MA3203 - Problem sheet 3

Problem 1. Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver $1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$ with relations $\rho = \{\beta\alpha\}$ and k is a field. Find the radicals and tops of representations of Λe_i for different possible values of i.

Problem 2. Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with relations $\rho = \{\alpha\beta\}$ and k is a field. Let J be the ideal in $k\Gamma$ generated by the arrows.

(a) Show that there is some t such that $J^t \subset \langle\rho\rangle \subset J^2$. What is the dimension of Λ over k?

(b) Find the representations of Λe_i for different possible values of i and find their radicals and tops.

(c) Find the radical of Λ.

Problem 3. Let $\mathfrak{r}$ be the radical of a ring Λ, and let

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence of Λ-modules and homomorphisms (i.e. $\operatorname{Im} f = \operatorname{Ker} g$ and g is onto; f does not need to be mono).

Show that the sequence

$$A/\mathfrak{r}A \xrightarrow{\bar{f}} B/\mathfrak{r}B \xrightarrow{\bar{g}} C/\mathfrak{r}C \to 0$$

also is exact, where the maps $\bar{f}$ and $\bar{g}$ are induced by $\bar{f}(a + \mathfrak{r}A) = f(a) + \mathfrak{r}B$ and $\bar{g}(b) = g(b) + \mathfrak{r}C$.

Problem 4. Let k be a field and let Γ be the quiver

$$\begin{array}{ccccc} 1 & & & & \\ & \searrow & & & \\ & & 3 & \longrightarrow & 4 \\ & \nearrow & & & \\ 2 & & & & \end{array}$$

For an ordered pair (i, j) of elements in k, let M_{ij} be the representation given by

$$\begin{array}{ccccc} k & & & & \\ & \searrow^{\binom{1}{0}} & & & \\ & & k^2 & \xrightarrow{(i\ j)} & k \\ & \nearrow_{\binom{0}{1}} & & & \\ k & & & & \end{array}$$

(a) determine for which (i, j) the representation M_{ij} is indecomposable and for which (i, j) it decomposes.

(b) Prove that if M_{ij} and M_{rs} are indecomposable then they are isomorphic.
Is the same true if M_{ij} and M_{rs} decomposes?

Problem 5. Let Λ_c be the algebra over $\mathbb{C}$ with basis $\{e_0, e_1, e_2, e_3\}$ over $\mathbb{C}$, where c is a given complex number. The multiplication is given by the following multiplication table:

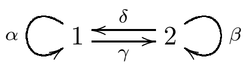
	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_3	e_3	0
e_2	e_2	$-e_3$	ce_3	0
e_3	e_3	0	0	0

For which c and c' are the algebras Λ_c and $\Lambda_{c'}$ isomorphic?

Challenge 1. Find a quiver with relations ρ_c over $\mathbb{C}$ such that $\Lambda_c \cong \mathbb{C}\Gamma/\langle \rho_c \rangle$.

Challenge 2. Show that there exists an infinite number of non-isomorphic indecomposable modules over Λ_c for any value of c in $\mathbb{C}$.

Problem 6. Let k be a field and Γ the quiver



with relations $\rho = \{\delta\gamma - \alpha^2, \alpha^3 - \alpha^2, \gamma\delta - \beta^2, \beta^3 - \beta^2, \alpha\delta - \delta\beta, \gamma\alpha - \beta\gamma\}$.

(a) Show that the dimension of $k\Gamma/\langle \rho \rangle$ over k is 12.

(b) Show that the subspace of $k\Gamma/\langle \rho \rangle$ spanned by $\alpha^2, \gamma\alpha^2, \alpha^2\delta, \beta^2$ is a ring which is isomorphic to $M_2(k)$-ring of 2×2-matrices over k.

Problem 7. We say that a ring Λ is local if the nonunits of Λ (elements in Λ without multiplicative invers) form an ideal in Λ.

Show that if Λ is local, then 0 and 1 are the only idempotents in Λ.