

MA3203 - PROBLEM SHEET 2

Problem 1. Let k be a field. Find the representations corresponding to the modules Λe_i for the different possible values of i and for the different cases of Λ listed below.

- (a) $\Lambda = k\Gamma$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

- (b) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\rho = \{\beta\alpha\}$.

- (c) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \begin{matrix} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{matrix} 3$$

and $\rho = \{\beta\alpha\}$.

- (d) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} 2 \begin{matrix} \xrightarrow{\gamma} \\ \xrightarrow{\gamma} \end{matrix} 2$$

and $\rho = \{\gamma\alpha, \gamma^3\}$.

Problem 2. Find a composition series for the following representations:

- (a) Λe_1 where Λ is as in (c) above.
 (b) Λe_1 where Λ is as in (d) above.

Problem 3.

- (a) Given a ring Λ . Show that a Λ -module M is decomposable if and only if its endomorphism ring $\text{End}_{\Lambda}(M) = \{f: M \rightarrow M \mid f \text{ } \Lambda\text{-homomorphism}\}$ contains a nontrivial idempotent (i.e. there is an f in $\text{End}_{\Lambda}(M)$ such that $f^2 = f$ and $f \neq 0, 1$).
- (b) Use (a) to show that Λe_1 where Λ is as in (b) in Problem 1 is indecomposable. Here we will use without proof that the endomorphism ring of the module Λe_1 is isomorphic to the endomorphism ring of the representation corresponding to Λe_1 .
- (c) Given a ring Λ with an idempotent e . Show that the endomorphism ring $\text{End}_{\Lambda}(\Lambda e)^{\text{op}}$ is isomorphic to $e\Lambda e$.

Let $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is a quiver with vertices $\{1, \dots, n\}$ and ρ is a set of relations. Assume that $J^t \subset \langle\rho\rangle \subset J^2$ for some t . Conclude (using (a)) that Λe_i is indecomposable for each i .

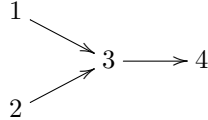
- (d) Given a ring Λ and two simple Λ -modules S and S' . Show that if $f: S \rightarrow S'$ is a nonzero Λ -homomorphism, then f is an isomorphism.

Problem 4. Let Γ be the quiver with relations as in (b) in Problem 1, and let V be its representation over k given by: $V(1) = k$, $V(2) = k^2$, $V(3) = k^2$, $f_\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $f_\beta = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

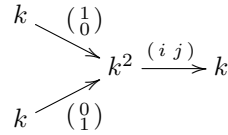
Determine if V is decomposable, and if it is, find its decomposition into a direct sum of indecomposable representations.

Furthermore, find a composition series for V .

Problem 5. Let k be a field and let Γ be the quiver



For an ordered pair (i, j) of elements in k , let M_{ij} be the representation given by



- determine for which (i, j) the representation M_{ij} is indecomposable and for which (i, j) it decomposes.
- Prove that if M_{ij} and M_{rs} are indecomposable then they are isomorphic.
Is the same true if M_{ij} and M_{rs} decomposes?

Problem 6. Let Λ_c be the algebra over \mathbb{C} with basis $\{e_0, e_1, e_2, e_3\}$ over \mathbb{C} , where c is a given complex number. The multiplication is given by the following multiplication table:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	e_3	0
e_2	e_2	$-e_3$	ce_3	0
e_3	e_3	0	0	0

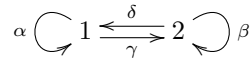
For which c and c' are the algebras Λ_c and $\Lambda_{c'}$ isomorphic?

Challenge 1. Find a quiver with relations ρ_c over \mathbb{C} such that $\Lambda_c \cong \mathbb{C}\Gamma/\langle\rho_c\rangle$.

Challenge 2. Show that there exists an infinite number of non-isomorphic indecomposable modules over Λ_c for any value of c in \mathbb{C} .

Challenge 3. Show that there exists an infinite number of non-isomorphic indecomposable modules over Λ_c for any value of c in \mathbb{C} .

Problem 7. Let k be a field and Γ the quiver



with relations $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$.

- Show that the dimension of $k\Gamma/\langle\rho\rangle$ over k is 12.
- Show that the subspace of $k\Gamma/\langle\rho\rangle$ spanned by $\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2$ is a ring which is isomorphic to $M_2(k)$ -ring of 2×2 -matrices over k .

Problem 8. We say that a ring Λ is local if the nonunits of Λ (elements in Λ without multiplicative invers) form an ideal in Λ .

Show that if Λ is local, then 0 and 1 are the only idempotents in Λ .