NTNU
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TMA4267 Linear Statistical Models V2014 (5)

The Chi-square distribution [2.1], The F distribution [2.3] Orthogonality [2.4] with tranformation formula [2.2] Normal sample mean and variance [2.5]

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ANalysis Of VAriance (ANOVA)

Bingham and Fry (2010): Chapter 2

- The Chi-square distribution and the F-distribution [2.1, 2.3]
- Orthogonality and multivariate tranformation formula [2.2, 2.4]
- Normal sample mean and variance [2.5]
- One-way ANOVA [2.6]
- Two-way ANOVA [2.7-2.8]

Concrete aggregates example

— Aggregates are inert granular materials such as sand, gravel, or crushed stone that, along with water and portland cement, are an essential ingredient in concrete.

— For a good concrete mix, aggregates need to be clean, hard, strong particles free of absorbed chemicals or coatings of clay and other fine materials that could cause the deterioration of concrete.

— We could like to examine 5 different aggregates, and measure the absorption of moisture after 48hrs exposure (to moisture).

— A total of 6 samples are tested for each aggregate.

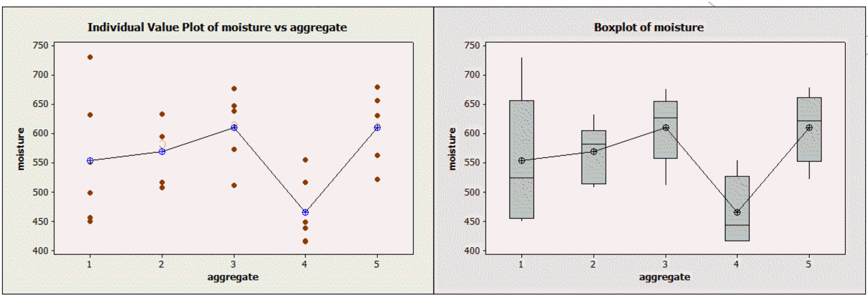
— Research question: Is there a difference between the aggregates with respect to absorption of moisture?

Concrete aggregates data

Aggregate:	1	2	3	4	5	
	551	595	639	417	563	
	457	580	615	449	631	
	450	508	511	517	522	
	731	583	573	438	613	
	499	633	648	415	656	
	632	517	677	555	679	
Total	3320	3416	3663	2791	3664	16,854
Mean	553.33	569.33	610.50	465.17	610.67	561.80

Table 13.1 of WMMY.

Concrete aggregates example



Comparing means by analysing variability

Fisher (1918), Rothamsted Experimental Station.
Fisher wanted to compare crop yields using different fertilzers. He realised that if there was more variability between groups from different fertilizers than within groups, then this would be evidence against believing that the fertilizers have the same effect. Fisher:

compare means by analysing variability!

The Chi-square distribution [2.1]

pdf χ^2_n:

$$f(y) = \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{(-y/2)} \text{ for } y > 0$$

MGF χ^2_n:

$$M_Y(t) = \frac{1}{(1-2t)^{n/2}}$$

Addition property:
Let $X_1 \sim \chi^2_n$ and $X_2 \sim \chi^2_m$, and let X_1 and X_2 be independent. Then $X_1 + X_2 \sim \chi^2_{n+m}$.
Subtraction property:
Let $X = X_1 + X_2$ with $X_1 \sim \chi^2_n$ and $X \sim \chi^2_{n+m}$. Assume that X_1 and X_2 are independent. Then $X_2 \sim \chi^2_m$.

TMA 4267 - lecture 5

Chapter 2: ANalysis Of VAriance (ANOVA)

The χ^2-distribution [2.1]

Let X_i be i.i.d $N(0,1)$. Then   Let $i = 1, .., n$

$$Y = X_1^2 + X_2^2 + \dots + X_n^2$$

is χ^2-distributed with n degrees of freedom.

From TMA4245 we remember finding the pdf of X_1^2 from $N(0,1)$ using the transformation formula [p 217 WMMY]

$$f_Y(y) = \frac{1}{\sqrt{2}} \frac{1}{\Gamma(1/2)} y^{\frac{1}{2}-1} e^{-y/2} \quad \text{for } y > 0$$

We may also use MGF: $X \sim N(0,1) \Rightarrow Y = X^2$

$$M_{X^2}(t) = \int_{-\infty}^{\infty} e^{tX^2} \cdot \phi(x)\, dx \qquad \phi = \text{pdf } N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{1}{2}x^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)x^2}\, dx$$

only convergence $t < \frac{1}{2}$

$$y = \sqrt{1-2t}\, x$$
$$dy = \sqrt{1-2t}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{1-2t}}\, dy$$

$$= \frac{1}{\sqrt{1-2t}} \cdot 1 = \frac{1}{\sqrt{1-2t}} \quad \text{for } t < \frac{1}{2}$$

When $X_1, \dots, X_n$ are identical and independent

$Y = \sum X_i^2$, $M_Y(t) = \left[M_{X^2}(t)\right]^n$

$$M_Y(t) = \left[\frac{1}{(1-2t)^{1/2}}\right]^n = \underline{\underline{\frac{1}{(1-2t)^{n/2}}}}$$

R: $\uparrow$ chisq

r

d

p

q

<u>What about E(Y) and Var(Y) ?</u>

$$E(Y) = E(X_1^2) + \dots + E(X_n^2) = n$$

$$E(X_i^2) = \underbrace{Var(X_i)}_{1} + \underbrace{E(X_i)}_{0}^2 = 1$$

$$Var(X_i^2) = \underbrace{E(X_i^4)}_{3} - \underbrace{E(X_i^2)}_{1}^2 = 3 - 1 = \underline{\underline{2}}$$

it is known that this is 3 in $N(0,1)$

$$Var(Y) = \sum_{i=1}^{n} Var(X_i^2) = \underline{\underline{2n}}$$

but may also find this

$$\left.\frac{d^4 M_X(t)}{dt^4}\right|_{t=0} = \mu_4$$

The pdf of χ^2_n is

$$f(y) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \cdot y^{\frac{n}{2}-1} e^{-\frac{1}{2}y}$$

χ^2 addition property

$X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$. X_1 and X_2 are independent.

This $X_1 + X_2 \sim \chi^2_{n_1+n_2}$.

Proof: follows directly by writing $X_1 = U_1^2 + \ldots + U_{n_1}^2$ and $X_2 = U_{n_1+1}^2 + \ldots + U_{n_1+n_2}^2$ and using def of χ^2.

χ^2 subtraction property

$X = X_1 + X_2$ with $X_1 \sim \chi^2_{n_1}$ and $X \sim \chi_{n_1+n_2}$.

Assume that X_1 and X_2 is independent, then

$X_2 \sim \chi^2_{n_2}$.

Proof: use MGF.

The Fisher distribution [2.3]

"Tabeller og formeler i statistikk":
If Z_1 and Z_2 are independent and χ^2-distributed with ν_1 and ν_2 degrees of freedom, then

$$F = \frac{Z_1/\nu_1}{Z_2/\nu_2}$$

is F(isher)-distributed with ν_1 and ν_2 degrees of freedom.

- The expected value of F is $\mathrm{E}(F) = \frac{\nu_2}{\nu_2 - 2}$.
- The mode is at $\frac{\nu_1 - 2}{\nu_1}\frac{\nu_2}{\nu_2 + 2}$.
- Identity:

$$f_{1-\alpha,\nu_1,\nu_2} = \frac{1}{f_{\alpha,\nu_2,\nu_1}}$$

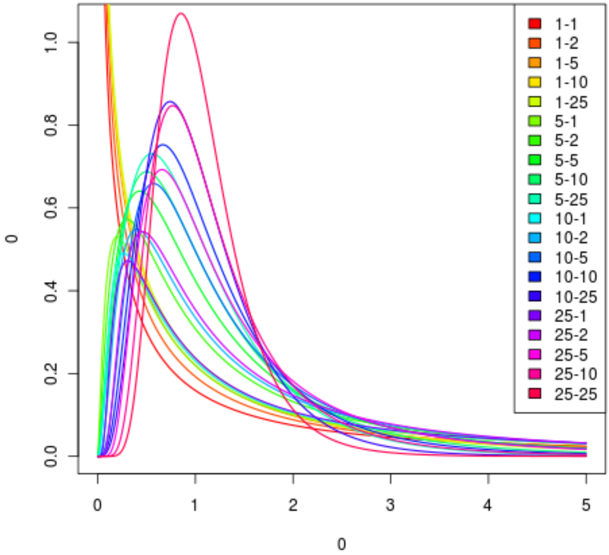
The F-distribution [2.8]

$$\left.\begin{array}{l} U \sim \chi^2_m \\ V \sim \chi^2_n \end{array}\right\} \text{independent} \Rightarrow F = \frac{\frac{U}{m}}{\frac{V}{n}} \sim F_{m,n}$$

Fisher distribution with m and n degrees of freedom

$$f(x) = \frac{m^{m/2}\, n^{n/2}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \cdot \frac{x^{\frac{m-2}{2}}}{(mx+n)^{\frac{m+n}{2}}}$$

Beta function $B(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$



The Fisher distribution with different degrees of freedom ν_1 and ν_2 (given in the legend).

[2.2 + 2.4] Tools -

Transformation formula fra TMA4295

1) $Y = u(X)$ one-to-one function u

$X = u^{-1}(Y) = w(Y)$

$$f_Y(y) = f_X(w(y)) \cdot \underbrace{|w'(y)|}_{\text{abs}}$$

$w'(y)$: $\frac{dx}{dy}$

2) Now, multivariate version

$$\underset{n\times 1}{X} \rightarrow \underset{n\times 1}{Y}$$

$f_X(x)$ $f_Y(y)$

joint pdfs

Let $y = u(x)$ and $x = w(y)$ as before

$$f_Y(y) = f_X(w(y)) \cdot \text{abs}(J)$$

where $J = \det\left\{\left[\frac{\partial x_i}{\partial y_j}\right]\right\}$

$n \times n$ matrix

In particular:

$$\overset{u}{\overbrace{\underset{n\times 1}{Y} = \underset{n\times n}{A}\,\underset{n\times 1}{X} + \underset{n\times 1}{b}}} \iff \overset{w}{\overbrace{X = A^{-1}Y - A^{-1}b}}$$

$$J = \det\left\{\frac{\partial x_i}{\partial y_j}\right\} = \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

$$A \cdot A^{-1} = I$$

$$\det(A \cdot A^{-1}) = \det(I) = 1$$

$$\det(A) \cdot \det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

And, if A is orthogonal: $A^T = A^{-1}$

$$\det(A^{-1}) = \det(A^T) \underset{\substack{\uparrow \\ \text{cofactor expansion}}}{=} \det(A)$$

such that

$$\det(A \cdot A^{-1}) = 1 \quad \text{(prev. page)}$$

$$\det(A)^2 = 1$$

$$\det(A) = \pm 1$$

In formula $\text{abs}(J) = 1$

Orthogonality theorem [Teo 2.2]

$X_1, X_2, \dots, X_n$ i.i.d $N(0, \sigma^2)$.

$$\underset{n\times 1}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Let $\underset{n\times 1}{Y} = \underset{n\times n}{A}\ \underset{n\times 1}{X}$ where A is orthogonal.

Then $Y_i = \{Y\}_i$ are i.i.d $N(0, \sigma^2)$.

means: element nr i of Y-vector

Proof via Jacobian

We know that $f_Y(y) = f_X(w(y)) \cdot \text{abs}(J)$

where $X = w(y) = A^{-1}Y = A^T Y$

and $J = \det(A^{-1}) = \pm 1$ and $\text{abs}(J) = 1$.

$$f_X(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \cdot \exp\left\{-\frac{1}{2\sigma^2} x_i^2\right\}$$

$$= (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \Sigma x_i^2\right\}$$

$$\Sigma x_i^2 = X^T X = (A^T Y)^T (A^T Y) = Y^T \underbrace{A A^T}_{I} Y = Y^T Y = \Sigma y_i^2$$

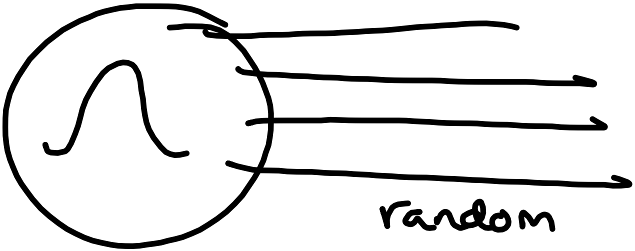
$$f_Y(y) = f_X(w(y)) \cdot 1$$

$$= (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \Sigma y_i^2\right\}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot y_i^2\right\}$$

that is $Y_1, \ldots, Y_n$ are i.i.d $N(0, \sigma^2)$

$\bar{X}$ and S^2 in normal samples [2.5]



random sample

pop

$X_1, X_2, \dots, X_n$

i.i.d $N(\mu, \sigma^2)$

$X_i \sim N(\mu, \sigma^2)$

Estimators: $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Normal samples [2.5]

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Further, let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
Then the following holds:

— $\bar{X}$ and S^2 are independent.

— $\bar{X} \sim N(\mu, \sigma^2/n)$.

— $nS^2/\sigma^2 \sim \chi^2_{n-1}$.