MA3203 - Problem sheet 2

Problem 1. Let k ba a field. Find the representations corresponding to the modules Λe_i for the different possible values of i and for the different cases of Λ listed below.

(a) $\Lambda = k\Gamma$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

(b) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\rho = \{\beta\alpha\}$.

(c) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3$$

and $\rho = \{\beta\alpha\}$.

(d) $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver:

$$1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \circlearrowleft \gamma$$

and $\rho = \{\gamma\alpha, \gamma^3\}$.

Problem 2. Find a composition series for the following representations:

(a) Λe_1 where Λ is as in (c) above.

(b) Λe_1 where Λ is as in (d) above.

Problem 3.

(a) Given a ring Λ. Show that a Λ-module M is decomposable if and only if its endomorphism ring $\operatorname{End}_\Lambda(M) = \{f\colon M \to M \mid f\ \Lambda\text{-homomorphism}\}$ contains a nontrivial idempotent (i.e. there is an f in $\operatorname{End}_\Lambda(M)$ such that $f^2 = f$ and $f \neq 0, 1$).

(b) Use (a) to show that Λe_1 where Λ is as in (b) in Problem 1 is indecomposable.

(c) Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is a quiver with vertices $\{1, \ldots, n\}$ and ρ is a set of relations. Assume that $J^t \subset \langle\rho\rangle \subset J^2$ for some t.

Show that the endomorphism ring $\operatorname{End}_\Lambda(\Lambda e_i)^{\mathrm{op}}$ is isomorphic to $e_i\Lambda e_i$. Conclude (using (a)) that Λe_i is indecomposable for each i.

(d) Given a ring Λ and two simple Λ-modules S and S'. Show that if $f\colon S \to S'$ is a nonzero Λ-homomorphism, then f is an isomorphism.

Problem 4. Let Γ be the quiver with relations as in (b) in Problem 1, and let V be its representation over k given by: $V(1) = k$, $V(2) = k^2$, $V(3) = k^2$, $f_\alpha = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $f_\beta = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$.

Determine if V is decomposable, and if it is, find its decomposition into a direct sum of indecomposable representations.

Furthermore, find a composition series for V.