MA3203 - Problem sheet 1

Problem 1.

(a) Given a quiver $\Gamma\colon\ 1 \longrightarrow 2 \longrightarrow 3$ and a field k, let $\Lambda = k\Gamma$. Let S be the k-algebra $\left(\begin{smallmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{smallmatrix}\right)$. Show that Λ and S are isomorphic as k-algebras.

(b) Given the representation $(V, f)\colon\ k^2 \xrightarrow{(1\ 1)} k \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} k^2$ and the representation $(V', f')\colon\ k \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} k^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} k^2$ of Γ over k, describe homomorphisms from (V, f) to (V', f').

(c) Describe homomorphisms from (V', f') to (V, f) where (V, f) and (V', f') are as in (b).

(d) We say that a representation (V, f) of some quiver Γ over some field k is a subrepresentation of a representation (V', f') of Γ over k if there is a monomorphism from (V, f) to (V', f').

Let (V', f') be the representation $k \xrightarrow{1} k \xrightarrow{1} k$ of the quiver Γ from (a) over the field k. Find all subrepresentations of (V', f').

(e) Vi say that a representation (V, f) of some quiver Γ over some field k is a factor of a representation (V', f') of Γ over k if there is an epimorphism from (V', f') to (V, f).

Let (V', f') be as in (d). Find all factor-representations of (V', f').

Problem 2.

(a) Given the quiver $\Gamma\colon\ 1 \circlearrowright \alpha$ with a relation $\sigma = \alpha^2$ and a field k, let $\Lambda = k\Gamma/\langle \alpha^2 \rangle$. Show that Λ and $k[x]/(x^2)$ are isomorphic as k-algebras.

(b) Find all homomorphisms from the representation $(V, f)\colon\ k^3 \circlearrowright \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ to itself.

(c) Find all indecomposable representations of (Γ, σ) over k. Hint: Only two!

Problem 3. Let $R = \left(\begin{smallmatrix} k & 0 \\ k[x]/(x^2) & k \end{smallmatrix}\right) \subset \left(\begin{smallmatrix} k[x]/(x^2) & 0 \\ k[x]/(x^2) & k[x]/(x^2) \end{smallmatrix}\right)$, where k is a field and addition and multiplication are the usual addition and multiplication of matrices.

(a) Show that R is a ring, and that it is an algebra over k when $\alpha \cdot r = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) r$ for $\alpha \in k$ and $r \in R$. Is R a left artinian ring?

(b) Let $I = \left(\begin{smallmatrix} 0 & 0 \\ k[x]/(x^2) & 0 \end{smallmatrix}\right)$. Show that I is an ideal in R. Is R a semisimple ring? Show that $R/I \cong k \bigoplus k$ as rings.

(c) Show that R is isomorphic to the path algebra of the quiver $\Gamma\colon\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$.

(d) Let M_λ be the representation of the quiver Γ over k given by $V(1) = k, V(2) = k, f_\alpha = 1$ and $f_\beta = \lambda$ (i.e. $f_\alpha(a) = a, f_\beta(a) = \lambda a, \forall a \in k$) where $\lambda \in k$.

Show that $M_\lambda \not\cong M_{\lambda'}$ if $\lambda \neq \lambda'$.

(e) Show that M_λ is indecomposable for each λ in k.