# MA3203 - Problem sheet 2

**Problem 1.** Let $k$ ba a field. Find the representations corresponding to the modules $\Lambda e_i$ for the different possible values of $i$ and for the different cases of $\Lambda$ listed below.

(a) $\Lambda = k\Gamma$, where $\Gamma$ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

(b) $\Lambda = k\Gamma/\langle\rho\rangle$, where $\Gamma$ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\rho = \{\beta\alpha\}$.

(c) $\Lambda = k\Gamma/\langle\rho\rangle$, where $\Gamma$ is the quiver:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\gamma"'] & 3 \end{tikzcd}$$

and $\rho = \{\beta\alpha\}$.

(d) $\Lambda = k\Gamma/\langle\rho\rangle$, where $\Gamma$ is the quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

and $\rho = \{\gamma\alpha, \gamma^3\}$.

**Problem 2.** Find a composition series for the following representations:

(a) $\Lambda e_1$ where $\Lambda$ is as in (c) above.
(b) $\Lambda e_1$ where $\Lambda$ is as in (d) above.

**Problem 3.**

(a) Given a ring $\Lambda$. Show that a $\Lambda$-module $M$ is decomposable if and only if its endomorphism ring $\operatorname{End}_\Lambda(M) = \{f \colon M \to M \mid f \ \Lambda\text{-homomorphism}\}$ contains a nontrivial idempotent (i.e. there is an $f$ in $\operatorname{End}_\Lambda(M)$ such that $f^2 = f$ and $f \neq 0, 1$).

(b) Use (a) to show that $\Lambda e_1$ where $\Lambda$ is as in (b) in Problem 1 is indecomposable.

(c) Given $\Lambda = k\Gamma/\langle\rho\rangle$, where $\Gamma$ is a quiver with vertices $\{1, \ldots, n\}$ and $\rho$ is a set of relations. Assume that $J^t \subset \langle\rho\rangle \subset J^2$ for some $t$.

Show that the endomorphism ring $\operatorname{End}_\Lambda(\Lambda e_i)^{\mathrm{op}}$ is isomorphic to $e_i\Lambda e_i$. Conclude (using (a)) that $\Lambda e_i$ is indecomposable for each $i$.

(d) Given a ring $\Lambda$ and two simple $\Lambda$-modules $S$ and $S'$. Show that if $f \colon S \to S'$ is a nonzero $\Lambda$-homomorphism, then $f$ is an isomorphism.

**Problem 4.** Let $\Gamma$ be the quiver with relations as in (b) in Problem 1, and let $V$ be its representation over $k$ given by: $V(1) = k$, $V(2) = k^2$, $V(3) = k^2$, $f_\alpha = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $f_\beta = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$.

Determine if $V$ is decomposable, and if it is, find its decomposition into a direct sum of indecomposable representations.

Furthermore, find a composition series for $V$.

**Problem 5.** Let $k$ be a field and let $\Gamma$ be the quiver

$$\begin{tikzcd}[row sep=small]
1 \arrow[dr] & & \\
& 3 \arrow[r] & 4 \\
2 \arrow[ur] & &
\end{tikzcd}$$

For an ordered pair $(i, j)$ of elements in $k$, let $M_{ij}$ be the representation given by

$$\begin{tikzcd}[row sep=small]
k \arrow[dr, "\binom{1}{0}"] & & \\
& k^2 \arrow[r, "(i\ j)"] & k \\
k \arrow[ur, "\binom{0}{1}"'] & &
\end{tikzcd}$$

(a) determine for which $(i, j)$ the representation $M_{ij}$ is indecomposable and for which $(i, j)$ it decomposes.
(b) Prove that if $M_{ij}$ and $M_{rs}$ are indecomposable then they are isomorphic. Is the same true if $M_{ij}$ and $M_{rs}$ decomposes?

**Problem 6.** Let $\Lambda_c$ be the algebra over $\mathbb{C}$ with basis $\{e_0, e_1, e_2, e_3\}$ over $\mathbb{C}$, where $c$ is a given complex number. The multiplication is given by the following multiplication table:

| | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
| $e_1$ | $e_1$ | $e_3$ | $e_3$ | 0 |
| $e_2$ | $e_2$ | $-e_3$ | $ce_3$ | 0 |
| $e_3$ | $e_3$ | 0 | 0 | 0 |

For which $c$ and $c'$ are the algebras $\Lambda_c$ and $\Lambda_{c'}$ isomorphic?

*Challenge 1.* Find a quiver with relations $\rho_c$ over $\mathbb{C}$ such that $\Lambda_c \cong \mathbb{C}\Gamma/\langle\rho_c\rangle$.

*Challenge 2.* Show that there exists an infinite number of non-isomorphic indecomposable modules over $\Lambda_c$ for any value of $c$ in $\mathbb{C}$.

**Problem 7.** Let $k$ be a field and $\Gamma$ the quiver

$$\begin{tikzcd}
1 \arrow[loop left, "\alpha"] \arrow[r, shift right, "\gamma"'] & 2 \arrow[l, shift right, "\delta"'] \arrow[loop right, "\beta"]
\end{tikzcd}$$

with relations $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$.

(a) Show that the dimension of $k\Gamma/\langle\rho\rangle$ over $k$ is 12.
(b) Show that the subspace of $k\Gamma/\langle\rho\rangle$ spanned by $\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2$ is a ring which is isomorphic to $M_2(k)$-ring of $2 \times 2$-matrices over $k$.

**Problem 8.** We say that a ring $\Lambda$ is local if the nonunits of $\Lambda$ (elements in $\Lambda$ without multiplicative invers) form an ideal in $\Lambda$.

Show that if $\Lambda$ is local, then 0 and 1 are the only idempotents in $\Lambda$.