MA3203 - Problem sheet 5

Problem 1. Given a ring Λ, let $F, G\colon \operatorname{mod}\Lambda \to Ab$ be two functors, F covariant and G contravariant. Then we have the following definitions:

(i) If $0 \to F(A) \to F(B) \to F(C)$ ($G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C$ is, we say that F is *left exact* functor (G is *right exact* functor).

(ii) If $F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C$ is, we say that F is *right exact* functor (G is *left exact* functor).

(iii) If $0 \to F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C \to 0$ is, we say that F is *exact* functor (G is *exact* functor).

Let X be in $\operatorname{mod}\Lambda$.

(a) Show that the functor $F = \operatorname{Hom}_\Lambda(X, -)$ is a left exact (covariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(X, -)$ is an exact functor if and only if X is a projective Λ-module.

(b) Show that the functor $F = \operatorname{Hom}_\Lambda(-, X)$ is a left exact (contravariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(-, X)$ is an exact functor if and only if X is an injective Λ-module.

Problem 2. Let Λ be an artin R-algebra. Assume that

$$\Lambda \simeq P_1^{n_1} \oplus \cdots \oplus P_t^{n_t}$$

where P_i is indecomposable, $P_i \not\simeq P_j$ for $i \neq j$, and $n_i > 0$. Let $\Sigma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$, which is basic.

Show that $e_P\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Sigma$ is an equivalence of R-categories.

Hints:

(1) Observe that $e_P\colon \operatorname{add} P \to \mathcal{P}$ is an equivalence (Proposition 42 (c)).

(2) e_P dense: Given C in $\operatorname{mod}\Sigma$, there exists an exact sequence $F_1 \xrightarrow{f} F_0 \to C \to 0$. Use (1) to show that f up to isomorphisms are given as $e_P(f')$ for some $f'\colon F_1' \to F_0'$ for F_i' in $\operatorname{add} P$. Furthermore, show that $C \simeq e_P(\operatorname{Coker} f')$.

(3) e_P full and faitful: Given X in $\operatorname{mod}\Lambda$, there exists and exact sequence $\eta\colon P_1 \to P_0 \to X \to 0$ with P_i in $\operatorname{add} P$. Then $e_P(\eta)\colon e_P(P_1) \to e_P(P_0) \to e_P(X) \to 0$ is also exact. Apply $\operatorname{Hom}_\Lambda(-, Y)$ to the exact sequence η and apply $\operatorname{Hom}_\Sigma(-, e_P(Y))$ to the exact sequence $e_P(\eta)$, and then compare them. Using (1) conclude that e_P is full and faithful.