1} p^{n-m}$. Now $x = y + z$ with $y \in \mathbb{Z}[1/p]$ and $z \in \mathcal{Z}_p$, and since $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathcal{Z}_p)$ it follows that $g = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in T\,\mathrm{SL}(2, \mathcal{Z}_p)$.

Case 2. Suppose $a = 0$ and $b \neq 0$. Then

$$g = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in T\mathrm{SL}(2, \mathcal{Z}_p).$$

Case 3. Suppose $a = p^m u$ and $b = p^n v$ with u, v units in $\mathcal{Z}_p$. We may assume $m \geq n$, if not we multiply by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ as in Case 2. So $p^{-n} a \in \mathcal{Z}_p$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ v^{-1} d & p^{-n} ad - vc \end{pmatrix} \begin{pmatrix} p^{-n} a & v \\ -v^{-1} & 0 \end{pmatrix}.$$

The second matrix on the right hand side is in $\mathrm{SL}(2, \mathcal{Z}_p)$, while the first has determinant equal to $ad - bc$ which by assumption is in $\pm p^{\mathbb{Z}}$, so by Case 1 this matrix is in $T\,\mathrm{SL}(2, \mathcal{Z}_p)$. □

Theorem 5.6. *Let* $Q = \mathrm{GL}(2, \mathbb{Z}[p^{-1}])$ *and* $R = \mathrm{SL}_{\pm}(2, \mathbb{Z})$. *Then*

(i) $\overline{R} = \varprojlim R/R(p^n) = \mathrm{SL}_{\pm}(2, \mathcal{Z}_p)$;

(ii) $\overline{Q} = \bigcup_{q \in Q/R} q\mathrm{SL}_{\pm}(2, \mathcal{Z}_p) = \{ g \in \mathrm{GL}(2, \mathbb{Q}_p) \mid \det(g) \in \pm p^{\mathbb{Z}} \}$,

with the topology coming from $\overline{R} = \mathrm{SL}_{\pm}(2, \mathcal{Z}_p)$.

---

[2] jq: Ok, I just have to say that my ability to check all this is quite limited. Are you indicating that we should mention this lack of understanding? Which result in [7] is relevant here?

Proof. Since $Q = TR$ we get $\overline{Q} = T\overline{R}$, which by Proposition 5.5 equals the right hand side. That $(\overline{NQ}, \overline{MR})$ satisfies [4, Theorem 3.8] now follows from the above lemmas and density of $\mathrm{SL}_\pm(2,\mathbb{Z})$ in $\mathrm{SL}_\pm(2,\mathcal{Z}_p)$ (see [5, Proposition IV.6.3]). ☐

Now look at the case where $Q = \mathrm{GL}(2,\mathbb{Q})$ and $R = \mathrm{SL}_\pm(2,\mathbb{Z})$. We first need a version of Proposition 5.5:

Proposition 5.7. *Let* $T = \{\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \mid a, c, d \in \mathbb{Q}, ad \neq 0\}$. *Then* $T\,\mathrm{SL}(2,\mathcal{Z}) = \{g \in \mathrm{GL}(2,\mathcal{A}_f) \mid \det(g) \in \mathbb{Q}\}$.

Proof. Again one inclusion is obvious, so suppose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathcal{A}_f)$ with $\det g \in \mathbb{Q}$, in fact without loss of generality we may assume $\det g = 1$. For each prime p let $g_p = \left(\begin{smallmatrix} a_p & b_p \\ c_p & d_p \end{smallmatrix}\right)$ be the corresponding matrix in $\mathrm{GL}(2,\mathbb{Q}_p)$. For all but finitely many p we will have $g_p \in \mathrm{SL}(2,\mathcal{Z}_p)$. In these cases take $k_p = g_p$.

In the other cases we can not have both a_p and b_p zero, so by Proposition 5.5 there is a matrix $k_p \in \mathrm{SL}(2,\mathcal{Z}_p)$ such that $g_p k_p^{-1} \in T \cap \mathrm{GL}(2,\ \mathbb{Z}[1/p])$. So $k = (k_p) \in \mathrm{SL}(2,\mathcal{Z})$ and $gk^{-1} \in T$ as claimed. ☐

Theorem 5.8. *Let* $Q = \mathrm{GL}(2,\mathbb{Q})$ *and* $R = \mathrm{SL}_\pm(2,\mathbb{Z})$. *Then*

(i) $\overline{R} = \mathrm{SL}_\pm(2,\mathcal{Z})$*;*
(ii) $\overline{Q} = \bigcup_{q \in Q/R} q\mathrm{SL}_\pm(2,\mathcal{Z}) = \{g \in \mathrm{GL}(2,\mathcal{A}_f) \mid \det(g) \in \mathbb{Q}\}$,

with the topology coming from $\overline{R} = \mathrm{SL}_\pm(2,\mathcal{Z})$.

Proof. From [5, Proposition IV.6.3] (the hard part is hidden there) it follows that

$$\begin{aligned} \overline{R} &= \varprojlim R/R(s) = \varprojlim \mathrm{SL}_\pm(2,\mathbb{Z})/\mathrm{SL}_\pm(2,s\mathbb{Z}) \\ &= \varprojlim \mathrm{SL}_\pm(2,\mathbb{Z}_s) = \mathrm{SL}_\pm(2,\mathcal{Z}). \end{aligned}$$

Since $\overline{Q} = T\overline{R}$, (ii) follows from Proposition 5.7.

Note that the topology on $\overline{Q}$ is not the relative topology from $\mathrm{GL}(2,\mathcal{A}_f)$, in contrast with Theorem 5.6. ☐

This is ***[[[essentially]]]*** the same result as [7, Proposition 2.5]. **new stuff
Since R is not normal in Q we can not use Theorem 4.1, but it would be interesting to get a description of the C^*-algebra $p_R A p_H A p_R$ in these cases. Also *cf.*[2]. However, note that we are not using the exact same algebra, since in both [2] and [7] the action of Q is by left multiplication on $\mathrm{M}_2(K)$.

Example 5.9. Much recent work on Hecke algebras started with the study of the affine group over $\mathbb{Q}$ in [1]. Other number fields have also been extensively studied, as in, *e.g.*, [2] and [6]. For a survey, see

[2, Section 1.4]. We shall here illustrate how our approach works for a quadratic extension of $\mathbb{Q}$. For details about the number theory used here we refer to the book [9].

Let d be a square-free integer and let $N = \mathbb{Q}(\sqrt{d})$, $M = \mathbb{Z}[\sqrt{d}]$, $Q = \mathbb{Q}(\sqrt{d})^\times$, and $R = \{r \in Q \mid r, r^{-1} \in M\}$. [3]

So

$$R = \{m + n\sqrt{d} \mid m, n \in \mathbb{Z}, m^2 - dn^2 = \pm 1\}$$

is the group of units in the field N. An alternative matrix description is as follows:

$$\begin{aligned} N &= \mathbb{Q}^2, \qquad M = \mathbb{Z}^2, \\ Q &= \left\{ \begin{pmatrix} a & db \\ b & a \end{pmatrix} \middle| \, a, b \in \mathbb{Q}, a^2 - db^2 \neq 0 \right\}, \\ R &= \left\{ \begin{pmatrix} m & dn \\ n & m \end{pmatrix} \middle| \, m, n \in \mathbb{Z}, m^2 - dn^2 = \pm 1 \right\}. \end{aligned}$$

So we get $\overline{N} = \mathcal{A}_f^2$ and $\overline{M} = \mathcal{Z}^2$.

Here Theorem 4.1 applies, so

$$pAp \sim_{MR} C_0(\mathcal{A}_f^2/\overline{R}) \rtimes Q/R.$$

In this way we obtain [6, Proposition 3.2] for the field $\mathbb{Q}(\sqrt{d})$ without using the theory of semigroup crossed products, and this will also work in more generality.

The structure of these crossed products can be studied by the Mackey-Takesaki orbit method as in [8]; note that the orbit closures in $\overline{N}/\overline{R}$ under the action of Q/R are basically the same as the orbit closures in $\overline{N}$ under the action of Q.

To determine $\overline{R}$ and its topology we need some more information. First, if $d < 0$ then R is finite (of order 2 or 4). So let us concentrate on the case with $d > 1$. Then, by [9, Theorem 7.26] we have $R \cong \{\pm 1\} \times \mathbb{Z}$, and in fact there exists $r_0 \in R$ such that $R = \{\pm r_0^n \mid n \in \mathbb{Z}\}$. For instance if $d = 2$ one can take $r_0 = 1 + \sqrt{2}$.

Let us look at $R(s)$. There is a smallest integer $n_s > 0$ such that $r_0^{n_s} \equiv 1 \bmod s$. From this we get $\overline{R} = \varprojlim R/R(s) = \{\pm 1\} \times \varprojlim \mathbb{Z}/\mathbb{Z}_{n_s}$. However, examples show that the behavior of the numbers n_s is complicated, so a more exact description of $\overline{R}$ is difficult.

Perhaps counterintuitively, in general it turns out that

$$\overline{R} \subsetneq \{m + n\sqrt{d} \mid m, n \in \mathcal{Z}, m^2 - dn^2 = \pm 1\}.$$

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[3] For number theory experts: If $d \equiv 1 \mod 4$ the above is not quite right. For instance, if $d = 5$ one should instead use $M = \mathbb{Z}[(1 + \sqrt{5})/2]$, *etc.* (see [9, Theorem 9.20]).

This is because under the homomorphism $\mathbb{Z}[\sqrt{d}] \mapsto \mathbb{Z}_s[\sqrt{d}]$ the units R in $\mathbb{Z}[\sqrt{d}]$ are in general mapped onto a proper subgroup of the units in $\mathbb{Z}_s[\sqrt{d}]$. For instance 4 is a unit in $\mathbb{Z}_{17}[\sqrt{2}]$, but $\pm(1+\sqrt{2})^n \not\equiv 4$ mod 17 for all n.

Example 5.10. We shall here give a slightly different treatment of the Heisenberg group than in [4]. Take

$$\begin{aligned} N &= \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}, & M &= \{0\} \times \mathbb{Z}, \\ Q &= \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \;\middle|\; q \in \mathbb{Q} \right\}, & R &= \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \;\middle|\; r \in \mathbb{Z} \right\}, \end{aligned}$$

with the obvious action of Q on N. If $x = \left(\begin{smallmatrix} 1 & 1/n \\ 0 & 1 \end{smallmatrix}\right)$ with $n \in \mathbb{N}$ one checks that $M \cap xMx^{-1} = \{0\} \times n\mathbb{Z}$. So we have

$$\overline{N} = \mathbb{Q}/\mathbb{Z} \times \mathcal{A}_f = \mathcal{A}_f/\mathcal{Z} \times \mathcal{A}_f \qquad \text{and} \qquad \overline{M} = \{0\} \times \mathcal{Z}.$$

If $n = \left(\begin{smallmatrix} a \\ b/m \end{smallmatrix}\right)$ with $b, m \in \mathbb{Z}$ and $r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$, then $rnr^{-1} - n \in M$ if and only if $r \in m\mathbb{Z}$. Thus

$$\overline{Q} = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \;\middle|\; q \in \mathcal{A}_f \right\} \qquad \text{and} \qquad \overline{R} = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \;\middle|\; r \in \mathcal{Z} \right\}.$$

We have $\widehat{\overline{N}} = \mathcal{Z} \times \mathcal{A}_f$ and $\overline{M}^{\perp} = \mathcal{Z} \times \mathcal{Z}$. Moreover, the dual action of $\overline{Q}$ on $\widehat{\overline{N}}$ is given by

$$(z, w) \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = (z, qz + w).$$

Lemma 5.11.

$$\begin{aligned} \Omega := \bigcup_{q \in \overline{Q}} q\overline{M}^{\perp} &= \{(z, qz + w) \mid z, w \in \mathcal{Z}, q \in \mathcal{A}_f\} \\ &= \{(z, u) \in \mathcal{Z} \times \mathcal{A}_f \mid z_p = 0 \implies u_p \in \mathcal{Z}_p\}. \end{aligned}$$

Proof. Clearly if $(z, w) \in \Omega$ and $z_p = 0$, then $w_p \in \mathcal{Z}_p$.

Conversely, suppose (z, u) is an element of the right hand side. If $u_p \in \mathcal{Z}_p$, take $q_p = 1$ and $w_p = u_p - z_p \in \mathcal{Z}_p$. For the finitely many p with $u_p \notin \mathcal{Z}_p$, we have $u_p = x_p + v_p$ with $x_p \in \mathbb{Q}^\times$ and $v_p \in \mathcal{Z}_p$, and by assumption $z_p \neq 0$. Take $q_p = z_p^{-1} x_p \in \mathbb{Q}_p$, so $q_p z_p + w_p = u_p$. Thus with $q := (q_p) \in \mathcal{A}_f$ and $w := (w_p) \in \mathcal{Z}$, we have $qz + w = u$. ☐

So here Ω is open but not closed, hence the projection p_∞ defined in Section 4 is not in $M(B^R)$.

The orbits under the action of R can be described as follows: $(0, w)$ is always a fixed point. If $z \neq 0$, then the R-orbit of (z, w) is $(z, w + z\mathcal{Z})$.

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