MA3203 - Problem sheet 4

Problem 1. Let $\Lambda = k\Gamma$ for a field k, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Find the projective covers and the kernel of the projective covers of the following representations :

(1) $k \xrightarrow{0} k \xrightarrow{0} 0$.

(2) $k \xrightarrow{1} k \xrightarrow{0} k$.

(3) $k^2 \xrightarrow{(1\ 0)} k \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} k^2$.

Problem 2. Given $\Lambda = k\Gamma/\langle\rho\rangle$ for a field k, where Γ is the quiver:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\gamma"'] & 3 \end{tikzcd}$$

and the relations $\rho = \{\beta\alpha\}$. Find the projective covers and the kernel of the projective covers of the following representations:

(1) $\begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, shift left, "0"] \arrow[r, shift right, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & k^2 \end{tikzcd}$.

(2) $\begin{tikzcd} k \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & k^2 \arrow[r, shift left, "(1\ 0)"] \arrow[r, shift right, "(1\ 1)"'] & k \end{tikzcd}$.

(3) $\begin{tikzcd} 0 \arrow[r, "0"] & k^2 \arrow[r, shift left, "\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)"] \arrow[r, shift right, "\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)"'] & k \end{tikzcd}$.

Problem 3. Let $f\colon A \to B$ and $g\colon B \to C$ be two essential epimorphisms of left Λ-modules. Show that gf is an essential epimorphism.

Problem 5.

(i) Consider the following commutative diagram with exact rows in $\operatorname{Mod}\Lambda$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, "g"] & C \arrow[d, "h"] \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

Show that there exists an $f\colon A \to A'$ such that the diagram is commutative. Also show that if g and h are isomorphisms, then f is also an isomorphism.

(ii) Consider now the following commutative diagram with exact rows in $\operatorname{Mod}\Lambda$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & 0 \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

Show that there exists an $h : C \to C'$ such that the diagram is commutative. Also show that if f and g are isomorphisms, then h is also an isomorphism.

Problem 5. Let Λ be an artin algebra, M a finitely generated Λ-module and P an indecomposable projective Λ-module.

Show that $\mathrm{Hom}_\Lambda(P, M) \neq (0)$ if and only if $P/\mathfrak{r}P$ is a composition factor of M.

Problem 6. Let Λ be an artin algebra and S a simple Λ-module. Let e be a primitive idempotent in Λ.

Show that there is a projective cover $\Lambda e \longrightarrow S$ if and only if $eS \neq 0$.