EXAMPLE 6 Matrices with Complex Entries

If

$$A = \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1+i & 1-2i \\ 3-2i & 8-i \end{bmatrix}, \qquad A - B = \begin{bmatrix} 1-i & -1 \\ -1+4i & -i \end{bmatrix}$$

$$iA = \begin{bmatrix} i & -i^2 \\ i+i^2 & 4i-i^2 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1+i & 1+4i \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix}\begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot i + (-i) \cdot (2-3i) & 1 \cdot (1-i) + (-i) \cdot 4 \\ (1+i) \cdot i + (4-i) \cdot (2-3i) & (1+i) \cdot (1-i) + (4-i) \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3-i & 1-5i \\ 4-13i & 18-4i \end{bmatrix}$$ ◆

EXERCISE SET 10.1

1. In each part, plot the point and sketch the vector that corresponds to the given complex number.
   (a) $2+3i$ (b) -4 (c) $-3-2i$ (d) $-5i$
2. Express each complex number in Exercise 1 as an ordered pair of real numbers.
3. In each part, use the given information to find the real numbers x and y.
   (a) $x - iy = -2 + 3i$ (b) $(x+y) + (x-y)i = 3 + i$
4. Given that $z_1 = 1 - 2i$ and $z_2 = 4 + 5i$, find
   (a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $4z_1$
   (d) $-z_2$ (e) $3z_1 + 4z_2$ (f) $\frac{1}{2}z_1 - \frac{3}{2}z_2$
5. In each part, solve for z.
   (a) $z + (1 - i) = 3 + 2i$ (b) $-5z = 5 + 10i$
   (c) $(i - z) + (2z - 3i) = -2 + 7i$
6. In each part, sketch the vectors z_1, z_2, $z_1 + z_2$, and $z_1 - z_2$.
   (a) $z_1 = 3 + i,\ z_2 = 1 + 4i$ (b) $z_1 = -2 + 2i,\ z_2 = 4 + 5i$
7. In each part, sketch the vectors z and kz.
   (a) $z = 1 + i,\ k = 2$ (b) $z = -3 - 4i,\ k = -2$ (c) $z = 4 + 6i,\ k = \frac{1}{2}$
8. In each part, find real numbers k_1 and k_2 that satisfy the equation.
   (a) $k_1 i + k_2(1 + i) = 3 - 2i$ (b) $k_1(2 + 3i) + k_2(1 - 4i) = 7 + 5i$
9. In each part, find $z_1 z_2$, z_1^2, and z_2^2.
   (a) $z_1 = 3i,\ z_2 = 1 - i$ (b) $z_1 = 4 + 6i,\ z_2 = 2 - 3i$
   (c) $z_1 = \frac{1}{3}(2 + 4i),\ z_2 = \frac{1}{2}(1 - 5i)$
10. Given that $z_1 = 2 - 5i$ and $z_2 = -1 - i$, find
    (a) $z_1 - z_1 z_2$ (b) $(z_1 + 3z_2)^2$ (c) $[z_1 + (1 + z_2)]^2$ (d) $iz_2 - z_1^2$

In Exercises 11–18 perform the calculations and express the result in the form $a + bi$.

11. $(1 + 2i)(4 - 6i)^2$
12. $(2 - i)(3 + i)(4 - 2i)$

13. $(1-3i)^3$

14. $i(1+7i)-3i(4+2i)$

15. $\left[(2+i)\left(\frac{1}{2}+\frac{3}{4}i\right)\right]^2$

16. $(\sqrt{2}+i)-i\sqrt{2}(1+\sqrt{2}i)$

17. $(1+i+i^2+i^3)^{100}$

18. $(3-2i)^2-(3+2i)^2$

19. Let

$$A=\begin{bmatrix}1 & i\\ -i & 3\end{bmatrix},\qquad B=\begin{bmatrix}2 & 2+i\\ 3-i & 4\end{bmatrix}$$

Find

(a) $A+3iB$ (b) BA (c) AB (d) B^2-A^2

20. Let

$$A=\begin{bmatrix}3+2i & 0\\ -i & 2\\ 1+i & 1-i\end{bmatrix},\qquad B=\begin{bmatrix}-i & 2\\ 0 & i\end{bmatrix},\qquad C=\begin{bmatrix}-1-i & 0 & -i\\ 3 & 2i & -5\end{bmatrix}$$

Find

(a) $A(BC)$ (b) $(BC)A$ (c) $(CA)B^2$ (d) $(1+i)(AB)+(3-4i)A$

21. Show that

(a) $\operatorname{Im}(iz)=\operatorname{Re}(z)$ (b) $\operatorname{Re}(iz)=-\operatorname{Im}(z)$

22. In each part, solve the equation by the quadratic formula and check your results by substituting the solutions into the given equation.

(a) $z^2+2z+2=0$ (b) $z^2-z+1=0$

23. (a) Show that if n is a positive integer, then the only possible values for i^n are $1, -1, i$, and $-i$.

(b) Find i^{2509}.

24. Prove: If $z_1z_2=0$, then $z_1=0$ or $z_2=0$.

25. Use the result of Exercise 24 to prove: If $zz_1=zz_2$ and $z\neq 0$, then $z_1=z_2$.

26. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1+z_2=z_2+z_1$ (b) $z_1+(z_2+z_3)=(z_1+z_2)+z_3$

27. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1z_2=z_2z_1$ (b) $z_1(z_2z_3)=(z_1z_2)z_3$

28. Prove that $z_1(z_2+z_3)=z_1z_2+z_1z_3$ for all complex numbers z_1, z_2, and z_3.

29. In quantum mechanics the ***Dirac matrices*** are

$$\beta=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\qquad \alpha_x=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},$$

$$\alpha_y=\begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{bmatrix},\qquad \alpha_z=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}$$

(a) Prove that $\beta^2=\alpha_x^2=\alpha_y^2=\alpha_z^2=I$.

(b) Two matrices A and B are called ***anticommutative*** if $AB=-BA$. Prove that any two distinct Dirac matrices are anticommutative.

30. Describe the set of all complex numbers $z=a+bi$ such that $a^2+b^2=1$. Show that if z_1, z_2 are such numbers, then so is z_1z_2.

4. Given that $z_1 = 1 - 5i$ and $z_2 = 3 + 4i$, find
   (a) z_1/z_2 (b) $\bar{z}_1/z_2$ (c) $z_1/\bar{z}_2$
   (d) $\overline{(z_1/z_2)}$ (e) $z_1/|z_2|$ (f) $|z_1/z_2|$

5. In each part, find $1/z$.
   (a) $z = i$ (b) $z = 1 - 5i$ (c) $z = \dfrac{-i}{7}$

6. Given that $z_1 = 1 + i$ and $z_2 = 1 - 2i$, find
   (a) $z_1 - \left(\dfrac{z_1}{z_2}\right)$ (b) $\dfrac{z_1 - 1}{z_2}$ (c) $z_1^2 - \left(\dfrac{iz_1}{z_2}\right)$ (d) $\dfrac{z_1}{iz_2}$

In Exercises 7–14 perform the calculations and express the result in the form $a + bi$.

7. $\dfrac{i}{1+i}$ 8. $\dfrac{2}{(1-i)(3+i)}$ 9. $\dfrac{1}{(3+4i)^2}$

10. $\dfrac{2+i}{i(-3+4i)}$ 11. $\dfrac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}$ 12. $\dfrac{1}{i(3-2i)(1+i)}$

13. $\dfrac{i}{(1-i)(1-2i)(1+2i)}$ 14. $\dfrac{1-2i}{3+4i} - \dfrac{2+i}{5i}$

15. In each part, solve for z.
   (a) $iz = 2 - i$ (b) $(4 - 3i)\bar{z} = i$

16. Use Theorem 10.2.3 to prove the following identities:
   (a) $\overline{\bar{z} + 5i} = z - 5i$ (b) $\overline{iz} = -i\bar{z}$ (c) $\overline{\dfrac{i+\bar{z}}{i-z}} = -1$

17. In each part, sketch the set of points in the complex plane that satisfies the equation.
   (a) $|z| = 2$ (b) $|z - (1+i)| = 1$ (c) $|z - i| = |z + i|$ (d) $\mathrm{Im}(\bar{z} + i) = 3$

18. In each part, sketch the set of points in the complex plane that satisfies the given condition(s).
   (a) $|z + i| \le 1$ (b) $1 < |z| < 2$ (c) $|2z - 4i| < 1$ (d) $|z| \le |z + i|$

19. Given that $z = x + iy$, find
   (a) $\mathrm{Re}(\overline{i\bar{z}})$ (b) $\mathrm{Im}(\overline{i\bar{z}})$ (c) $\mathrm{Re}(i\bar{z})$ (d) $\mathrm{Im}(i\bar{z})$

20. (a) Show that if n is a positive integer, then the only possible values for $(1/i)^n$ are $1, -1, i$, and $-i$.
   (b) Find $(1/i)^{2509}$.
   Hint See Exercise 23(b) of Section 10.1.

21. Prove:
   (a) $\dfrac{1}{2}(z + \bar{z}) = \mathrm{Re}(z)$ (b) $\dfrac{1}{2i}(z - \bar{z}) = \mathrm{Im}(z)$

22. Prove: $z = \bar{z}$ if and only if z is a real number.

23. Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, find
   (a) $\mathrm{Re}\left(\dfrac{z_1}{z_2}\right)$ (b) $\mathrm{Im}\left(\dfrac{z_1}{z_2}\right)$

24. Prove: If $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.

25. Prove that $|z| = |\bar{z}|$.

26. Prove:
   (a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (c) $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$ (d) $\bar{\bar{z}} = z$

27. (a) Prove that $\overline{z^2} = (\bar{z})^2$.
   (b) Prove that if n is a positive integer, then $\overline{z^n} = (\bar{z})^n$.
   (c) Is the result in part (b) true if n is a negative integer? Explain.

In Exercises 28–31 solve the system of linear equations by Cramer's rule.

28. $\begin{aligned} ix_1 - ix_2 &= -2 \\ 2x_1 + x_2 &= i \end{aligned}$ **29.** $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2i \end{aligned}$

30. $\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 2 + 2i \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$ **31.** $\begin{aligned} ix_1 + 3x_2 + (1+i)x_3 &= -i \\ x_1 + ix_2 + \quad 3x_3 &= -2i \\ x_1 + x_2 + \quad x_3 &= 0 \end{aligned}$

In Exercises 32 and 33 solve the system of linear equations by Gauss–Jordan elimination.

32. $\begin{bmatrix} -1 & -1-i \\ -1+i & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **33.** $\begin{bmatrix} 2 & -1-i \\ -1+i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

34. Solve the following system of linear equations by Gauss–Jordan elimination.

$$\begin{aligned} x_1 + \qquad\qquad ix_2 - \ ix_3 &= 0 \\ -x_1 + \quad (1-i)x_2 + 2ix_3 &= 0 \\ 2x_1 + (-1+2i)x_2 - 3ix_3 &= 0 \end{aligned}$$

35. In each part, use the formula in Theorem 1.4.5 to compute the inverse of the matrix, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $A = \begin{bmatrix} i & -2 \\ 1 & i \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & i \\ 1 & 0 \end{bmatrix}$

36. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial for which the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real. Prove that if z is a solution of the equation $p(z) = 0$, then so is $\bar{z}$.

37. Prove: For any complex number z, $|\text{Re}(z)| \le |z|$ and $|\text{Im}(z)| \le |z|$.

38. Prove that

$$\frac{|\text{Re}(z)| + |\text{Im}(z)|}{\sqrt{2}} \le |z|$$

Hint Let $z = x + iy$ and use the fact that $(|x| - |y|)^2 \ge 0$.

39. In each part, use the method of Example 4 in Section 1.5 to find A^{-1}, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $A = \begin{bmatrix} 1 & 1+i & 0 \\ 0 & 1 & i \\ -i & 1-2i & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} i & 0 & -i \\ 0 & 1 & -1-4i \\ 2-i & i & 3 \end{bmatrix}$

40. Show that $|z - 1| = |\bar{z} - 1|$. Discuss the geometric interpretation of the result.

41. (a) If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, find $|z_1 - z_2|$ and interpret the result geometrically.

(b) Use part (a) to show that the complex numbers 12, $6 + 2i$, and $8 + 8i$ are vertices of a right triangle.

42. Use Theorem 10.2.3 to show that if the coefficients a, b, and c in a quadratic polynomial are real, then the solutions of the equation $az^2 + bz + c = 0$ are complex conjugates. What can you conclude if a, b, and c are complex?

10.3 POLAR FORM OF A COMPLEX NUMBER

In this section we shall discuss a way to represent complex numbers using trigonometric properties. Our work will lead to an important formula for powers of complex numbers and to a method for finding nth roots of complex numbers.

Polar Form

If $z = x + iy$ is a nonzero complex number, $r = |z|$, and θ measures the angle from the positive real axis to the vector z, then, as suggested by Figure 10.3.1,

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

or, equivalently,

$$\overline{z} = re^{-i\theta} \tag{14}$$

In the special case where $r = 1$, the polar form of z is $z = e^{i\theta}$, and (14) yields the formula

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{15}$$

EXERCISE SET 10.3

1. In each part, find the principal argument of z.
   (a) $z = 1$ (b) $z = i$ (c) $z = -i$
   (d) $z = 1 + i$ (e) $z = -1 + \sqrt{3}i$ (f) $z = 1 - i$

2. In each part, find the value of $\theta = \arg(1 - \sqrt{3}i)$ that satisfies the given condition.
   (a) $0 < \theta \leq 2\pi$ (b) $-\pi < \theta \leq \pi$ (c) $-\frac{\pi}{6} \leq \theta < \frac{11\pi}{6}$

3. In each part, express the complex number in polar form using its principal argument.
   (a) $2i$ (b) -4 (c) $5 + 5i$
   (d) $-6 + 6\sqrt{3}i$ (e) $-3 - 3i$ (f) $2\sqrt{3} - 2i$

4. Given that $z_1 = 2(\cos \pi/4 + i \sin \pi/4)$ and $z_2 = 3(\cos \pi/6 + i \sin \pi/6)$, find a polar form of
   (a) $z_1 z_2$ (b) $\frac{z_1}{z_2}$ (c) $\frac{z_2}{z_1}$ (d) $\frac{z_1^5}{z_2^2}$

5. Express $z_1 = i$, $z_2 = 1 - \sqrt{3}i$, and $z_3 = \sqrt{3} + i$ in polar form, and use your results to find $z_1 z_2 / z_3$. Check your results by performing the calculations without using polar forms.

6. Use Formula (6) to find
   (a) $(1 + i)^{12}$ (b) $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{-6}$ (c) $(\sqrt{3} + i)^7$ (d) $(1 - i\sqrt{3})^{-10}$

7. In each part, find all the roots and sketch them as vectors in the complex plane.
   (a) $(-i)^{1/2}$ (b) $(1 + \sqrt{3}i)^{1/2}$ (c) $(-27)^{1/3}$
   (d) $(i)^{1/3}$ (e) $(-1)^{1/4}$ (f) $(-8 + 8\sqrt{3}i)^{1/4}$

8. Use the method of Example 4 to find all cube roots of 1.

9. Use the method of Example 4 to find all sixth roots of 1.

10. Find all square roots of $1 + i$ and express your results in polar form.

11. Find all solutions of the equation $z^4 - 16 = 0$.

12. Find all solutions of the equation $z^4 + 8 = 0$ and use your results to factor $z^4 + 8$ into two quadratic factors with real coefficients.

13. It was shown in the text that multiplying z by i rotates z counterclockwise by 90°. What is the geometric effect of dividing z by i?

14. In each part, use (6) to calculate the given power.
    (a) $(1 + i)^8$ (b) $(-2\sqrt{3} + 2i)^{-9}$

15. In each part, find $\text{Re}(z)$ and $\text{Im}(z)$.
    (a) $z = 3e^{i\pi}$ (b) $z = 3e^{-i\pi}$ (c) $\overline{z} = \sqrt{2}e^{\pi i/2}$ (d) $\overline{z} = -3e^{-2\pi i}$

16. (a) Show that the values of $z^{1/n}$ in Formula (10) are all different.
    (b) Show that integer values of k other than $k = 0, 1, 2, \ldots, n - 1$ produce values of $z^{1/n}$ that are duplicates of those in Formula (10).

17. Show that Formula (7) is valid if $n = 0$ or n is a negative integer.

18. **(For Readers Who Have Studied Calculus)** To prove Formula (11), recall that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

(a) By substituting $x = i\theta$ in this series and simplifying, show that

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

(b) Use the result in part (a) to obtain Formula (11).

19. Derive Formula (5).

20. When $n = 2$ and $n = 3$, Equation (7) gives

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Use these two equations to obtain trigonometric identities for $\cos 2\theta$, $\sin 2\theta$, $\cos 3\theta$, and $\sin 3\theta$.

21. Use Formula (11) to show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

22. Show that if $(a + bi)^3 = 8$, then $a^2 + b^2 = 4$.

23. Show that Formula (6) is valid for negative integer exponents if $z \neq 0$.

10.4 COMPLEX VECTOR SPACES

In this section we shall develop the basic properties of vector spaces with complex scalars and discuss some of the ways in which they differ from real vector spaces. However, before going farther, the reader should review the vector space axioms given in Section 5.1.

Basic Properties

Recall that a vector space in which the scalars are allowed to be complex numbers is called a ***complex vector space***. Linear combinations of vectors in a complex vector space are defined exactly as in a real vector space except that the scalars are allowed to be complex numbers. More precisely, a vector $\mathbf{w}$ is called a ***linear combination*** of the vectors of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are complex numbers.

The notions of ***linear independence***, ***spanning***, ***basis***, ***dimension***, and ***subspace*** carry over without change to complex vector spaces, and the theorems developed in Chapter 5 continue to hold with R^n changed to C^n.

Among the real vector spaces the most important one is R^n, the space of n-tuples of real numbers, with addition and scalar multiplication performed coordinatewise. Among the complex vector spaces the most important one is C^n, the space of n-tuples of complex numbers, with addition and scalar multiplication performed coordinatewise. A vector $\mathbf{u}$ in C^n can be written either in vector notation,

$$\mathbf{u} = (u_1, u_2, \ldots, u_n)$$

or in matrix notation,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$