MA3203 - Problem sheet 5

Problem 1. Given a ring Λ, let $F, G\colon \operatorname{mod}\Lambda \longrightarrow Ab$ be two functors, F covariant and G contravariant.

(a) If $0 \to F(A) \to F(B) \to F(C)$ ($G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C$ is, we say that F is *left exact* functor (G is *right exact* functor).

(b) If $F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A)$) is an exact sequence whenever $A \to B \to C \to 0$ is, we say that F is *right exact* functor (G is *left exact* functor).

(c) If $0 \to F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C \to 0$ is, we say that F is *exact* functor (G is *exact* functor).

Let $X \in \operatorname{mod}\Lambda$.

(1) Show that the functor $F = \operatorname{Hom}_\Lambda(X, \ )$ is a left exact (covariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(X, \ )$ is an exact functor if and only if X is a projective Λ-module.

(2) Show that the functor $F = \operatorname{Hom}_\Lambda(\ , X)$ is a left exact (contravariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(\ , X)$ is an exact functor if and only if X is an injective Λ-module.

Problem 2. Let $\Lambda = k\Gamma$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Find the indecomposable injective Λ-modules and find the socles and injective envelopes of the following representations:

(1) $k \xrightarrow{0} k \xrightarrow{0} 0$

(2) $k \xrightarrow{1} k \xrightarrow{0} 0$

(3) $k^2 \xrightarrow{(1\ 0)} k \xrightarrow{\binom{1}{1}} k^2$

Problem 3. Let $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3,$$

$\rho = \{\beta\alpha\}$ and k is a field. Find all indecomposable injective Λ-modules and find the socles and injective envelopes of the following representations :

(1) $k \xrightarrow{1} k \underset{\binom{0}{1}}{\overset{0}{\rightrightarrows}} k^2$.

(2) $k \xrightarrow{\binom{0}{1}} k^2 \underset{(1\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} k$.

(3) $0 \xrightarrow{0} k^2 \underset{\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)}{\rightrightarrows}} k^2$.

Problem 4. Let $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

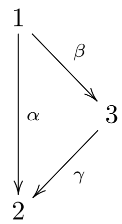
$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2,$$

$\rho = \{\alpha\beta\}$ and k is a field. Find the injective envelopes of the simple Λ-modules. Also, for each indecomposable module I_i, find representations of (Γ, ρ) corresponding to $I_i/\operatorname{soc} I_i$ for each i.

Problem 5. Let Λ be an artin algebra, I an indecomposable injective and M an arbitrary module in $\operatorname{mod}\Lambda$.

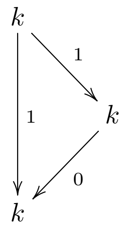
Prove that $\operatorname{Hom}_\Lambda(M, I) \neq (0)$ if and only if $\operatorname{soc} I$ is a composition factor of M.

Problem 6 (Challenge). Let k be a field, Γ the quiver

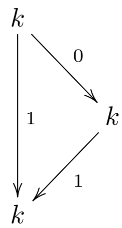


and $\Lambda = k\Gamma$.

Let M be the representation



and let N be the representation



(a) Find the radical and the socle of M and N.

(b) Given a ring R and a left R-module A, we define the annihilator of A by $\operatorname{Ann}_R(A) = \{r \in R \mid ra = 0, \forall a \in A\}$. It is a two-sided ideal in R.

Find the annihilator, $\operatorname{Ann}_\Lambda(M)$ and $\operatorname{Ann}_\Lambda(N)$, of M and N, respectively.

(c) Prove that M is a projective $(\Lambda/\operatorname{Ann}_\Lambda(M))$-module and that N is an injective $(\Lambda/\operatorname{Ann}\Lambda(N))$-module.

(d) **Challenge:** For a general artin algebra Λ, show that

$$\operatorname{soc} M = \{m \in M \mid \mathfrak{r}m = (0)\},$$

where $\mathfrak{r}$ is the radical of Λ.