NTNU
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TMA4267 Linear Statistical Models V2014 (7)

One-way ANOVA [2.6]

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wiki.math.ntnu.no/emner/tma4267/2014v/start/

ANalysis Of VAriance (ANOVA)

Bingham and Fry (2010): Chapter 2

- The Chi-square distribution and the F-distribution [2.1, 2.3]
- Orthogonality and multivariate transformation formula [2.2, 2.4]
- Normal sample mean and variance [2.5]
- One-way ANOVA [2.6]
- Two-way ANOVA [2.7-2.8]

Today

Primary aim chapter 2: ANalysis Of VAriance – comparing means by studying variability.

One-way ANOVA : sums of squares and F-distribution, concrete example.

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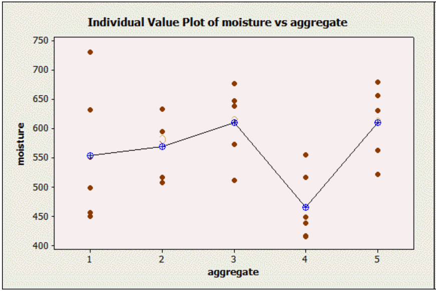
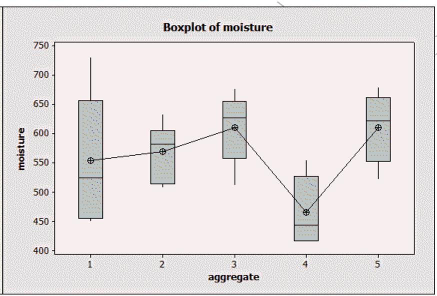
Concrete aggregates data

Aggregate:	1	2	3	4	5	
	551	595	639	417	563	
	457	580	615	449	631	
	450	508	511	517	522	
	731	583	573	438	613	
	499	633	648	415	656	
	632	517	677	555	679	
Total	3320	3416	3663	2791	3664	16,854
Mean	553.33	569.33	610.50	465.17	610.67	561.80

Table 13.1 of WMMY.

Concrete aggregates example





One factor: unequal sample sizes

Model

$$X_{ij} = \mu_i + \varepsilon_{ij} \text{ for } i = 1, 2, ..., r \text{ and } j = 1, 2, ..., n_i$$

The sample sizes for each group, n_i may vary. $\varepsilon_{ij} \sim N(0, \sigma^2)$.
Let $n = \sum_{i=1}^{r} n_i$ be the total number of observations.
Group means:

$$X_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$$

and grand mean:

$$X_{..} = \frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^{r} n_i X_{i.}$$

where the latter is a weighted average over the group means.

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One-way ANOVA

$$X_{ij} = \mu_i + \varepsilon_{ij} \qquad \varepsilon_{ij} \text{ i.i.d } N(0,\sigma^2)$$

$$X_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \qquad i = 1, 2, \dots, r \quad j = 1, 2, \dots, n_i$$

$$X_{..} = \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}$$

grand mean (estimator population grand mean $\mu = \frac{1}{n}\sum_{i=1}^{r} n_i \mu_i$)

$$\underbrace{\sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{..})^2}_{SS} = \sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{i.} - X_{..})^2 + \sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{i.})^2$$

$$= \underbrace{\sum_{i=1}^{r} n_i (X_{i.} - X_{..})^2}_{SSA} + \underbrace{\sum_{i=1}^{r} n_i \cdot S_i^2}_{SSE}$$

SS

SSA (SST_r treatment)

SSE (error)

One factor: unequal sample sizes

ANOVA decomposition:

$$X_{ij} - X_{..} = (X_{ij} - X_{i.}) + (X_{i.} - X_{..})$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{..})^2 = \sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{i.})^2 + \sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{i.} - X_{..})^2$$

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{..})^2 = \sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij} - X_{i.})^2 + \sum_{i=1}^{r} n_i(X_{i.} - X_{..})^2$$

$$\text{SS} = \text{SSE} + \text{SSA}$$

One factor: unequal sample sizes

ANOVA decomposition: what happened to the cross-term?

$$2\sum_{i=1}^{r}\sum_{j=1}^{n_i}(X_{ij}-X_{i.})(X_{i.}-X_{..}) = 2\sum_{i=1}^{r}(X_{i.}-X_{..})\sum_{j=1}^{n_i}(X_{ij}-X_{i.}) \qquad =$$

$$\sum_{j=1}^{n_i}(X_{ij}-X_{i.}) = \sum_{j=1}^{n_i}X_{ij} - \sum_{j=1}^{n_i}X_{i.} = n_i X_{i.} - n_i X_{i.} = 0$$

We want to test if the treatment means differ.

$H_0: \mu_1 = \mu_2 = \cdots = \mu_r$ vs H_1: at least two means differ

Next: look at distribution of SS, SSA and SSE when A) H_0 true and B) H_0 false

A) H_0 true : $\mu_1 = \mu_2 = \cdots = \mu_r = \mu$ so that X_{ij} i.i.d $N(\mu, \sigma^2)$

1) $SS = \sum\sum (X_{ij} - X_{..})$

Teo 2.4 iii $\qquad \dfrac{SS}{\sigma^2} = \dfrac{\sum\sum (X_{ij} - X_{..})}{\sigma^2} \sim \chi^2_{n-1}$

$$E\left(\frac{SS}{\sigma^2}\right) = n-1$$

$$E(SS) = \sigma^2 \cdot (n-1) \Rightarrow E\left(\frac{SS}{n-1}\right) = \sigma^2$$

2) SSE:

$$\frac{n_i \cdot S_i^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{j=1}^{n_i} (X_{ij} - X_{i\cdot})^2$$

When H_0 is true (or false) $X_{i1}, X_{i2}, \ldots, X_{in_i}$ i.i.d $N(\mu_i, \sigma^2)$

$$\frac{n_i \cdot S_i^2}{\sigma^2} \sim \chi^2_{n_i - 1} \qquad \text{from Teo 2.4iii}$$

Further

$$\frac{SSE}{\sigma^2} = \frac{\sum n_i S_i^2}{\sigma^2} \sim \chi^2_{n-r}$$

due to χ^2-addition property where

$$\sum_{i=1}^{r} (n_i - 1) = \underbrace{\sum n_i}_{n} - r = n - r$$

Then $E\left(\frac{SSE}{\sigma^2}\right) = n - r$

and $E\left(\frac{SSE}{n-r}\right) = \sigma^2$

3) $SSA = \sum_{i=1}^{r} n_i (X_{i.} - X_{..})^2$

Remember $SSE = \sum_{i=1}^{r} n_i S_i^2$

From Teo 2.4 i S_i^2 is independent of $X_{i.}$
Also S_i^2 is independent of $X_{j.}$ for $j \neq i$, since they are based on different samples. Then S_i^2 must also be independent of $X_{..}$ and of SSA.
Therefore SSA and SSE are independent.

Since:

$$\underbrace{\frac{SS}{\sigma^2}}_{\chi^2_{n-1}} = \underbrace{\frac{SSE}{\sigma^2}}_{\chi^2_{n-r}} + \frac{SSA}{\sigma^2}$$

and SSE and SSA are independent, χ^2 subtr. prop. gives

$$\frac{SSA}{\sigma^2} \sim \chi^2_{\underbrace{(n-1)-(n-r)}_{r-1}}$$

$$E\left(\frac{SSA}{\sigma^2}\right) = r-1$$

$$E\left(\frac{SSA}{r-1}\right) = \sigma^2$$

$$E\left(\frac{SSA}{r-1}\right) = \sigma^2$$

$$E\left(\frac{SSE}{n-r}\right) = \sigma^2$$

$$E\left(\frac{SS}{n-1}\right) = \sigma^2$$

B : H_0 is false

2) SSE: we have already seen that

$$\frac{SSE}{\sigma^2} \sim \chi^2_{n-r}$$ when H_0 is both true and false.

$$E\left(\frac{SSE}{n-r}\right) = \sigma^2$$

3) If the treatments differ (H_0 false) $\frac{SSA}{r-1}$ should be large compared $\frac{SSE}{n-r}$. How large?

$$SSA = \sum_{i=1}^{r} n_i (X_{i.} - X_{..})^2$$

$$= \sum_{i=1}^{r} n_i X_{i.}^2 - 2X_{..} \underbrace{\sum_{i=1}^{r} n_i X_{i.}}_{n X_{..}} + X_{..}^2 \underbrace{\sum_{i=1}^{r} n_i}_{n}$$

$$= \sum_{i=1}^{r} n_i X_{i.}^2 - n X_{..}^2$$

$$E(SSA) = \ldots \geq (r-1)\sigma^2$$

$$E(SSA) = \sum_{i=1}^{r} n_i E(X_{i\cdot}^2) - nE(X_{\cdot\cdot}^2)$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \sum_{i=1}^{r} n_i \left[Var(X_{i\cdot}) + E(X_{i\cdot})^2 \right] - n \left[Var(X_{\cdot\cdot}) + E(X_{\cdot\cdot})^2 \right]$$

$X_{ij} \sim N(\mu_i, \sigma^2)$

$X_{i\cdot} = \frac{1}{n_i} \sum X_{ij}$

$E(X_{i\cdot}) = \mu_i$

$Var(X_i) = \frac{\sigma^2}{n_i}$

$X_{\cdot\cdot} = \frac{1}{n} \sum\sum X_{ij}$

$E(X_{\cdot\cdot}) = \mu$

$Var(X_{\cdot\cdot}) = \frac{\sigma^2}{n}$

$$= \sum_{i=1}^{r} n_i \left(\frac{\sigma^2}{n_i} + \mu_i^2 \right) - n \left(\frac{\sigma^2}{n} + \mu^2 \right)$$

$$= r\sigma^2 + \sum n_i \mu_i^2 - \sigma^2 - n\mu^2$$

$$= (r-1)\sigma^2 + \sum_{i=1}^{r} n_i (\mu_i - \mu)^2$$

$= 0$ when all $\mu_i = \mu$

else > 0

When H_0 is true

$$F = \frac{\dfrac{SSA/\sigma^2}{r-1}}{\dfrac{SSE/\sigma^2}{n-r}} = \frac{\dfrac{SSA}{r-1}}{\dfrac{SSE}{n-r}} = \frac{MSA}{MSE}$$

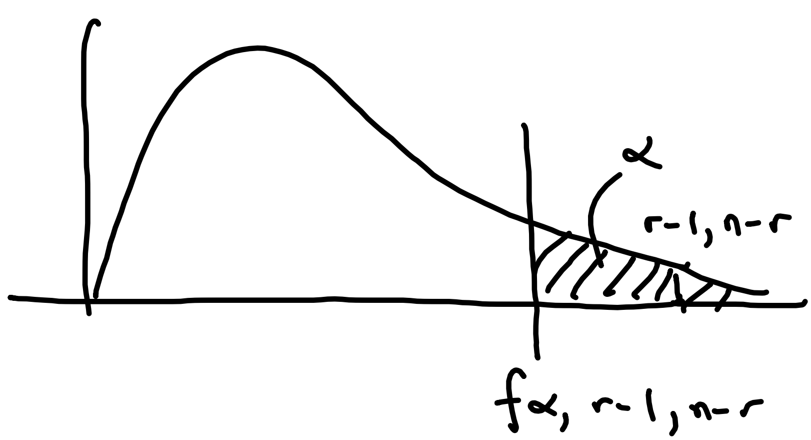
where $\dfrac{SSA}{\sigma^2} \sim \chi^2_{r-1}$ and $\dfrac{SSE}{\sigma^2} \sim \chi^2_{n-r}$

$$\sim F_{r-1,\, n-r}$$

But when H_0 is false the expected value of the numerator increases.

Hypothesis testing strategy:

Reject H_0 if $F_{obs} = \dfrac{SSA/(r-1)}{SSE/(n-r)} \geq f_{\alpha, r-1, n-r}$



One factor: unequal sample sizes

$H_0 : \mu_1 = \mu_2 = \cdots = \mu_r = 0$ vs. $H_1 :$ At least one pair of μ_i different

is then tested based on

$$F = \frac{\frac{\text{SSA}}{r-1}}{\frac{\text{SSE}}{n-r}}$$

Where H_0 is rejected if $f_{\text{obs}} > f_{\alpha}, (r-1), (n-r)$.

One-way ANOVA table

Source	df	SS	MS	F	p-value
Treatment	$r-1$	SSA	$\frac{SSA}{r-1}$	$\frac{MSA}{MSE}$	
Error	$n-r$	SSE	$\frac{SSE}{n-r}$		
Total	$n-1$	SS	$\frac{SS}{n-1}$		

Numerically (concrete data)

	df	SS	MS	F	p
Treatment	4	85356	21339	43	0.00873
Error	25	124020	496		

Treatment of tennis elbow (exam V2012, 3b)

The term *tennis elbow* is used to describe a state of inflammation in the elbow, causing pain. This injury is common in people who play racquet sports, however, any activity that involves repetitive twisting of the wrist (like using a screwdriver) can lead to this condition. The condition may also be due to constant computer keyboard and mouse use.

In a randomized clinical study the aim was to compare three different methods for treatment of tennis elbow,

- A: physiotherapy intervention,
- B: corticosteroid injections and
- C: wait-and-see (the patients in the wait-and-see group did not get any treatment but was told to use the elbow as little as possible).

Treatment of tennis elbow (cont.)

We will look at the short-term effect of treatment by studying measurements at 6 weeks. All patients participating in the study only had one affected arm.
We will look at the outcome measure called *pain-free grip force*. This was measured by a digital grip dynamometer and normalized to the grip force of the unaffected arm. A pain-free grip force of 100 would mean that the affected and the unaffected arm performed equally good.

Summary statistics for each of the treatment groups.

Treatment	Sample size	Average	Standard deviation
A (physiotherapy)	63	70.2	25.4
B (injection)	65	83.6	22.9
C (wait-and-see)	60	51.8	23.0
Total	188	69.0	

We would like to investigate if the expected pain-free grip force varies between the treatment groups. Write down the null- and alternative hypothesis and perform one hypothesis test using the summary statistics in the table above.
What are the assumptions you need to make to use this test?
What is the conclusion from the test?