MA3203 - Problem sheet 4

Problem 1. Let $\Lambda = k\Gamma$ for a field k, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Find the projective covers and the kernel of the projective covers of the following representations :

(1) $k \xrightarrow{0} k \xrightarrow{0} 0$.

(2) $k \xrightarrow{1} k \xrightarrow{0} k$.

(3) $k^2 \xrightarrow{(1\ 0)} k \xrightarrow{\binom{1}{1}} k^2$.

Problem 2. Given $\Lambda = k\Gamma/\langle\rho\rangle$ for a field k, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$$

and the relations $\rho = \{\beta\alpha\}$. Find the projective covers and the kernel of the projective covers of the following representations:

(1) $k \xrightarrow{1} k \underset{\binom{0}{1}}{\overset{0}{\rightrightarrows}} k^2$.

(2) $k \xrightarrow{\binom{0}{1}} k^2 \underset{(1\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} k$.

(3) $0 \xrightarrow{0} k^2 \underset{\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\rightrightarrows}} k$.

Problem 3. Let $f\colon A \to B$ and $g\colon B \to C$ be two essential epimorphisms of left Λ-modules. Show that gf is an essential epimorphism.

Problem 4. Let (Γ, ρ) be a quiver with relations such that $J^t \subseteq \langle\rho\rangle \subseteq J^2$ for some $t \geq 2$. Let $\Lambda = k\Gamma/\langle\rho\rangle$. Let $F\colon \operatorname{mod}\Lambda \to \operatorname{Rep}(\Gamma,\rho)$ be the equivalence defined in the lectures. Show that M in $\operatorname{mod}\Lambda$ is indecomposable if and only if $F(M)$ is.

Problem 5.

(i) Consider the following commutative diagram with exact rows in $\operatorname{Mod}\Lambda$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

Show that there exists a unique $f\colon A \to A'$ such that the diagram is commutative. Also show that if g and h are isomorphisms, then f is also an isomorphism.

(ii) Consider now the following commutative diagram with exact rows in $\operatorname{Mod}\Lambda$:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Show that there exists a unique $h : C \to C'$ such that the diagram is commutative. Also show that if f and g are isomorphisms, then h is also an isomorphism.

Problem 6. Let Λ be an artin algebra, M a finitely generated Λ-module and P an indecomposable projective Λ-module.

Show that $\operatorname{Hom}_\Lambda(P, M) \neq (0)$ if and only if $P/\mathfrak{r}P$ is a composition factor of M.

Problem 7. Let Λ be an artin algebra and S a simple Λ-module. Let e be a primitive idempotent in Λ.

Show that there is a projective cover $\Lambda e \longrightarrow S$ if and only if $eS \neq 0$.